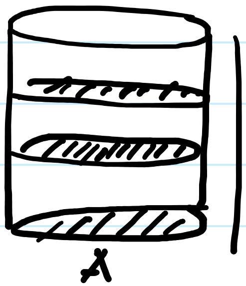
6.2. Volumes

Building Blocks: Cylinders



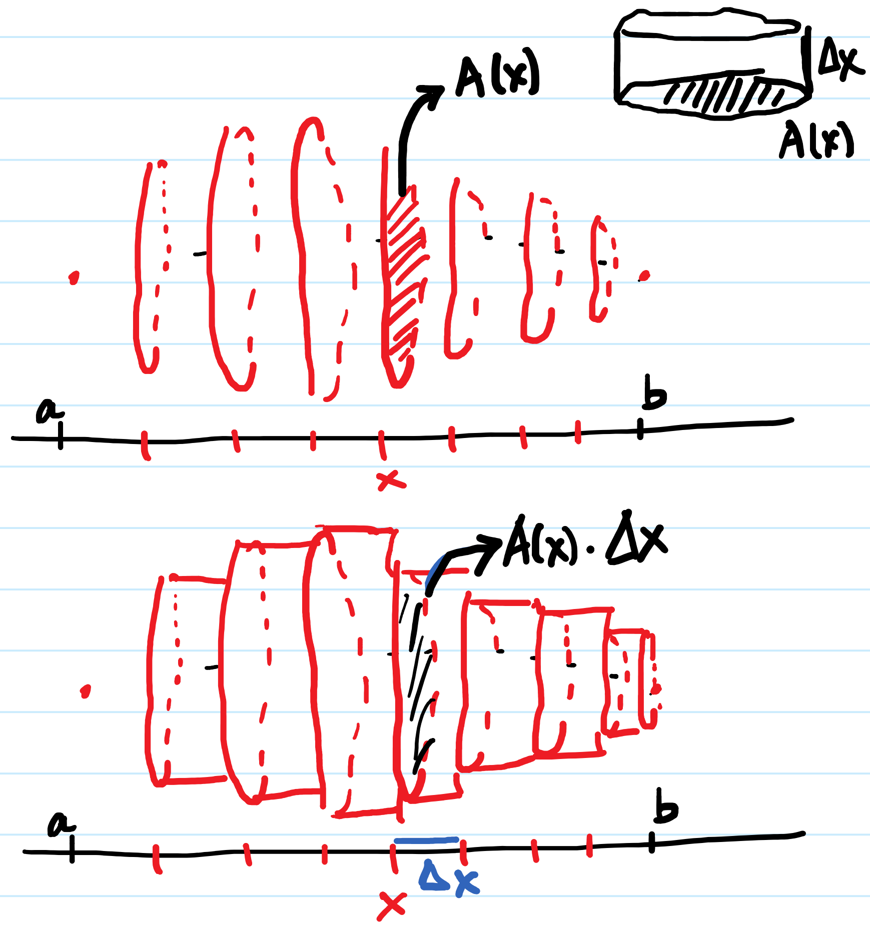
Volume $= \pi r^2 h$

(Area of the base) $\times$ height

Volume

$=$ (area of base) $\times$ height

General Solids



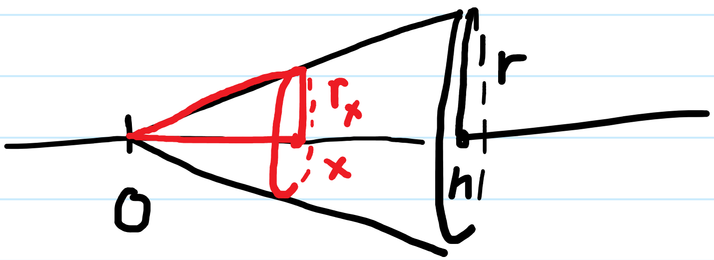
S: Solid between $x=a$ and $x=b$

A(x): cross-sectional area at x

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x = \int_a^b A(x)\,dx$$

Applications.

1. Volume of a cone of height h and base radius r.



$A(x)$: Cross-sectional area at x

r_x : radius of the cross-section at x.

$$A(x) = \pi (r_x)^2$$

Similar triangles : $\dfrac{r_x}{r} = \dfrac{x}{h}$

$$r_x = \frac{r}{h} \cdot x$$

$$A(x) = \pi \left(\frac{r}{h} x\right)^2 = \frac{\pi r^2}{h^2} x^2.$$

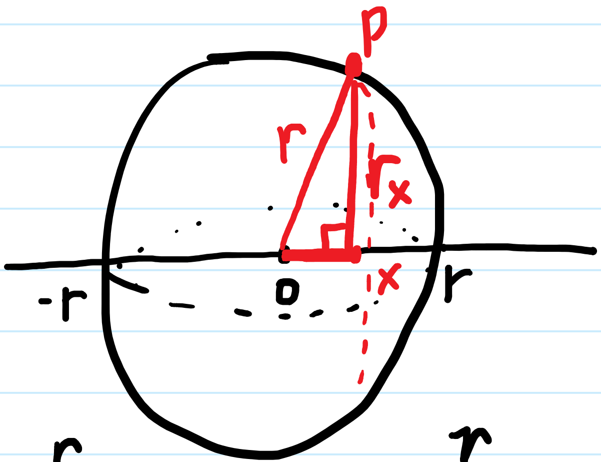
$$V = \int_0^h A(x)\,dx = \int_0^h \frac{\pi r^2}{h^2} x^2\,dx$$

$$= \frac{\pi r^2}{h^2} \int_0^h x^2\,dx$$

$$= \frac{\pi r^2}{h^2} \left.\frac{x^3}{3}\right|_0^h$$

$$= \frac{\pi r^2}{\cancel{h^2}} \cdot \frac{h^{\cancel{3}}}{3} = \frac{1}{3} \pi r^2 h$$

2. Volume of a sphere of radius r.



$$V = \int_{-r}^{r} A(x)\,dx = 2\int_0^r A(x)\,dx$$

$A(x)$: cross-sectional area at x.
r_x : radius of the cross-section at x

$$r_x^2 + x^2 = r^2$$

So, $r_x = \sqrt{r^2 - x^2}$

$$A(x) = \pi (r_x)^2$$
$$= \pi\left(\sqrt{r^2-x^2}\right)^2 = \pi(r^2-x^2)$$

$$V = 2\int_0^r A(x)\,dx = 2\int_0^r \pi(r^2-x^2)\,dx$$

$$= 2\pi \int_0^r (r^2-x^2)\,dx$$

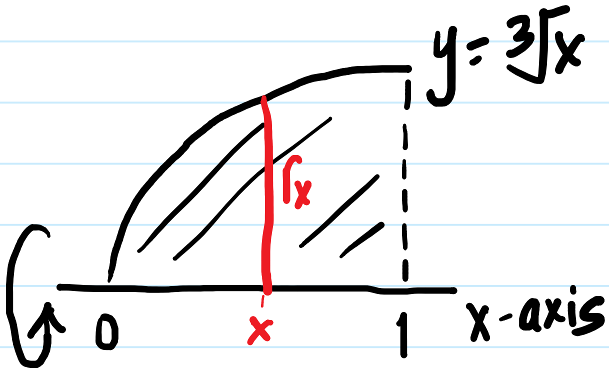
$$= 2\pi\left[r^2x - \frac{x^3}{3}\Big|_0^r\right]$$

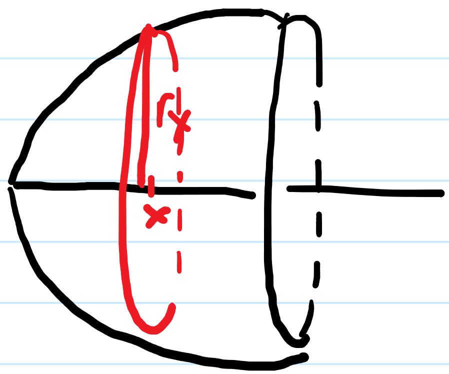
$$= 2\pi\left[r^3 - \frac{r^3}{3}\right]$$

$$= \frac{4}{3}\pi r^3 \,//$$

Volumes of revolution

3. Find the volume of the solid obtained by revolving $y = \sqrt[3]{x}$ between 0 and 1 about the x-axis





$A(x)$: cross-sectional area

$r_x = \sqrt[3]{x}$

$A(x) = \pi (r_x)^2 = \pi (\sqrt[3]{x})^2$

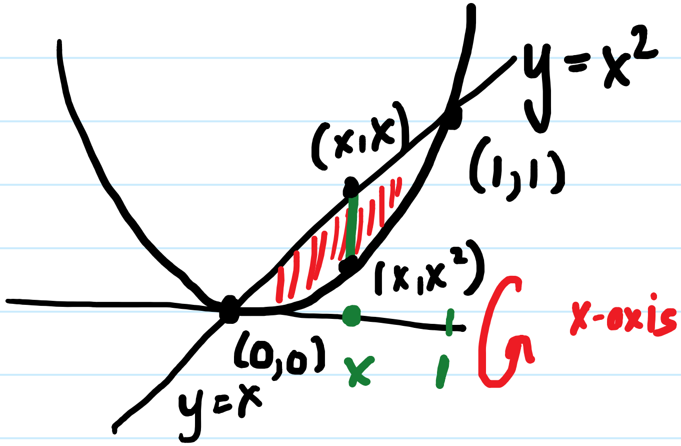
$= \pi x^{2/3}.$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x^{2/3}\,dx$$

$$= \pi \left.\frac{x^{5/3}}{5/3}\right|_0^1$$

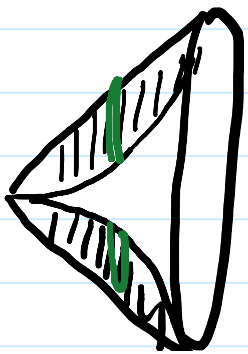
$$= \frac{3\pi}{5}$$

4. Find the volume of the solid obtained by rotating the region bounded by the curves $y=x^2$ and $y=x$ about the x-axis.

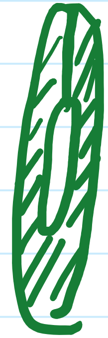


Points of intersection : $x=x^2$

$$x-x^2=0$$
$$x(1-x)=0$$
$$x=0,1$$



conical cup with a curved interior



$A(x)=$ cross-sectional area at x.

r_{out} : outside radius at x

r_{in} : inside radius at x

$$r_{out}=x$$
$$r_{in}=x^2$$

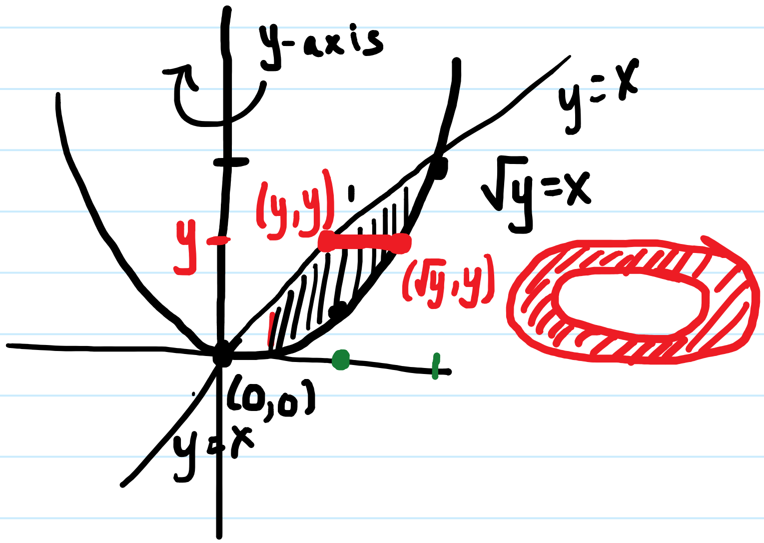
$$A(x)=\pi(r_{out})^2-\pi(r_{in})^2$$
$$=\pi(x)^2-\pi(x^2)^2$$
$$=\pi(x^2-x^4).$$

$$V=\int_0^1 \pi(x^2-x^4)\,dx=\pi\left(\frac{x^3}{3}-\frac{x^5}{5}\Big|_0^1\right)$$
$$=\pi\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{2\pi}{15}.$$

a curved interior

$$= \pi\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{2\pi}{15}$$

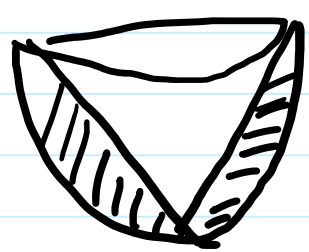
5. Find the volume of the solid obtained by rotating the region bounded by the curves $y=x^2$ and $y=x$ about the y-axis.



$A(y)$ = cross-sectional area at y

$r_{out} = \sqrt{y}$

$r_{in} = y$



$$A(y) = \pi(r_{out})^2 - \pi(r_{in})^2$$

$$= \pi(\sqrt{y})^2 - \pi(y)^2$$

$$= \pi(y - y^2).$$

$$V = \int_0^1 A(y)\,dy$$

$$= \int_0^1 \pi(y-y^2)\,dy$$

$$= \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\Big|_0^1\right]$$

$$= \pi\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{\pi}{6}$$