5.5. The Substitution Rule

Nov. 30, 2015

Dec. 02, WEDNESDAY

Course Evaluations
+
Quiz 4.

Undoing the Chain Rule!

Indefinite integrals

$$\int f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = \int f(u)\,du.$$

$u = g(x)$

$\frac{du}{dx} = g'(x)$

$du = g'(x)\,dx$

Skill: Writing the given integral as the LHS.

Examples

Should write this

$u = x^4$

$\frac{du}{dx} = 4x^3$

$\frac{1}{4}du = x^3 dx$

1. $\int x^3 \sin(x^4)\,dx.$

$= \int \sin(u)\left(\frac{du}{4}\right)$

$= \frac{1}{4}\int \sin u\,du = -\frac{\cos u}{4} + C$

Sub $u = x^4$ back!

$= -\frac{\cos(x^4)}{4} + C$

Check: $\frac{d}{dx}\left[-\frac{\cos(x^4)}{4} + C\right]$

$= \frac{-1}{\cancel{4}} \cdot -\sin(x^4) \cdot \cancel{4}x^3$

$= x^3 \sin(x^4).$

2. $\int \sqrt{1-2x}\,dx.$

$u = 1-2x$

$\frac{du}{dx} = -2$

$-\frac{1}{2}du = dx$

$= \int \sqrt{u}\left(-\frac{du}{2}\right)$

$= -\frac{1}{2}\frac{u^{3/2}}{3/2} + C$

$= -\frac{1}{3}u^{3/2} + C$

$= -\frac{1}{3}(1-2x)^{3/2} + C.$

Check: $\frac{d}{dx}\left(-\frac{1}{3}(1-2x)^{3/2} + C\right)$

$= -\frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}}(1-2x)^{1/2} \cdot (-\cancel{2}) = \sqrt{1-2x}.$ ✓

3. $\int \frac{(\ln(x)+1)^3}{x}\,dx$

$u = \ln(x)+1$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{dx}{x}$

$= \int u^3\,du$

$= \frac{u^4}{4} + C.$

$= \frac{(\ln(x)+1)^4}{4} + C.$

Check: $\frac{d}{dx}\left(\frac{(\ln(x)+1)^4}{4} + C\right)$

$= \frac{\cancel{4}(\ln(x)+1)^3}{\cancel{4}} \cdot \frac{1}{x} = \frac{(\ln(x)+1)^3}{x}$ ✓

4.

$$\int \left(e^{3x+2} + x\right) dx$$

$$= \int e^{3x+2}\,dx + \int x\,dx$$

$u = 3x+2$

$\frac{du}{dx} = 3$

$\frac{du}{3} = dx$

$$= \int e^{u} \frac{du}{3} + \frac{x^2}{2} + C$$

$$= \frac{e^{u}}{3} + \frac{x^2}{2} + C = \frac{e^{3x+2}}{3} + \frac{x^2}{2} + C$$

Check: $\frac{d}{dx}\left(\frac{e^{3x+2}}{3} + \frac{x^2}{2} + C\right) = \frac{1}{\cancel{3}} e^{3x+2} \cdot \cancel{3} + \frac{\cancel{2}x}{\cancel{2}}$ ✓

Definite Integral

$$\int_a^b f(\overbrace{g(x)}^{u})\,\overbrace{g'(x)dx}^{du} = \int_{g(a)}^{g(b)} f(u)\,du.$$

when $x=a$, $u=g(a)$

when $x=b$, $u=g(b)$

Example:

$$\int_0^1 (1-3x)^2\,dx$$

$u = 1-3x$

$-\frac{1}{3}du = dx$

at $x=0$, $u = 1-3\cdot 0 = 1$

$x = 1$

$u = 1-3\cdot 1 = -2.$

$$= \int_1^{-2} (u)^2 \left(-\frac{du}{3}\right) = -\frac{1}{3}\frac{u^3}{3}\Big|_1^{-2}$$

$$= -\frac{1}{9}\left[(-2)^3 - (1)^3\right] = -\frac{1}{9}(-9)$$

$$= \underline{\underline{1.}}$$

② $$\int_0^{\pi/4} \tan(x)\,dx = \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx$$

$u = \cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$

at $x = 0$, $u = \cos(0) = 1$

at $x = \frac{\pi}{4}$, $u = \cos(\pi/4) = \frac{1}{\sqrt{2}}$

$$= \int_1^{1/\sqrt{2}} \frac{-du}{u} \overset{\text{flip}}{\underset{\text{limits}}{=}} \int_{1/\sqrt{2}}^{1} \frac{du}{u}$$

$$= \ln(u)\Big|_{1/\sqrt{2}}^{1}$$

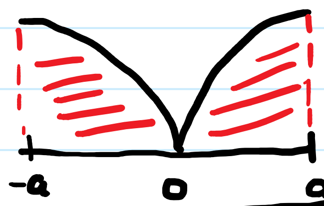
$$= \ln(1) - \ln(1/\sqrt{2})$$

$$= \ln(\sqrt{2}).$$

Exploiting symmetries to compute integrals.

Even Functions

$f(-x) = f(x)$ OR symmetric about the y-axis.

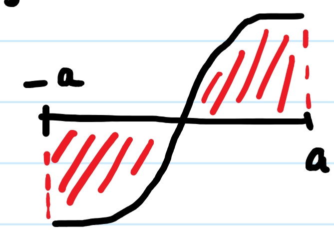


Examples:
1. x^n; n is even
2. $\cos x$

$$\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \ f \text{ is even.}}$$

odd functions

$f(-x) = -f(x)$ OR symmetric about $y = -x$.



Examples: x^n, n odd
$\sin x$, $\tan x$.

$$\boxed{\int_{-a}^{a} f(x)\,dx = 0, \quad f \text{ is odd}}$$

Proof: $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$\displaystyle = \int_{-a}^{0} -f(-x)\,dx + \int_{0}^{a} f(x)\,dx$

$u = -x$

$du = -dx$

$x = -a,\ u = a$

$x = 0,\ u = 0$

$\displaystyle = \int_{a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$\displaystyle = -\int^{a} f(x)\,dx + \int^{a} f(x)\,dx = 0.$

Examples

① $\displaystyle\int_{-1}^{1} \underbrace{\frac{\tan x}{1+x^4}}_{f(x)}\,dx = 0.$

$\displaystyle f(-x) = \frac{\tan(-x)}{1+(-x)^4}$

$\displaystyle = -\frac{\tan(x)}{1+x^4}$

$= -f(x)$

So, f is odd.

② $\displaystyle\int_{-1}^{1} \underbrace{\sqrt[3]{x}\,\ln(3+x^2)}_{f(x)}\,dx = 0.$

$f(-x)$

$= \sqrt[3]{-x}\,\ln(3+(-x)^2)$

$= -\sqrt[3]{x}\,\ln(3+x^2)$

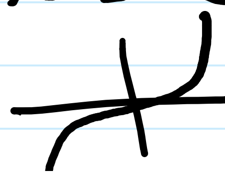
$= -f(x)$

So, f is an odd fn.

$\sqrt[3]{-8} = \sqrt[3]{(-2)^3} = -2$

$= -\sqrt[3]{8}$

$= -2$

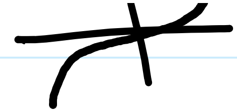


inverse of an odd fn is

$$= -\int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx = \underline{0}.$$

$= -\sqrt[3]{8}$

$= -2$



inverse of an odd fn is odd!

3. $\displaystyle\int_{-2}^{2} \overbrace{(1+x)\sqrt{4-x^2}}^{f(x)}\,dx$

$f(-x) = (1-x)\sqrt{4-(-x)^2}$
$= (1-x)\sqrt{4-x^2}$

$\neq f(x)$
$\neq -f(x)$.

Neither even nor odd!

$$= \int_{-2}^{2} \sqrt{4-x^2} + x\sqrt{4-x^2}\,dx$$

$$= \int_{-2}^{2} \sqrt{4-x^2}\,dx + \int_{-2}^{2} \underbrace{x\sqrt{4-x^2}}_{g(x)}\,dx$$

– try substitution

OR

– $g(-x) = -x\sqrt{4-(-x)^2}$
$= -x\sqrt{4-x^2}$
$= -g(x)$

g is an odd fn

$y = \sqrt{4-x^2} \rightarrow y^2 + x^2 = 4$

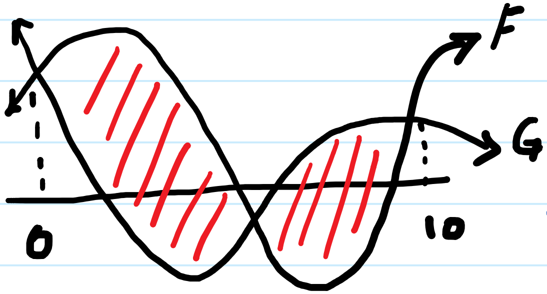
$= \frac{1}{2}\pi(2)^2 + 0$

$= 2\pi.$

6.1 AREAS BETWEEN CURVES

Chapter 5: View the definite integral as the 'area' under a graph.

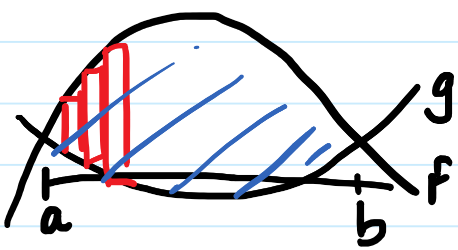
WANT: Area captured between 2 curves.



Application: IF F and G represent the velocity functions of two cars, then the shaded area is the distance between the two cars at 10 units of time

① $f \geq g$

$[a, b]$



$$\text{Area} = \int_a^b \big(f(x) - g(x)\big)\,dx$$

② In general, the region between two curves $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is

$$\int_a^b |f(x) - g(x)|\,dx.$$

Example: Find the area of the region bounded above by $y = e^x$, below by $y = x$ and on the sides by $x = 1$ and $x = 2$

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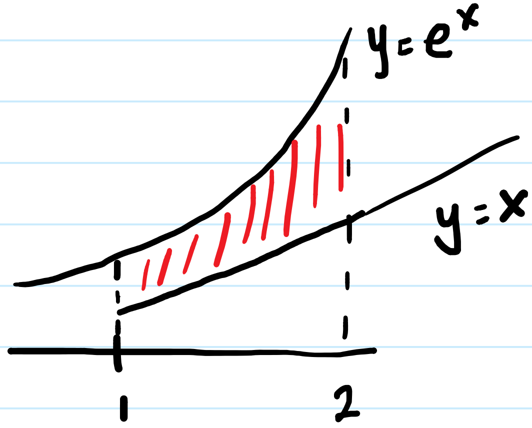


Area of the shaded region $= \int_1^2 (e^x - x)\,dx$

$$= e^x - \frac{x^2}{2}\Big|_1^2 = \left(e^2 - \frac{4}{2}\right) - \left(e - \frac{1}{2}\right)$$

$$= \underline{e^2 - e - 3/2}$$

Example:

Find the area of the region enclosed by the parabola $y=x^2$ $f(x)$ and the curve $y=x^3-x^2$ $g(x)$

LIMITS NOT GIVEN!

Solution

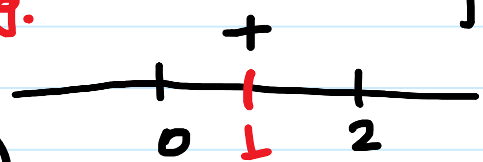
Step 1 Points of intersection:

$f(x)-g(x)=0$

$$x^2-(x^3-x^2)=0$$
$$2x^2-x^3=0$$
$$x^2(2-x)=0$$
$$x=2 \text{ and } x=0.$$

Step 2 Determine which function is larger in $[0,2]$.

Sign of $f-g$.



$f(x) \geq g(x)$ in $[0,2]$

$f(1)-g(1)$
$= 1^2-(1^3-1^2)$
$= 1$

The region is from $x=0$ to $x=2$

$$\text{Area} = \int_0^2 |f(x)-g(x)|\,dx$$

$$= \int_0^2 x^2 - (x^3 - x^2)\,dx$$

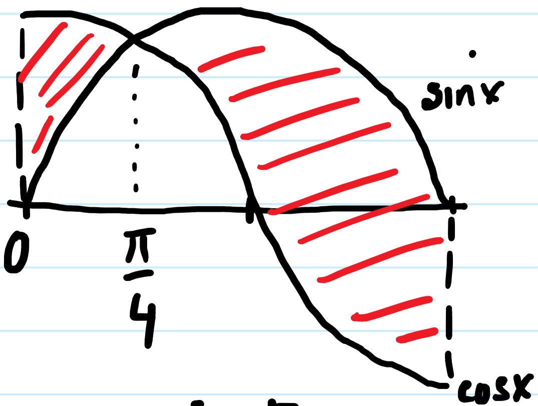
$$= \int_0^2 (2x^2 - x^3)\,dx$$

$$= \frac{2x^3}{3} - \frac{x^4}{4} \Big|_0^2$$

$$= \frac{16}{3} - \frac{16}{4} = \underline{\underline{\frac{16}{12}}}.$$

3. Sketch the region enclosed by

$y = \sin x, \quad y = \cos x, \quad x = 0, \; x = \pi.$



Point of intersection $\pi/4$.

Determine the upper and lower curves.

$$|\sin(x) - \cos(x)| = \begin{cases} \cos(x) - \sin x, & [0, \pi/4] \\ \sin x - \cos x, & [\pi/4, \pi] \end{cases}$$

So, the required area

$$= \int_0^{\pi} |\sin(x) - \cos x| \, dx$$

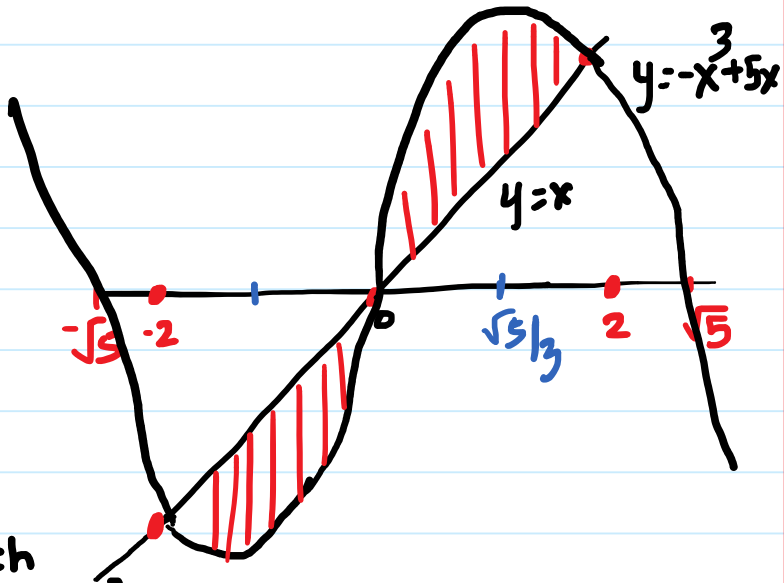
$$= \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\,dy$$

$$= \sin x + \cos x \Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi}$$

$$= \sin\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right) - \cancel{\sin(0)} - \cos(0)$$
$$+ \left(-\cos(\pi) - \cancel{\sin(\pi)}\right) - \left(-\cos\left(\tfrac{\pi}{4}\right) - \sin\left(\tfrac{\pi}{4}\right)\right)$$

$$= \frac{2}{\sqrt{2}} \cancel{-1} \cancel{+1} + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}}.$$

4. Sketch the area enclosed by $y=x$ and $y=-x^3+5x$. Find its area.



sketch

$y=-x^3+5x = -x(x^2-5)$

1. intersects the x-axis at $x=0,\pm\sqrt{5}$

2. Points of intersection with $y=x$

$x=-x^3+5x$

$x^3-4x=0 \to x(x^2-4)=0$

$x=0,\pm 2$.

3. $y'=-3x^2+5$

$y'=0$ when $x^2=5/3$ or $x=\pm\sqrt{5/3}$ $=\pm 1.$

Sign of y'

	dec		inc		dec
	$-$		$+$		$-$
-2	$-\sqrt{5/3}$		0	$+\sqrt{5/3}$	2

$-3(-2)^2+5 < 0$

$\downarrow$ local min

$-3(0)^2+5$

$\downarrow$ local max.

$-3(2)^2+5 = -7$

4. $y''=-6x$, $y''=0$ when $x=0$

sign of y'': $+$ inf. point $-$

c.up 0 c.d.

5. Symmetry: x and $-x^3+5x$ are both odd - so only need to compute one of the two areas.

$$A = \int_{-2}^{2} \left| -x^3+5x - x \right| dx$$

Symmetry

$$= 2\int_0^2 \underbrace{-x^3+5x}_{\text{larger fn.}} - x \, dx$$

$$= 2\int_0^2 (-x^3+4x)\,dx$$

$$= -\frac{x^4}{4} + \frac{4x^2}{2} \Big|_0^2$$

$$= 2\left(-\frac{2^4}{4} + \frac{4(2)^2}{2}\right)$$

$$= 2\left(-\frac{16}{4} + \frac{16}{2}\right) = 2(-4+8)$$

$$= \underline{8.}$$