5.3. FTOC Continued Nov. 25, '15

f is continuous on $[a,b]$.

Last time: (FTOC, Part 1) How to find an anti-derivative of $f(x)$?

Set $g(x) = \int_a^x f(t)\,dt$,

then $g'(x) = f(x)$.

FTOC, Part 2: Suppose $F(x)$ is anti-derivative of $f(x)$ on $[a,b]$ — i.e.

$F'(x) = f(x)$, then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

$$F(b) + C - (F(a) + C) = F(b) - F(a)$$

Remark: This works for any anti-derivative of $f(x)$

Evaluate:

1. $\int_1^3 \underbrace{e^{2x}}_{f(x)}\,dx$ ($b = 3$, $a = 1$)

Guess $F(x)$

try $F(x) = e^{2x}$

$F'(x) = \frac{d}{dx} e^{2x} = 2e^{2x}$

$\frac{1}{2}\frac{d}{dx} e^{2x} = e^{2x}$

$\frac{d}{dx}\left(\frac{e^{2x}}{2}\right) = e^{2x}$

$F(x)$ "$\frac{e^{2x}}{2}$ is an anti-der of e^{2x}

$$\overset{FTOC}{=} \left.\frac{e^{2x}}{2}\right|_1^3$$

$$= \underbrace{\frac{e^{2\cdot 3}}{2}}_{F(b)} - \underbrace{\frac{e^{2\cdot 1}}{2}}_{F(a)}$$

$$= \frac{1}{2}\left[e^6 - e^2\right].$$

$F(b)+C-(F(a)+C)$

$= F(b)-F(a)$

2.
$$\int_0^4 (4-t)\sqrt{t}\,dt$$

$x^n \xrightarrow[n\neq -1]{\text{anti-der}} \dfrac{x^{n+1}}{n+1}$

$x^{-1} \rightarrow \ln|x|$

$$= \int_0^4 \underbrace{\left(4t^{1/2} - t^{3/2}\right)}\,dt \qquad F(b)-F(a)$$

$$= \frac{4t^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1}\Bigg|_0^4$$

$$= \frac{8}{3}t^{3/2} - \frac{2}{5}t^{5/2}\Bigg|_0^4$$

$$= \frac{8}{3}\left(4^{1/2}\right)^3 - \frac{2}{5}\left(4^{1/2}\right)^5 - \left(\frac{8}{3}(0)^{3/2} - \frac{2}{5}0^{5/2}\right)$$

$$= \frac{8\cdot 2^3}{3} - \frac{2\cdot 2^5}{5} = \frac{64}{3} - \frac{64}{5}$$

$$= 64\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{128}{15}.$$

3.
$$\int_0^{\pi/4} \sin(t) + \frac{1}{1+t^2}\,dt$$

SEE SEC. 4.9

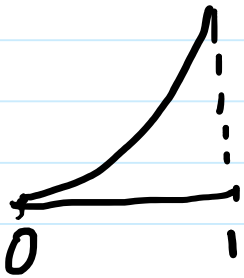
$$\overset{FTOC}{=} -\cos(t) + \tan^{-1}(t)\Bigg|_0^{\pi/4}$$

$$= -\cos\left(\frac{\pi}{4}\right) + \tan^{-1}\left(\frac{\pi}{4}\right) - \left(-\cos(0) + \tan^{-1}(0)\right)$$

$$= -\frac{1}{\sqrt{2}} + \tan^{-1}\left(\frac{\pi}{4}\right) - (-1+0)$$

$$= -\frac{1}{\sqrt{2}} + \tan^{-1}\left(\frac{\pi}{4}\right) + 1$$

4. Find the area under the graph of $y=x^2$ b/w $x=0$ and $x=1$.



Sol^n

$$\text{Area} = \int_0^1 x^2\,dx$$

$$\overset{FTOC}{=} \left.\frac{x^3}{3}\right|_0^1$$

$$= \frac{1}{3} - \frac{0}{3} = 1/3$$

much simpler way of computing areas!

WARNING:

$$\int_{-1}^{2} \frac{1}{x^4}\,dx = \int_{-1}^{2} x^{-4}\,dx$$

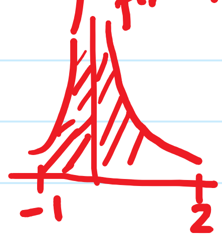
$$\cancel{\overset{FTOC}{=}} \left.\frac{x^{-4+1}}{-4+1}\right|_{-1}^{2}$$

$$= \left.\frac{x^{-3}}{-3}\right|_{-1}^{2}$$

$$= \frac{(2)^{-3}}{-3} - \frac{(-1)^{-3}}{-3}$$

$$= \frac{-1}{3}\left[\frac{1}{8} - (-1)\right] = \frac{-9}{3\cdot 8} = \frac{-3}{8}$$

area not finite!



Observation

$\frac{1}{x^4} > 0$

$\int_{-1}^{2} \frac{dx}{x^4} > 0.$

$\frac{1}{x^2}$ is not continuous at 0 which is in $[-1, 2]$!

5.3 Recap.

f is continuous in $[a,b]$

1. $g(x)=\int_a^x f(t)\,dt$, then $g'(x)=f(x)$.

2. $g(x)=\int_{a(x)}^{b(x)} f(t)\,dt$ then

$$g'(x)=f(b(x))\,b'(x)-f(a(x))\,a'(x).$$

3. If $F'(x)=f(x)$, then

$$\int_a^b f(x)\,dx = F(b)-F(a).$$

Recall:

$$\ast \quad \int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$$

$$F(b)-F(a) \qquad F(\cancel{c})-F(a)+F(b)-\cancel{F(c)}$$

$$=F(b)-F(a)$$

1. Differentiate $g(x)=\int_{\sin x}^{x^3} e^{-\tan t}\,dt$

$b(x)=x^3$, $a(x)=\sin x$, $f(t)=e^{-\tan t}$

$$= e^{-\tan(x^3)}\cdot 3x^2 - e^{-\tan(\sin x)}\cdot\cos x.$$

2. $\int_2^x f(t)\,dt = 1$, $\int_2^4 f(t)\,dt = 1$. Compute

$$\int_x^4 f(t)\,dt \overset{\text{by } \ast}{=} \int_x^2 f(t)\,dt + \int_2^4 f(t)\,dt$$

$b=4$, $a=x$

flip limits $-1 + 1 = 0.$

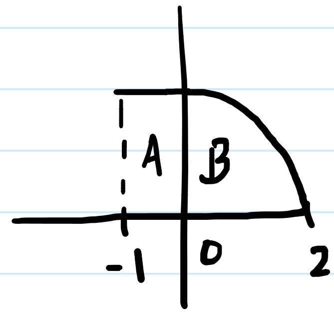
a

flip limits $\quad -1 \quad + \quad 1 = 0.$

3. Find the area under the graph of $f(x)$ in $[-1,2]$ where
$f(x) = 2$ in $[-1,0]$ and
$f(x) = \sqrt{4-x^2}$ in $[0,2]$.

Solution

$$\text{Area} = \int_{-1}^{2} f(x)\,dx$$



$$= \underbrace{\int_{-1}^{0} 2\,dx}_{A} + \underbrace{\int_{0}^{2} \sqrt{4-x^2}\,dx}_{B}$$

$$\overset{FTOC}{=} 2x\Big|_{-1}^{0} + \frac{1}{4}\pi(2)^{\cancel{2}}$$

$$= 2[0-(-1)] + \pi = \overset{A}{2} + \overset{B}{\pi}$$

4. $f(x) = \int_0^x (1-t^2)e^{t^2}\,dt$. On what intervals is $f(x)$ increasing?

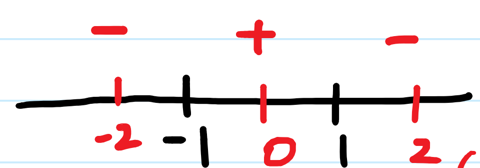
Solution: f is inc $\leftrightarrow f' > 0$.

$f'(x) = (1-x^2)e^{x^2}$.

Good idea to compute critical points

$f'(x) = 0 \rightarrow (1-x^2)e^{x^2} = 0$
$\rightarrow 1-x^2 = 0$
$\rightarrow x = \pm 1$

Sign of f'



@ -2, $f'(-2) = (1-(-2)^2)e^{(-2)^2} = -3e^4$
@ 0, $f'(0) = (1-0)\cdot e^0 = 1$
@ 2, $f'(2) = -3e^4$

So, $f(x)$ is increasing in $(-1,1)$.

5.4. Indefinite Integrals.

NOTATION!

indefinite integral

NO LIMITS $\int f(x)dx$ = SHORT FOR MOST GENERAL ANTI-DERIVATIVE OF $f(x)$.

$$= F(x)+C$$

where $F'(x) = f(x)$.

Example $\int x^2 dx = \frac{x^3}{3} + C$

Compute : $\int \left(\sin(t) + \frac{1+\sqrt{t}-3t}{t^2}\right)dt$

$$= \int \sin t\, dt + \int \left(t^{-2} + t^{-3/2} - \frac{3}{t}\right)dt$$

$$= -\cos t + \frac{t^{-2+1}}{-2+1} + \frac{t^{-3/2+1}}{-3/2+1} - 3\ln|t| + C$$

$$= -\cos t - \frac{1}{t} - 2t^{-1/2} - 3\ln|t| + C.$$

IGNORE: $\sinh x$ $\cosh x$

CAUTION $\int f(x)dx \neq \int_a^b f(x)dx$

$\downarrow$ functions $\downarrow$ number!

Ex. $\int x^2 dx = \frac{x^3}{3} + C$

$$\int_1^2 x^2 dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

NET CHANGE THEOREM:

What is an anti-derivative of $F'(x)$?

$F(x)$.

$$\int_a^b F'(x)\,dx = \overbrace{F(b)-F(a)}^{\text{net change}}$$

Integral of the rate of change of a function from a to b is the net change of the function from a to b.

Example: $F'(x) = \left(\frac{1+x}{x}\right)^2$

$F(1) = 2$

Find $F(2)$.

Solution: By the NCT,

$$\int_1^2 F'(x)\,dx = F(2) - F(1)$$

$$= F(2) - 2$$

So, $F(2) = 2 + \int_1^2 F'(x)\,dx$

$$= 2 + \int_1^2 \left(\frac{1+x}{x}\right)^2 dx$$

$$= 2 + \int_1^2 \left(\frac{1}{x}+1\right)^2 dx$$

$$= 2 + \int_1^2 \left(\frac{1}{x^2} + \frac{2}{x} + 1\right) dx$$

$$= 2 + \left.\frac{x^{-2+1}}{-2+1} + 2\ln x + x\right|_1^2$$

$$= 2 + \left[\frac{-1}{2} + 2\ln 2 + 2 - (-1 + 2\ln(1) + 1)\right]$$

Answer $\boxed{5/2 + 2\ln 2}$

Ex. A particle is moving along a line and has velocity

$$v(t) = t^3 - \sqrt{t} + 1 \text{ m/s}$$

at t seconds. What is the displacement in the interval $1 \le t \le 3$?

<u>Solution</u>: $v(t) = s'(t) = t^3 - \sqrt{t} + 1$

To find: $s(3) - s(1)$

By the NCT.

$$\int_1^3 s'(t)\,dt = s(3) - s(1)$$

So, $$\int_1^3 \left(t^3 - t^{1/2} + 1\right) dt = s(3) - s(1)$$

So, $$s(3) - s(1) = \left. \frac{t^4}{4} - \frac{t^{3/2}}{3/2} + t \right|_1^3$$

$$s(3) - s(1) = \frac{3^4}{4} - \frac{3^{3/2}}{3/2} + 3 - \left(\frac{1}{4} - \frac{1}{3/2} + 1\right)$$

$$= \frac{81}{4} - \frac{2 \cdot \cancel{3}\sqrt{3}}{\cancel{3}} + 3 - \frac{1}{4} + \frac{2}{3} - 1$$

$$= 20 + \frac{2}{3} + 2 - 2\sqrt{3} \text{. m}$$

is the displacement in $[1, 3]$.