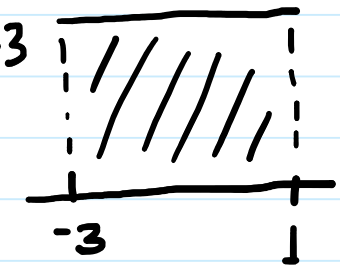
5.2 Definite Integrals. Nov. 23, '15

Computing definite integrals as areas.

1. $\int_{-3}^{1} 3\,dx = (1-(-3))\cdot 3$

$= 4\cdot 3$

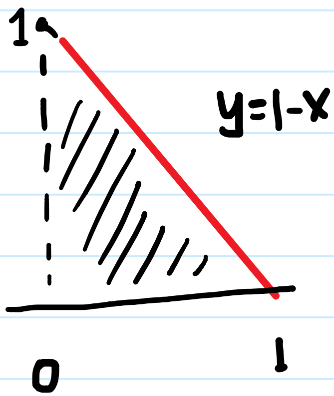
$= 12.$



2. $\int_{0}^{1} (1-x)\,dx$

$= \frac{1}{2}\cdot 1\cdot 1$

$= \frac{1}{2}$



3. $\int_{0}^{3} (3-3x)\,dx$

$3-3x=0$

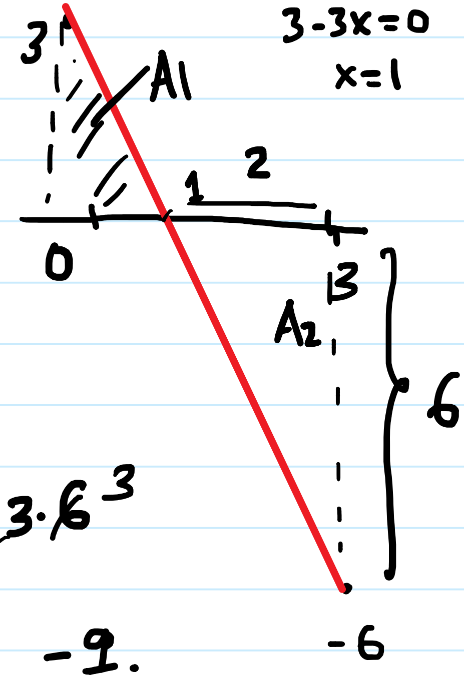
$x=1$

$=$ net area

$= A_1 - A_2$

$= \frac{1}{2}\cdot 1\cdot 3 - \frac{1}{\cancel{2}}\cdot 3\cdot \cancel{6}^{\,3}$

$= 3/2 - 9 = -\frac{9}{2}.$

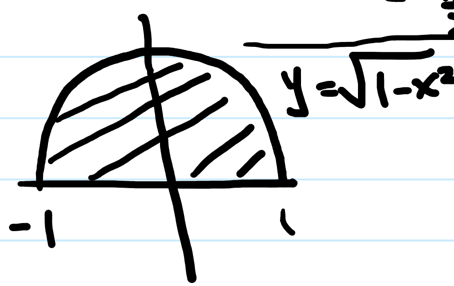


4. $\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$

$y=\sqrt{1-x^2}$

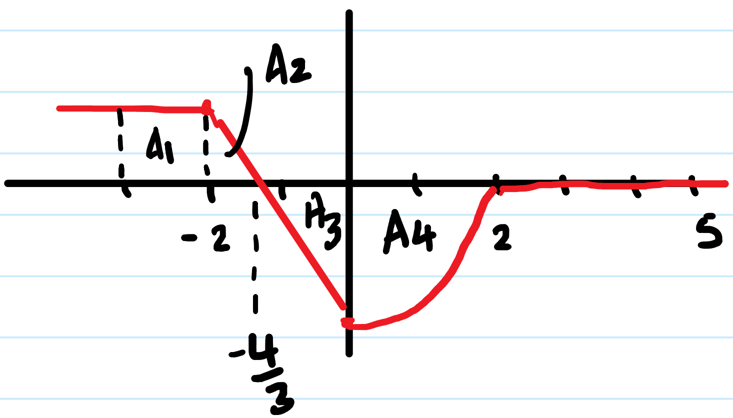
$y^2 = 1-x^2$

$x^2+y^2=1$



5. $f(x) = \begin{cases} 1, & x \le -2 \\ -2 - \frac{3}{2}x, & -2 \le x \le 0 \\ -\sqrt{4-x^2}, & 0 \le x \le 2 \\ 0, & x \ge 2 \end{cases}$

Compute $\displaystyle\int_{-3}^{5} f(x)\,dx$ viewing this as a net area.



$y = -\sqrt{4-x^2}$

$y^2 = 4 - x^2 \rightarrow y^2 + x^2 = 4.$

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x-int. of $f(x)$

$-2 - \frac{3}{2}x = 0 \rightarrow 2 = -\frac{3}{2}x$

$\rightarrow x = -\frac{4}{3}.$

Solution:

$$\int_{-3}^{5} f(x)\,dx = A_1 + A_2 - A_3 - A_4$$

$$= 1\cdot 1 + \frac{1}{2}\cdot\left(\frac{-4}{3} - (-2)\right)\cdot 1$$

$$-\frac{1}{\cancel{2}}\left(\frac{4}{3}\right)(\cancel{2}) - \frac{1}{4}\pi(\cancel{2})^2$$

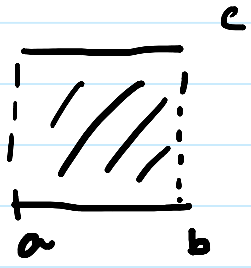
$$= 1 + \frac{1}{3} - \frac{4}{3} - \pi = -\pi.$$

Properties of definite integrals

1. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

2. $$\int_a^a f(x)\,dx = 0.$$

3. $$\int_a^b c\,dx = (b-a)\cdot c.$$



4. $$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

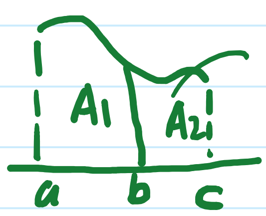
5. $$\int_a^b f(x)-g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

6. $$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx.$$

7. $$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

a, b and c are any real numbers

If $f \geq 0$, $a < b < c$



$$A = A_1 + A_2$$

Ex. Suppose $\int_{-1}^{1} f(x)\,dx = 2$,

$\int_{-1}^{2} g(x)\,dx = 3$ and $\int_{-1}^{2} f(x)+3g(x)\,dx = -1.$

Compute $\int_{1}^{2} f(x)\,dx.$

Solution: $$\int_{1}^{2} f(x)\,dx = \int_{1}^{-1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx$$

$$= -\int_{-1}^{1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx$$

$$= -2 + \int_{-1}^{2} f(x)\,dx$$ need to find

$$-1 = \int_{-1}^{2} \left(f(x) + 3g(x)\right) dx$$

$$= \int_{-1}^{2} f(x)\,dx + 3\int_{-1}^{2} g(x)\,dx$$

$$-1 = \int_{-1}^{2} f(x)\,dx + 3\cdot 3$$

So, $$\int_{-1}^{2} f(x)\,dx = -1-9$$
$$= -10$$

$$\int_{1}^{2} f(x)\,dx = -2-10$$
$$= \underline{-12}.$$

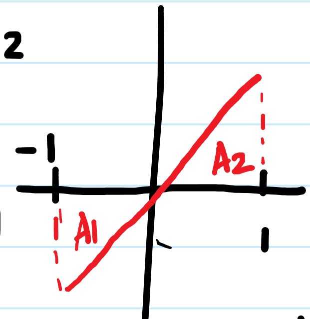
Ex. Compute $\displaystyle\int_{-1}^{1}(x-\sqrt{1-x^2})\,dx.$

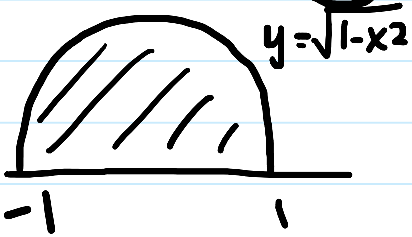
Solution : $\displaystyle\int_{-1}^{1}(x-\sqrt{1-x^2})\,dx$

$$= \int_{-1}^{1} x\,dx - \int_{-1}^{1}\sqrt{1-x^2}\,dx$$

$$= A_2 - A_1 - \frac{1}{2}\pi(1)^2$$

$$= 0 - \frac{\pi}{2} = -\frac{\pi}{2}$$





By summetry
$A_1 = A_2$

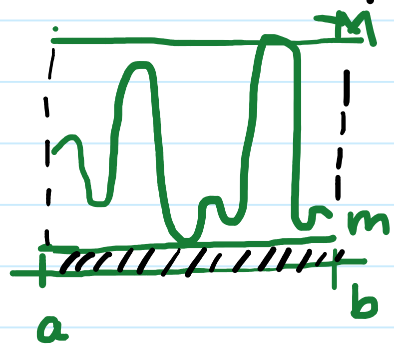
Some more properties

1. $f(x) \geq 0$ for $a \leq x \leq b$, $\displaystyle\int_a^b f(x)\,dx \geq 0.$

2. $f(x) \geq g(x)$ for $a \leq x \leq b$, then
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

3. $m \leq f(x) \leq M$, $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$



Ex. Provide upper and lower estimates for $\int_{-1}^{1} \frac{1}{x^2-4}\,dx$.

Solution: $f(x) = \frac{1}{x^2-4}$ is cont. in $[-1,1]$.

Max and min. values using the closed Interval Method.

$$f'(x) = -1(x^2-4)^{-2} \cdot 2x = \frac{-2x}{(x^2-4)^2}.$$

$f'(x) = 0$ when $x = 0$.

$f(-1) = \frac{1}{1^2-4} = -1/3$; $f(0) = \frac{1}{-4} = -\frac{1}{4}$.
$f(1) = -1/3$

So, $f(x)$ has an abs. max in $[-1,1]$ of $-\frac{1}{4}$. and has an abs. min in $[-1,1]$ of $-1/3$.

$$\underset{m}{-\frac{1}{3}} \le f(x) \le \underset{M}{-\frac{1}{4}} \text{ in } [-1,1]$$

$$-\frac{1}{3}(1-(-1)) \le \int_{-1}^{1} \frac{dx}{x^2-4} \le -\frac{1}{4}(2)$$

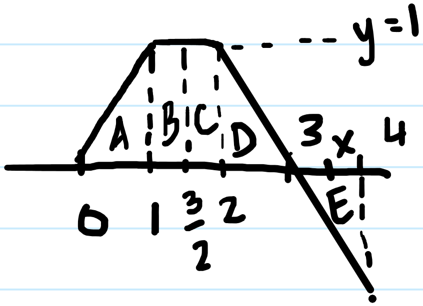
$$\boxed{-\frac{2}{3} \le \int_{-1}^{1} \frac{dx}{x^2-4} \le -\frac{1}{2}}$$

5.3 Fundamental Theorem of Calculus (FTOC).

Integration → Differentiation.

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1, & 1 \le x \le 2 \\ 3-x, & 2 \le x \le 4 \end{cases}$$

$$g(x) = \underline{\int_0^x f(t)\,dt}$$



Compute:

$g(0), g(1), g(3/2), g(2), g(4)$.

<u>Solution</u> $g(0) = \int_0^0 f(t)\,dt = 0.$

$$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}.$$

$$g(3/2) = \int_0^{3/2} f(t)\,dt = A+B = \frac{1}{2} + \frac{1}{2}\cdot 1$$
$$= 1.$$

$$g(2) = \int_0^2 f(t)\,dt = A+B+C$$
$$= \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}.$$

$$g(4) = \int_0^4 f(t)\,dt = A+B+C+D-E$$
$$= \frac{3}{2} + 0 = 3/2$$

D=E
by symmetry

Theorem (FTOC, Part 1).

f is continuous on $[a,b]$.

$$g(x) = \int_a^x f(t)\,dt.$$

Then,

$$g'(x) = f(x).$$

$\begin{cases} g \text{ is cont. in } [a,b] \\ g \text{ is diff. in } (a,b) \end{cases}$

i.e. $g(x)$ is an anti-derivative of $f(x)$.

Ex. Differentiate.

1. $g(x) = \int_0^x \sin^3 t\,dt.$

2. $\int_x^0 e^{-t^2}dt = g(x)$

3. $\int_0^2 \frac{dt}{t^2+4} = g(x).$

Solution: 1. $\sin^3 t$ is continuous

$$g'(x) = \sin^3(x).$$

2. $g(x) = \int_x^0 e^{-t^2}dt = -\int_0^x e^{-t^2}dt$

$= \int_0^x \underbrace{-e^{-t^2}}\,dt$

FTOC, $g'(x) = -e^{-x^2}.$

3. $g(x) = \int_0^2 \frac{dt}{t^2+4}.$ $g'(x) = 0.$

Ex. Find and classify the critical points of

$$g(x) = \int_0^x (t^2-4t+4)\,dt.$$

Solution g is continuous everywhere.

$$g'(x) \overset{FTOC}{=\!=} x^2-4x+4$$
$$= (x-2)^2$$

$g'(x) = 0$, when $(x-2)^2 = 0$
OR $x = 2$.

sign of g

$$\begin{array}{ccc} >0 & & >0 \\ \hline 0 & 2 & 4 \end{array}$$

$g'(0) = (-2)^2$
$g'(4) = (4-2)^2$

2 is neither a local max nor a local min. critical point of $g(x)$.

Ex. $$\lim_{x\to 0} \frac{\overbrace{\int_x^0 (1+t^2)\,dt}^{=f(x)}}{\underbrace{x^3+2x}_{=g(x)}}$$

Substitution
$$\frac{\int_0^0 (1+t^2)\,dt}{0^3+2\cdot 0} = \frac{0}{0}$$

$$\overset{L'H}{=\!=} \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to 0} \frac{-(1+x^2)}{3x^2+2}$$

$$= \frac{-(1+0)}{0+2}$$

$$= -\frac{1}{2}.$$

$$f(x) = \int_x^0 (1+t^2)\,dt$$
$$= -\int_0^x (1+t^2)\,dt$$
$$f'(x) = -(1+x^2)$$

Differentiate.

1. $g(x) = \int_1^{x^3} \sqrt{9-t^2}\, dt.$

$x^3 = u$

$$g(x) = \int_1^{u} \sqrt{9-t^2}\, dt$$

$$g'(x) = \frac{d}{du}\left(\int_1^{u} \sqrt{9-t^2}\, dt\right) \cdot \frac{du}{dx}$$

$\downarrow$ FTOC

$$= \sqrt{9-u^2} \cdot 3x^2$$

$$= \sqrt{9-(x^3)^2} \cdot 3x^2$$

$$= \sqrt{9-x^6} \cdot 3 \cdot x^2.$$

2. $h(x) = \int_{\tan(x)}^{e^x} \sin(t)\, dt$

$$= \int_{\tan x}^{0} \sin(t)\, dt + \int_0^{e^x} \sin(t)\, dt$$

$$= -\int_0^{\tan x} \sin(t)\, dt + \int_0^{e^x} \sin(t)\, dt$$

$$h'(x) = \frac{d}{du}\left(\int_0^{u} -\sin(t)\, dt\right) \frac{du}{dx}$$

$u = \tan x$

$v = e^x$

$$+ \frac{d}{dv}\left(\int_0^{v} \sin(t)\, dt\right) \frac{dv}{dx}$$

$$= -\sin(u)\sec^2 x + \sin(v)\, e^x$$

$$= -\sin(\tan x) \cdot \sec^2 x + \sin(e^x) \cdot e^x.$$

FORMULA:

$$g(x) = \int_{a(x)}^{b(x)} f(t)\,dt$$

FTOC

$$\boxed{g'(x) = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)}$$

Ex. Find the inflection points of

$$g(x) = \int_{x}^{x^2} (t-1)\,dt \qquad f(t) = t-1$$

$b(x) = x^2$

$a(x) = x$

Solution $g'(x) = (x^2-1)\cdot 2x - (x-1)$

$$g'(x) = 2x^3 - 2x - x + 1$$
$$= 2x^3 - 3x + 1$$

$$g''(x) = 6x^2 - 3.$$

So $g''(x) = 0 \quad x^2 = \frac{3}{6}$

$x^2 = 1/2$

$x = \pm \frac{1}{\sqrt{2}}$

Potential inflection pts.

Sign of g''

-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1
>0		<0		>0

$g''(-1) = 6(-1)^2 - 3 > 0$

$g''(0) = 0 - 3 < 0$

$g''(1) = 6 - 3 > 0$

So, $g(x)$ has inflection pts. at $x = \pm\frac{1}{\sqrt{2}}$.