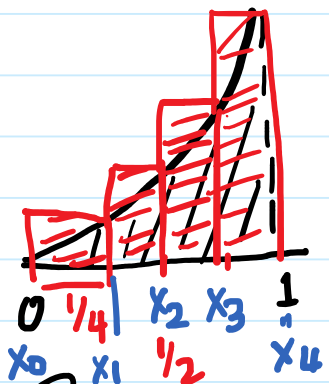
5.1. Continued. Nov. 18, '15.

Approximate the area of the region under the curve $y = x^2$ between 0 and 1.



▨ : R_4

$$= \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot(1)^2$$

R_n : sum of the area of n rectangles

- Divide [0,1] into n equal parts.

  Width of each part: $\frac{1}{n}$

- Let's say that the points that divide [0,1] into n equal parts are

$$0 = x_0,\ x_1,\ x_2,\ \dots,\ x_{n-1},\ x_n = 1$$
$$\quad \tfrac{1}{n} \quad \tfrac{1}{n}+\tfrac{1}{n} \quad\quad (n-1)\tfrac{1}{n} \quad \tfrac{n}{n}$$

j^{th} interval (part) : $[x_{j-1}, x_j]$

$$= \left[\frac{(j-1)}{n}, \frac{j}{n}\right]$$

right end-point

Base of the j^{th} rectangle : $1/n$

Height of " " " : $\left(\frac{j}{n}\right)^2$

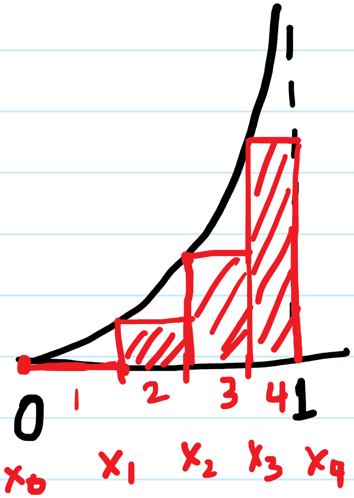
Area " " " " $= \frac{1}{n}\cdot\left(\frac{j}{n}\right)^2.$

Right sum

So $R_n = \sum$ all n rectangles

$$= \sum_{j=1}^{n} \frac{1}{n}\left(\frac{j}{n}\right)^2.$$

We saw $\lim_{n\to\infty} R_n = \frac{1}{3}.$



$L_4 =$

= sum of areas of rectangles

$$= \frac{1}{4}(0)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2$$

$$= \frac{1}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right)$$

$$= \frac{1}{4}\left(\frac{1+4+9}{16}\right)$$

$$= \frac{1}{4}\ \frac{14}{16} = \frac{14}{64}.$$

$A > L_4$

$A > \frac{14}{64}.$

L_n: Left sum with n rectangles

- Divide $[0,1]$ into n equal parts each of width : $\frac{1}{n}$.

So, the points dividing $[0,1]$ are

$$\underbrace{0,\ 0+\frac{1}{n}}_{1^{st}},\ \underbrace{0+\frac{1}{n},\ 0+\frac{2}{n}}_{2^{nd}},\ \dots,\ \underbrace{0+\frac{n-1}{n},\ \frac{n}{n}}_{n^{th}}$$

$$L_n = \sum_{j=1}^{n} \overbrace{\frac{1}{n}}^{\text{base}} \overbrace{\left(\frac{j-1}{n}\right)^2}^{\text{height}}$$

$$R_n = \sum_{j=1}^{n} \frac{1}{n}\left(\frac{j}{n}\right)^2$$

compare

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n}\left(\frac{j-1}{n}\right)^2$$

$$= \lim_{n\to\infty}\left(\sum_{j=1}^{n} \frac{1}{n^3}(j-1)^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{j=1}^{n}(j-1)^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{j=1}^{n} j^2-2j+1\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^3}\left(\sum_{j=1}^{n} j^2 - 2\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^3}\left(\frac{2n^3+3n^2+n}{6} - \frac{\cancel{2}(n^2+n)}{\cancel{2}}+n\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{\cancel{2n^3}}{{}_3\cancel{6n^3}} + \underbrace{\frac{3n^2}{6n^3}}_{\to 0}+\underbrace{\frac{n}{6n^3}}_{\to 0} - \underbrace{\frac{n^2}{n^3}}_{\to 0}-\underbrace{\frac{n}{n^3}}_{\to 0}+\underbrace{\frac{n}{n^3}}_{\to 0}\right)$$

$$= \frac{1}{3}.$$

Recall: $A \le \lim_{n\to\infty} R_n = \frac{1}{3}$

$A \ge \lim_{n\to\infty} L_n = \frac{1}{3}$

So, $\boxed{A = \frac{1}{3}}$

FACT: The area A of the region S that lies under the graph of the continuous function $f(x)$ between $x=a$ and $x=b$ is given by

$$A = \lim_{n\to\infty} R_n$$ Right sums

$$= \lim_{n\to\infty}\left(\sum_{j=1}^{n} \underbrace{\frac{b-a}{n}}_{\text{base}} \cdot \underbrace{f\left(a+j\frac{(b-a)}{n}\right)}_{\text{height}}\right)$$

OR

$$= \lim_{n\to\infty} L_n$$ Left sums

$$= \lim_{n\to\infty}\left(\sum_{j=1}^{n}\left(\frac{b-a}{n}\right) f\left(a+(j-1)\cdot\frac{b-a}{n}\right)\right)$$

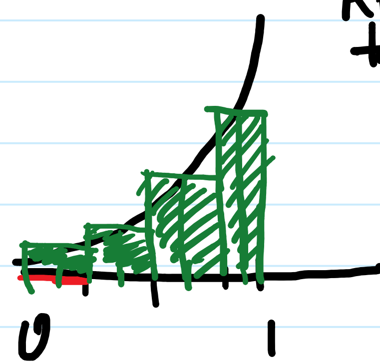
OR $$= \lim_{n\to\infty} M_n$$ midpoint sums

$$= \lim_{n\to\infty}\left[\Delta x\, f(x_1^*) + \Delta x\, f(x_2^*) + \dots + \Delta x\, f(x_n^*)\right]$$

$$= \lim_{n\to\infty}\left(\sum_{j=1}^{n} \Delta x \cdot f(x_j^*)\right)$$

$$\Delta x = \frac{b-a}{n}$$

x_j^* = midpoint of the j^{th} interval $[x_{j-1}, x_j]$



Rmk: The points chosen to determine the heights of the rectangles are called sample points

Example:

① let A be the area of the region under the graph of $y=2^x$ between $x=-1$ and $x=2$.

Write A as a limit, using right end-points.

Solution: $f(x)=2^x$; $a=-1$, $b=2$

$$R_n = \sum_{j=1}^{n} \frac{2-(-1)}{n} f\left(-1+j\frac{(2-(-1))}{n}\right)$$

$$= \sum_{j=1}^{n} \frac{3}{n} f\left(-1+\frac{3j}{n}\right)$$

$$= \sum_{j=1}^{n} \left(\frac{3}{n}\, 2^{\left(-1+\frac{3j}{n}\right)}\right)$$

Since 2^x is continuous in $[-1, 2]$,

$$A = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{3}{n}\cdot 2^{\left(-1+\frac{3j}{n}\right)}\right)$$

② Determine a region whose area is equal to

$$\lim_{n\to\infty} \sum_{j=1}^{n} \left(\underbrace{\frac{e^3}{3n}} \cdot \underbrace{\log\left(1 + \frac{e^3 j}{3n} \right)} \right).$$

Compare:

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{b-a}{n} \cdot f\left(a + j\left(\frac{b-a}{n}\right) \right)$$

Solution: Guess: $\frac{b-a}{n} = \frac{e^3}{3n}$

So $b-a = \frac{e^3}{3}$ —(i)

$f(x) = \log(x)$

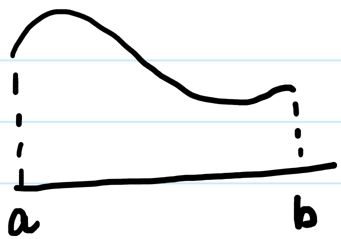
$a = 1$ —(ii)

$b = \frac{e^3}{3} + 1$

The region whose area is equal to the given limit is the region under the graph of $y = \log(x)$ between $x = 1$ and $x = \frac{e^3}{3} + 1$.

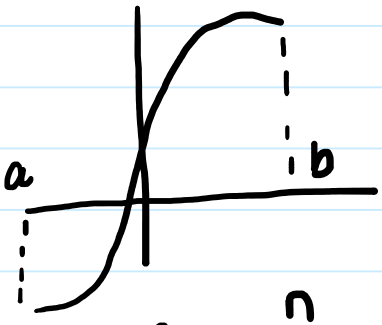
Remark: If $v(t)$ is the velocity of an object in time t seconds, then the area under the graph of $v(t)$ between $t = a$ sec. and $t = b$ secs. is the distance covered between a secs. and b secs.

5.2. DEFINITE INTEGRAL



$$A = \underbrace{\lim_{n\to\infty} \sum_{j=1}^{n} \Delta x\, f(x_j^*)}$$

Can we compute the same limits for a function like ← this?



$$\lim_{n\to\infty} \sum_{j=1}^{n} \Delta x\, f(x_j^*).$$

$\Delta x = \frac{b-a}{n}$, where x_j^* is some sample point in the j^{th} interval

If this limit exists for every choice of sample points, and is the same, then we defined

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{j=1}^{n} \Delta x\, f(x_j^*)}$$

integral of $f(x)$ from a to b.

TERMINOLOGY:

$f(x)$: integrand

a, b : limits of integration

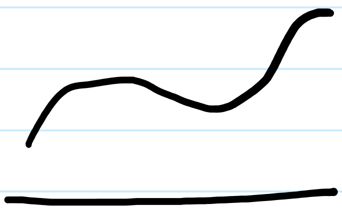
a : lower limit

b : upper limit.

$\sum_{j=1}^{n} \Delta x\, f(x_j^*)$: Riemann sum.

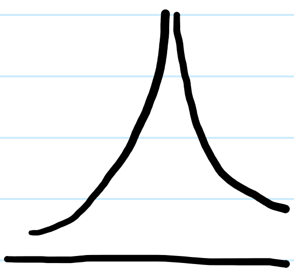
In the above case, f is called <u>integrable</u>.

FACT: If f is continuous or has only a finite number of jump discontinuities in $[a,b]$, then f is integrable.



✓

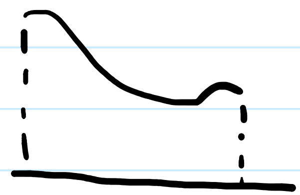
✓



DON'T KNOW!

If f is integrable, then one choice of sample points is enough!

$f \geq 0$



$\int_a^b f(x)\,dx$ = area under the graph between a and b.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\sum_{j=1}^{n}\left(\Delta x\, f(x_j^*)\right)\right)$$

$$= A_1 - A_2 = \text{NET AREA}$$

A_1: area under the graph above the x-axis

A_2: area below the x-axis

Ex: Evaluate the Riemann sum for $f(x) = x^2 - 2x$, taking the sample points to be right end-points and $a=0$, $b=3$ and $n=6$.

<u>Solution:</u> $f(x) = x^2 - 2x$ on $[0,3]$.

To compute R_6

width of each part: $\dfrac{b-a}{n} = \dfrac{3-0}{6} = \dfrac{1}{2}$

$$R_6 = \sum_{j=1}^{6} \frac{1}{2} \cdot f\left(0 + j\frac{1}{2}\right)$$

$$= \sum_{j=1}^{6} \frac{1}{2}\left[\left(\frac{j}{2}\right)^2 - 2\left(\frac{j}{2}\right)\right]$$

$$= \frac{1}{2}\sum_{j=1}^{6}\left[\frac{j^2}{4} - j\right]$$

$$= \frac{1}{2}\left(\frac{1}{4}\sum_{j=1}^{6} j^2 - \sum_{j=1}^{6} j\right)$$

App. E   App. E.

$$= \frac{1}{2}\left(\frac{1}{4} \cdot \frac{\cancel{6} \cdot 7 \cdot 13}{\cancel{6}} - \frac{\cancel{6}^{3} \cdot 7}{\cancel{2}}\right)$$

$$= \frac{1}{2}\left(\frac{91}{4} - 21\right) = \frac{1}{2}\left(\frac{91-84}{4}\right)$$

$$= \frac{7}{8}.$$