Appendix E: Sigma Notation — Nov. 16, '15

Simplify

1. $1 + 2 + \cdots + 100$

2. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4}$

3. $\underbrace{3 + 3 + \cdots + 3}_{105 \text{ times}}$

DEFINITION: $a_m, a_{m+1}, \ldots, a_n$ are $n-m+1$ real numbers and m, n are integers

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_n$$

"Summation of a_j, where j goes from m to n."

Examples

1. $1 + 2 + \cdots + 100 = \sum_{j=1}^{100} j$

2. $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} = \sum_{j=1}^{4} \frac{1}{3^j}$

$= \sum_{j=0}^{3} \frac{1}{3^{j+1}}$

both are correct.

3. $\underbrace{3 + \cdots + 3}_{105 \text{ times}} = \sum_{j=1}^{105} 3.$

Ex. Write the following in sigma notation.

1. $1+3+5+\cdots+2n-1.$

Solution

$$\sum_{j=1}^{n} 2j-1$$

2. $\frac{3}{7}+\frac{4}{8}+\frac{5}{9}+\cdots+\frac{23}{27}$

Solution $\frac{3}{7}+\frac{3+1}{7+1}+\frac{3+2}{7+2}+\cdots+\frac{3+20}{7+20}$

$$=\sum_{j=0}^{20}\frac{3+j}{7+j}$$

3. Alternating sum.

$$\underbrace{1-1+1-1+\cdots+1}_{33 \text{ times}}$$

$$(-1)^0+(-1)^1+(-1)^2+(-1)^3+\cdots+(-1)^{32}$$

$$=\sum_{j=0}^{32}(-1)^j$$

$$\underbrace{(1-1)}_{0}+\underbrace{(1-1)}_{0}+\underbrace{(1-1)}_{0}+\underbrace{\cdots+1}_{32^{nd}}\ 1$$

First 32 terms cancel to give a 0

$= 1.$

Some rules:

1. $\sum_{j=m}^{n} c a_j = c \sum_{j=m}^{n} a_j$ 　 c: Constant does not depend on j.

2. $\sum_{j=m}^{n} a_j + \sum_{j=m}^{n} b_j = \sum_{j=m}^{n} (a_j + b_j)$

3. $\sum_{j=m}^{n} a_j - \sum_{j=m}^{n} b_j = \sum_{j=m}^{n} a_j - b_j$

4. $\sum_{j=1}^{n} 1 = n$; $\sum_{j=1}^{n} c = cn$.

5. $1 + 2 + 3 + \cdots + n$

Solution : Gauss @ the age of 10

$$S = 1 + 2 + \cdots + n-1 + n$$

$$+ \quad S = n + n-1 + \cdots + 2 + 1$$

$$2S = (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

$$= n(n+1)$$

So, $S = \dfrac{n(n+1)}{2}$

$$\boxed{\sum_{j=1}^{n} j = \frac{n(n+1)}{2}}$$

Sum of first 100 integers

$= \dfrac{100(101)}{2}$

$= 50 \times 101$

$= 5050.$

$j=1$ $\qquad j=1$ $\qquad$ 10

$= 50 \times 101$

$= 5050.$

6. $1^2+2^2+3^2+\cdots+n^2 = \sum_{j=1}^{n} j^2$

Solution:

$$\sum_{i=1}^{n} (1+i)^3 - i^3$$

Telescoping sum

$$= \left((1+1)^3-1^3\right)+\left((1+2)^3-2^3\right)$$

$$+ \left((1+3)^3-3^3\right)+\cdots+\left((1+n)^3-n^3\right)$$

$$= \underbrace{\cancel{2^3}-1^3} + \underbrace{\cancel{3^3}-\cancel{2^3}} + \underbrace{\cancel{4^3}-\cancel{3^3}}$$

$$+\cdots+\underbrace{(1+n)^3-\cancel{n^3}}$$

$$-1^3+(1+n)^3$$

$$= -1+1+3n+3n^2+n^3$$

$$= n^3+3n^2+3n.$$

$$\sum_{i=1}^{n} (1+i)^3 - i^3$$

$$= \sum_{i=1}^{n} \left(1+3i+3i^2+\cancel{i^3}-\cancel{i^3}\right)$$

$$= \sum_{i=1}^{n} 3i+3i^2+\sum_{i=1}^{n} 1$$

$$= 3\sum_{i=1}^{n} i+3\sum_{i=1}^{n} i^2+n$$

$$= 3\frac{n(n+1)}{2}+3S+n.$$

$$n^3+3n^2+3n = \frac{3}{2}n(n+1)+n+3S$$

$$3S = n^3+3n^2+3n-n-\frac{3}{2}n^2-\frac{3}{2}n$$

$$\boxed{S = \frac{n(n+1)(2n+1)}{6}}$$

7. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \sum_{j=1}^{n} j^3$

Ex: Try $\sum_{j=1}^{n} (1+j)^4 - j^4$!

Alternate Solution:

To Prove

$$S_n : \sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}.$$

Step 0 : Call the above statement S_n.

Step 1 : Show S_1 is true

$\rightarrow$ L.H.S. $\sum_{j=1}^{1} j^3 = 1^3 = 1$

R.H.S $\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$

$\}$ S_0, S_1 is true!

Step 2: Assume S_k is true and show that S_{k+1} is also true.

Assume: $\sum_{j=1}^{k} j^3 = \frac{k^2(k+1)^2}{4}$

To show: $\sum_{j=1}^{k+1} j^3 = \frac{(k+1)^2(k+2)^2}{4}$

$$\sum_{j=1}^{k+1} j^3 = \sum_{j=1}^{k} j^3 + (k+1)^3$$

Step 2: Assume S_k is true and show that S_{k+1} is also true.

Assume: $$\sum_{j=1}^{k} j^3 = \frac{k^2(k+1)^2}{4}$$

To show: $$\sum_{j=1}^{k+1} j^3 = \frac{(k+1)^2(k+2)^2}{4}$$

$$\sum_{j=1}^{k+1} j^3 = \sum_{j=1}^{k} j^3 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= (k+1)^2\left(\frac{k^2}{4} + k+1\right)$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}.$$ So, S_{k+1} is true!

So, by mathematical induction,

$$\boxed{\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}}$$

Evaluate.

1. $\displaystyle\sum_{j=1}^{30}\left(\frac{1}{j}-\frac{1}{j+1}\right)$

Telescoping sum.

$$=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots+\left(\cancel{\frac{1}{29}}-\cancel{\frac{1}{30}}\right)+\left(\cancel{\frac{1}{30}}-\frac{1}{31}\right)$$

$$=1-\frac{1}{31}=\frac{30}{31}.$$

2. $\displaystyle\sum_{i=1}^{n}(2-5i)=\sum_{i=1}^{n}2-\sum_{i=1}^{n}5i$

$$=2\sum_{i=1}^{n}1-5\sum_{i=1}^{n}i$$

$$=2n-5\frac{n(n+1)}{2}$$

$$=2n-\frac{5n^2}{2}-\frac{5}{2}n$$

$$=-\frac{5n^2}{2}-\frac{n}{2}.$$

3. $\displaystyle\sum_{j=3}^{100} j(j+4) = \sum_{j=3}^{100} j^2 + 4j$

Recall $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

$$= \underbrace{\sum_{j=3}^{100} j^2 + 4\sum_{j=3}^{100} j}$$

$\displaystyle\sum_{j=3}^{100} j = \sum_{j=1}^{100} j - \sum_{j=1}^{2} j$

$$= \sum_{j=1}^{100} j^2 - 1^2 - 2^2 + 4\left(\sum_{j=1}^{100} j - 1 - 2\right)$$

$$= \frac{(\cancel{100}^{50})(101)(\cancel{201}^{67})}{\cancel{6}\,\cancel{3}} - 1 - 4 + 4\left(\frac{\cancel{100}^{50}\cdot 101}{\cancel{2}} - 3\right)$$

$$= (5050)(67) - 5 + 4(5050-3)$$

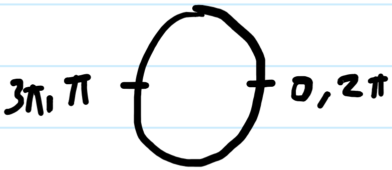
$$= 338350 - 5 + 4(5047)$$

$$= 338345 + 20188$$

$$= 358533.$$

3. $19 - 0 + 1$ terms $= 20$ terms

$$\sum_{i=0}^{19} \cos(i\pi) = \cos(0) + \cos(\pi) + \cos(2\pi) + \cdots + \cos(18\pi) + \cos(19\pi)$$



$$= 1 - 1 + 1 - 1 + \cdots + 1 - 1$$

$$= 0.$$

Geometric series

4. $$S = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n}$$

trick:

$$S = \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$$

$$- \quad \frac{1}{3}S = \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n} + \frac{1}{3^{n+1}}$$

$$\frac{2}{3}S = \frac{1}{3} - \frac{1}{3^{n+1}}$$

$$\frac{2}{3}S = \frac{3^n - 1}{3^{n+1}}$$

$$S = \frac{\cancel{3}}{2} \cdot \frac{3^n - 1}{3^{\cancel{n+1}}} = \frac{1}{2}\left(\frac{3^n - 1}{3^n}\right).$$

$$\sum_{j=0}^{n} r^j = 1 + r + r^2 + r^3 + \cdots + r^n$$

$$= \frac{1 - r^{n+1}}{1 - r}.$$

Evaluate:

1. $$\lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} \left(\frac{1}{n}\left(\left(\frac{i}{n}\right)^2 + \frac{1}{n}\right)\right)}$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \sum_{i=1}^{n} \left[\left(\frac{i}{n}\right)^2 + \frac{1}{n}\right]\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n}\left(\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 + \sum_{i=1}^{n}\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n}\left(\frac{1}{n^2}\sum_{i=1}^{n} i^2 + n\cdot\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n}\left[\frac{1}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + 1\right]\right]$$

$$= \lim_{n\to\infty} \left[\frac{n(2n^2+3n+1)}{n^3\cdot 6} + \frac{1}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{\frac{2n^3+3n^2+n}{n^3}}{\frac{6n^3}{n^3}} + \frac{1}{n}\right]$$

$$= \lim_{n\to\infty} \frac{2+\overset{0}{3/n}+\overset{0}{1/n^2}}{6} + \overset{0}{\lim_{n\to\infty}\frac{1}{n}} = \frac{1}{3}$$

Evaluate:

$$\lim_{n\to\infty}\left(\sum_{i=2}^{n}\frac{1}{n}\left(\frac{i}{n}+1\right)^{3}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=2}^{n}\frac{i^{3}}{n^{3}}+\frac{3i^{2}}{n^{2}}+\frac{3i}{n}+1\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{n}\left(\frac{1}{n^{3}}\sum_{i=2}^{n}i^{3}+\frac{3}{n^{2}}\sum_{i=2}^{n}i^{2}+\frac{3}{n}\sum_{i=2}^{n}i+\overbrace{\sum_{i=2}^{n}1}^{n-2+1=n-1\text{ terms.}}\right)\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{n}\left(\frac{1}{n^{3}}\left(\frac{n^{2}(n+1)^{2}}{4}-1\right)+\frac{3}{n^{2}}\left(\frac{n(n+1)(2n+1)}{6}-1\right)+\frac{3}{n}\left(\frac{n(n+1)}{2}-1\right)+n-1\right)\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{n}\left(\frac{n^{2}(n+1)^{2}-4}{n^{3}}+\frac{3}{n^{2}}\left(\frac{n(2n^{2}+3n+1)-6}{6}\right)+\frac{3}{n}\left(\frac{n^{2}+n-2}{2}\right)+n-1\right)\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n} \left(\frac{1}{n^3} \left(\frac{n^4+2n^3+n^2-4}{4} \right) + \frac{3}{n^2} \left(\frac{2n^3+3n^2+n-6}{6} \right) + \frac{3}{n} \left(\frac{n^2+n-2}{2} \right) + n-1 \right) \right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(\frac{n^4+2n^3+n^2-4}{4n^3} + \frac{2n^3+3n^2+n-6}{2n^3} + \frac{3n^2+3n-6}{2n} + n-1 \right)$$

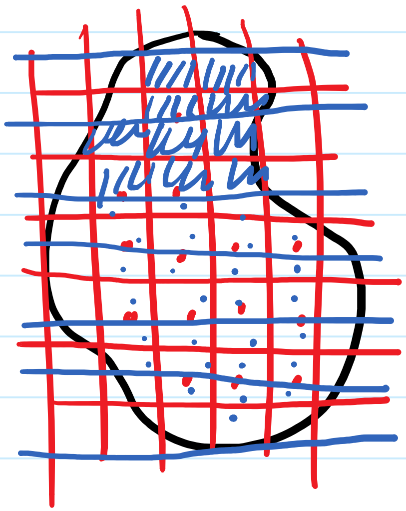
$$= \lim_{n\to\infty} \left(\frac{n^4+2n^3+n^2-4}{4n^4} + \frac{2n^3+3n^2+n-6}{2n^3} + \frac{3n^2+3n-6}{2n^2} + \frac{n-1}{n} \right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{4} + \underbrace{\frac{1}{2n}}_{\to 0} + \underbrace{\frac{1}{4n^2}}_{\to 0} - \underbrace{\frac{1}{n^4}}_{\to 0} + 1 + \underbrace{\frac{3}{2n}}_{\to 0} + \underbrace{\frac{1}{2n^2}}_{\to 0} - \underbrace{\frac{3}{n^3}}_{\to 0} + \frac{3}{2} + \underbrace{\frac{3}{2n}}_{\to 0} - \underbrace{\frac{3}{n^2}}_{\to 0} + 1 - \underbrace{\frac{1}{n}}_{\to 0} \right)$$

$$= \frac{1}{4} + 1 + \frac{3}{2} + 1 = \frac{1+4+6+4}{4} = \frac{15}{4}.$$

5.1 Areas and Distances

IDEA:



Each square is $1 cm^2$

$A \geq 14 cm^2$

Each rectangle $\frac{1}{2} cm^2$

$A \geq 36 \cdot \frac{1}{2}$

$18 cm^2$

Finer grids give better estimates

Example : $y = x^2$

Area under the graph between $x=0$ and $x=1$

- Divide $[0,1]$ into 4 parts.
- Take the right end-point of each part
- Draw a rectangle with height the y-value at that point and base, the length of the part.

Sum of area of rectangles $\geq A$

right

R_4 = no. of parts

$$\frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}(1)^2$$

$$\frac{1}{4}\left(\frac{1}{4^2} + \frac{1}{2^2} + \left(\frac{3}{4}\right)^2 + 1^2\right) \geq A$$

http://mathinsight.org/applet/area_right_riemann_sum

$$R_4 = \frac{1}{4}\left(\frac{1}{16}+\frac{1}{4}+\frac{9}{16}+1\right)$$

$$= \frac{1}{4}\left(\frac{1+4+9+16}{16}\right)$$

$$= \frac{30}{64} \geq A$$

right sum

R_n

$[0,1]$ into n equal parts

$$= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right]$$

base: $\frac{1-0}{n}$

height:

$$\left[0,\frac{1}{n}\right] \left[\frac{1}{n},\frac{2}{n}\right] \cdots \left[\frac{n-1}{n},1\right]$$

$$R_n = \frac{1}{n}\left[\sum_{j=1}^{n}\left(\frac{j}{n}\right)^2\right]$$

right-sums

$$\lim_{n\to\infty} R_n = A$$

$$\lim_{n\to\infty}\frac{1}{n}\left[\sum_{j=1}^{n}\frac{j^2}{n^2}\right] = \lim_{n\to\infty}\frac{1}{n^3}\sum_{j=1}^{n} j^2$$

$$= \lim_{n\to\infty}\frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3} = \frac{1}{3}$$