4.9. Anti- derivatives

Nov. 11. 2015

Definition: A function F is called an anti-derivative of the function f on the interval I if

$$F'(x) = f(x) \quad \text{on } I.$$

Examples: Find an anti-derivative:

① $f(x) = x^2$

Try: $F(x) = x^3$ Check: $F'(x) = 3x^2$

3 times what we want!

So, we try $F(x) = \frac{1}{3}x^3$

Check: $F'(x)$

$\frac{1}{\cancel{3}} \cdot \cancel{3}x^2 = x^2 = f(x)$

So, $\frac{1}{3}x^3$ is an anti-derivative of x^2.

② $f(x) = e^x$

So, $F(x) = e^x$ is an anti-derivative of $f(x)$.

But, what about $F(x) = e^x + 1$.

$$F'(x) = e^x$$

So, $e^x + 1$ is also an anti-derivative of e^x.

Theorem: If F is an anti-derivative of f then, the most general anti-derivative of f is

$$F(x) + c, \quad c \text{ is a constant}$$

Example: $e^x + c$ is the most general anti-derivative of $f(x) = e^x$.

anti-derivative of x.

Example: $e^x + C$ is the most general

anti-derivative of $f(x) = e^x$.

Examples Find the most general anti-derivative of

① $f(x) = \sin x.$

Solution $\frac{d}{dx}\cos(x) = -\sin x$

$-\frac{d}{dx}\cos x = \sin x$

$\frac{d}{dx}(-\cos x) = \sin x$

So, $-\cos x$ is one anti-derivative of $\sin x$. So, the most general anti-der. is

$-\cos x + C$, C is a constant.

② $f(x) = \frac{1}{x} = x^{-1}$ $x > 0$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$, when $x > 0$

So, the most general anti-der. of $\frac{1}{x}$ is $\ln x + C$, C is a constant.

③ $f(x) = x^n$ $(n \neq -1)$

$F(x) = \frac{x^{n+1}}{n+1}$

So, the most general anti-derivative of x^n is $\frac{x^{n+1}}{n+1} + C$.

Check $\frac{d}{dx} x^{n+1}$

$= n+1\, x^n$

$\frac{1}{n+1}\frac{d}{dx} x^{n+1} = x^n$

$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$

LIST OF ANTI-DERIVATIVES

F is an anti-der. of f
G " " " " g

function	an anti-derivative
$f+g$	$F+G$
$c \cdot f$	cF
x^{-1}	$\ln\|x\|$
x^n	$\dfrac{x^{n+1}}{n+1}$
$x^0 = 1$	x
e^x	e^x
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\dfrac{1}{1+x^2}$	$\tan^{-1}(x)$

Examples: Find all the functions g such that

$$g'(x) = \frac{4x\cos x + \sqrt{x} - 10x^2}{x}$$

$$= 4\cos x + x^{-1/2} - 10x$$

So, $g(x) = 4\sin x + \dfrac{x^{-1/2+1}}{-1/2+1} - \dfrac{10x^2}{2} + C$

$$= 4\sin x + 2\sqrt{x} - 5x^2 + C.$$

C is a constant.

$\cos x$	$\sin x$	c is a constant.

(2) Find all functions f such that

(i) $f'(x) = \sec x \tan x - 2e^x$

Solution: $f(x) = \sec x - 2e^x + C$.

C - constant.

(ii) $f'(x) = 2\cos x - \dfrac{3}{\sqrt{1-x^2}}$

Solution

$f(x) = 2\sin x - 3\sin^{-1}(x) + C$.

C: constant.

(iii) $f'(x) = 1 + 2\sin x + \dfrac{3}{\sqrt{x}}$

and $f(0) = 2$.

General expression for $f(x)$

$$= x + 2(-\cos x) + 3\frac{x^{-1/2+1}}{-\frac{1}{2}+1} + C$$

$$= x - 2\cos x + 6\sqrt{x} + C.$$

$f(0) = 2$, so we set $x = 0$

$$2 = 0 - 2\cos(0) + 6\sqrt{0} + C$$

$$2 = -2 + C$$

so, $\boxed{C = 4}$

$$f(x) = x - 2\cos x + 6\sqrt{x} + 4.$$

③ Find $f(x)$ if

$$f'(x) = 3(1+x^2)^{-1} + 2^x = \frac{3}{1+x^2} + 2^x$$

and $f(0) = 2$.

<u>Solution</u> The most general anti-derivative of $f'(x)$:

$$f(x) = 3\tan^{-1}(x) + \frac{2^x}{\ln(2)} + C.$$

$f(0) = 2$

$$3\tan^{-1}(0) + \frac{2^0}{\ln(2)} + C = 2$$

$$\frac{1}{\ln(2)} + C = 2$$

$$\boxed{\begin{aligned} &\frac{d}{dx}(2^x) = 2^x \ln 2 \\ &\frac{1}{\ln 2}\frac{d}{dx} 2^x = 2^x \\ &\frac{d}{dx}\left(\frac{2^x}{\ln 2}\right) = 2^x \end{aligned}}$$

So, $C = 2 - \frac{1}{\ln(2)}$.

Answer $f(x) = 3\tan^{-1}(x) + \frac{2^x}{\ln(2)} + 2 - \frac{1}{\ln(2)}$.

Find f.

① $f''(x) = x^{-2} + e^x$

$\hookrightarrow f'(x) = \frac{x^{-2+1}}{-2+1} + e^x + C. = -\frac{1}{x} + e^x + C$

$\hookrightarrow f(x) = -\ln|x| + e^x + Cx + D.$

C, D constants.

② Find f.

$$f''(x) = x^{-2} + x^2 \quad \text{— (i)}$$

$$f(2) = 3 \quad \text{— (ii)}$$

$$f'(1) = 2. \quad \text{— (iii)}$$

$$f'(x) = \frac{x^{-2+1}}{-2+1} + \frac{x^{2+1}}{2+1} + C.$$

$$= -\frac{1}{x} + \frac{x^3}{3} + C.$$

From (iii)

$$2 = f'(1) = -\frac{1}{1} + \frac{1^3}{3} + C$$

$$2 = -1 + \frac{1}{3} + C$$

So, $C = 2 + 1 - \frac{1}{3} = \frac{8}{3}$

So, $f'(x) = -\frac{1}{x} + \frac{x^3}{3} + \frac{8}{3}$

anti-der.

$$f(x) = -\ln|x| + \frac{1}{3}\frac{x^4}{4} + \frac{8}{3}x + D$$

Use (ii)

$$3 = f(2)$$

$$= -\ln|2| + \frac{2^4}{12} + \frac{8 \cdot 2}{3} + D$$

$$3 = -\ln(2) + \frac{16}{12} + \frac{16}{3} + D$$

$$D = 3 + \ln(2) - \frac{\cancel{16}\,4}{3}\left(\frac{5}{\cancel{4}}\right)$$

$$f(x) = -\ln|x| + \frac{x^4}{12} + \frac{8x}{3} + 3 + \ln(2) - \frac{20}{3}.$$

Example: Find the anti-derivative of

$$f(x) = e^{3x}$$ which takes the value 1 at $x=0$.

Solution: Guess: $F(x) = e^{3x}$

Check: $F'(x) = e^{3x} \cdot 3$

$$\frac{1}{3}\frac{d}{dx}(e^{3x}) = e^{3x}$$

So $\frac{d}{dx}\left(\frac{e^{3x}}{3}\right) = e^{3x}$

So the most general anti-derivative is $\frac{e^{3x}}{3} + C$.

at $x=0$, $\frac{e^{3\cdot 0}}{3} + C = 1$

$$\frac{1}{3} + C = 1$$

or $\boxed{C = \frac{2}{3}}$

Answer: $F(x) = \frac{e^{3x}}{3} + \frac{2}{3}$.

Example: 1 Find the derivative of

$$\frac{1}{1+x^2}.$$

2 Find $g(x)$ so that

$$g''(x) = \frac{-2x}{(1+x^2)^2} + \sin x$$

$g(0)=1$ and $g(1)=\frac{\pi}{4}$.]

Solution: $y = \frac{1}{1+x^2} = (1+x^2)^{-1}$

$$y' = -1(1+x^2)^{-2}(2x)$$

$$= \frac{-2x}{(1+x^2)^2}$$

2 $g''(x) = \frac{-2x}{(1+x^2)^2} + \sin x$

anti-der.

$$g'(x) = \frac{1}{1+x^2} - \cos x + C.$$

antider.

$$g(x) = \tan^{-1}(x) - \sin x + Cx + D.$$

$$1 = g(0) = \tan^{-1}(0) - \sin(0) + C\cdot 0 + D$$

$$\boxed{1 = D}$$

$$\frac{\pi}{4} = g(1) = \tan^{-1}(1) - \sin(1) + C\cdot 1 + 1$$

$$\cancel{\frac{\pi}{4}} = \cancel{\frac{\pi}{4}} - \sin(1) + C + 1$$

so, $C = \sin(1) - 1$

$$g(x) = \tan^{-1}(x) - \sin x + (\sin(1)-1)x + 1.$$

Rectilinear motion.

t : time or displacement.
$s(t)$: position function
$v(t)$: velocity "
$a(t)$: acceleration function.

$$a(t) = v'(t) = s''(t)$$
$$v(t) = s'(t)$$

Example: A particle moves in a straight line and has velocity given by

$$v(t) = t^3 - 4\sqrt[3]{t}$$

and its initial displacement is 3cm. What is its displacement in 1 sec.?

Solution: $v(t) = t^3 - 4\sqrt[3]{t}$
$s(0) = 3$

To find: $s(1)$.

Answer: $s(t)$ is an anti-der. of $v(t)$.

$$s(t) = \frac{t^4}{4} - 4\frac{t^{4/3}}{4/3} + C$$

$$= \frac{t^4}{4} - 3t^{4/3} + C$$

Given

$$3 = s(0) = 0 - 3\cdot 0 + C$$

$$\boxed{C = 3}$$

$$s(t) = \frac{t^4}{4} - 3t^{4/3} + 3$$

$$s(1) = \frac{1}{4} - \cancel{3\cdot 1} + \cancel{3} = \frac{1}{4} \text{ cm.}$$