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4.4 INDETERMINATE FORMS AND L'HOSPITAL'S RULE

Type 1 ($\frac{0}{0}$ form)

Substitution $x=a$ $\lim_{x\to a} \frac{f(x)}{g(x)}$ has $\frac{0}{0}$ form.

Examples

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

2. $\lim_{x\to 1} \frac{x^3-3x^2+3x-1}{x^2-1} = \lim_{x\to 1} \frac{(x-1)^3}{(x-1)(x+1)} = \lim_{x\to 1} \frac{(x-1)^2}{x+1} \overset{\text{sub}}{=} \frac{(1-1)^2}{1+1} = 0.$

3. $\lim_{x\to 1} \frac{x^2-1}{x^3-3x^2+3x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)^3} = \lim_{x\to 1} \frac{(x+1)_{>0}}{(x-1)^2_{>0}} = +\infty.$

If you have a $\frac{0}{0}$ form, you can get any limit.

Type 2 ($\frac{\infty}{\infty}$ form)

Arises when looking for asymptotes

Examples

1. $\lim_{x\to\infty} \dfrac{x^2-1}{2x^2+1}$ $\qquad \dfrac{\lim_{x\to\infty} x^2-1}{\lim_{x\to\infty} 2x^2+1} = \dfrac{\infty}{\infty}$

$= \lim_{x\to\infty} \dfrac{\frac{x^2-1}{x^2}}{\frac{2x^2+1}{x^2}} = \lim_{x\to\infty} \dfrac{1-\frac{1}{x^2}\to 0}{2+\frac{1}{x^2}\to 0}$

$= \frac{1}{2}$

2. $\lim_{x\to\infty} \dfrac{x^3-1}{2x^2+1}$ $\qquad \left(\dfrac{\lim_{x\to\infty} x^3-1}{\lim_{x\to\infty} 2x^2+1} = \dfrac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{\frac{x^3+1}{x^2}}{\frac{2x^2+1}{x^2}} = \lim_{x\to\infty} \dfrac{x+\frac{1}{x^2}\to 0}{2+\frac{1}{x^2}\to 0}$

$= \infty$

3. $\lim_{x\to\infty} \dfrac{x^2-1}{2x^3+1}$ $\qquad \left(\dfrac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{\frac{x^2-1}{x^3}}{\frac{2x^3+1}{x^3}} = \lim_{x\to\infty} \dfrac{\frac{1}{x}\to 0 - \frac{1}{x^3}\to 0}{2+\frac{1}{x^2}\to 0}$

$= \frac{0}{2} = 0.$

Examples of limits we can't compute with current techniques.

① $\lim_{x\to 0} \dfrac{e^x-1}{x}$ $\left(\dfrac{0}{0}\right)$ $\qquad$ ② $\lim_{x\to\infty} \dfrac{\ln(x)}{x-1}$ $\left(\dfrac{\infty}{\infty}\right)$

L'HOSPITAL'S RULE

Suppose f and g are differentiable and $g'(x) \neq 0$ for x near a (but it's ok if $g'(a) = 0$). Suppose

$\left(\frac{0}{0}\right)$ $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

OR

$\left(\frac{\infty}{\infty}\right)$ $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

Then,
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \underbrace{\lim_{x \to a} \frac{f'(x)}{g'(x)}}_{\text{if this limit exists in } (-\infty,\infty) \text{ or } \pm\infty.}.$$

Examples.

① $\lim_{x \to 0} \frac{e^x - 1}{x}$ $\begin{matrix} f(x) \\ g(x) \end{matrix}$ $\left(x=0 \quad \frac{e^0 - 1}{0} = \frac{0}{0} \checkmark \right)$

$\overset{L'H}{=} \lim_{x \to 0} \frac{e^x}{1} \overset{\text{sub}}{=} e^0 = \underline{\underline{1}}$

② $\lim_{x \to \infty} \frac{\ln(x)}{x-1}$ $\left(\frac{\infty}{\infty}\right)$ $\overset{L'H}{=} \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x}$

$= \underline{\underline{0}}$

③ $\lim_{x\to 1} \frac{\ln(x)}{x-1}$ $\left(\frac{\ln(1)}{1-1} = \frac{0}{0}\right)$

$\stackrel{L'H}{=} \lim_{x\to 1} \frac{1/x}{1} = \lim_{x\to 1} \frac{1}{x} = 1.$

④ $\lim_{\underline{x\to 0}} \frac{\tan x - x}{x^3}$ $g(x)$ $\left(\frac{\tan 0 - 0}{0^3} = \frac{0}{0} \text{ form}\right)$

$g'(x) = 3x^2$ which is non-zero near 0 (but not at 0)

$\stackrel{L'H}{=} \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$ $\left(\frac{\sec^2 0 - 1}{3\cdot 0^2} = \frac{1-1}{0} = \frac{0}{0}\right)$

$\stackrel{L'H}{=} \lim_{x\to 0} \frac{\cancel{2}\sec x \cdot \sec x \cdot \tan x}{3\,\cancel{6}x}$

$= \lim_{x\to 0} \frac{\sec^2 x \cdot \tan x}{3x}$ $\left(\frac{\sec^2 0 \tan 0}{0} = \frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{1}{3x} \frac{1}{\cos^2 x} \frac{\sin x}{\cos x}$

$= \frac{1}{3} \lim_{x\to 0} \frac{1}{\cos^3 x} \cdot \lim_{x\to 0} \frac{\sin x}{x}$

$= \frac{1}{3} \cdot \frac{1}{\cos^3(0)} \cdot 1 = \underline{\underline{\frac{1}{3}}}.$

EXAMPLES WHERE L'H. RULE DOES NOT WORK!

(1.) $\lim_{x\to\pi^+} \frac{\sin x}{1-\cos x}$ ~~$\overset{L'H}{=} \lim_{x\to\pi^+} \frac{\cos x}{\sin x} = \lim_{x\to\pi^+} \cot x = -\infty.$~~

$\left(x=\pi \quad \frac{\sin(\pi)}{1-\cos(\pi)} = \frac{0}{1-(-1)} = \frac{0}{2} \right)$ NOT INDETERMINATE!

$\underset{=}{\text{substitution}}$ $0.$

(2) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$ $\left(\frac{\infty}{\infty}\right)$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{1}{\cancel{\frac{1}{2}}(x^2+1)^{-1/2}\cdot \cancel{2}x} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$ $\left(\frac{\infty}{\infty}\right)$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{\cancel{2}\sqrt{x^2+1}}\cdot\cancel{2}x}{1} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}.$

$= \lim_{x\to\infty} \frac{\frac{x}{\sqrt{x^2}}}{\frac{\sqrt{x^2+1}}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{\frac{x}{x}}{\sqrt{1+\frac{1}{x^2}}}$ $\sqrt{x^2} = x \quad x>0$

$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1.$

③ $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x^2}{x^4}$ $\left(\dfrac{e^0-1-0^2}{0} = \dfrac{1-1}{0} = \dfrac{0}{0}\right)$

$\overset{L'H.}{=} \displaystyle\lim_{x\to 0} \frac{e^x - 2x}{4x^3}$ $\left(\dfrac{e^0 - 2\cdot 0}{4\cdot 0^3} = \dfrac{1}{0}\right)$

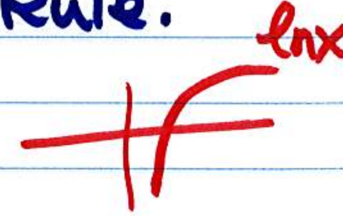
$\downarrow$ $+\infty$ OR $-\infty$

$\displaystyle\lim_{x\to 0^+} \frac{e^x - 2x^{>0}}{4x^3{}_{>0}} = +\infty$ $e^x \sim 1,\ 2x \sim 0$

$\displaystyle\lim_{x\to 0^-} \frac{e^x - 2x^{>0}}{4x^3{}_{<0}} = -\infty$

$\displaystyle\lim_{x\to 0} \frac{e^x - 2x}{4x^3}$ <u>DNE.</u>

So, $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x^2}{x^4}$ cannot be computed using L'H Rule.



Type 3 $(0\cdot\infty$ type$)$

Examples

① $\displaystyle\lim_{x\to 0^+} x\ln(x)$ $\left(\displaystyle\lim_{x\to 0^+} x = 0 \quad \lim_{x\to 0^+} \ln x = -\infty \quad 0\cdot -\infty\right)$

② $\displaystyle\lim_{x\to \infty} e^x\left(\tan^{-1}(x) - \frac{\pi}{2}\right)$ $\left(\displaystyle\lim_{x\to\infty} e^x = \infty \quad \lim_{x\to\infty} \tan^{-1}(x) - \frac{\pi}{2} = 0\right)$

$0 \cdot \infty$ $1 \cdot \infty$ $1 \cdot 0$ $\infty \cdot \infty$

INDET. ∞ 0 ∞

TRICK:

$$\underset{0}{f(x)} \cdot \underset{\infty}{g(x)} = f(x) \cdot \frac{1}{\frac{1}{g(x)}} = \frac{f(x)}{1/g(x)} \quad \frac{0}{0}$$

L'H

$$\underset{0}{f(x)} \cdot \underset{\infty}{g(x)} = \frac{1}{\frac{1}{f(x)}} \cdot g(x) = \frac{g(x)}{1/f(x)} \quad \frac{\infty}{\infty}$$

EXAMPLES

① $\lim_{x \to 0^+} x \ln(x)$ $(0 \cdot -\infty)$

$$= \lim_{x \to 0^+} \frac{\ln(x)}{1/x} \quad \left(\frac{-\infty}{\infty} \text{ form}\right)$$

$$\overset{L'H}{=} \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} \frac{-x^2}{x}$$

$$= \lim_{x \to 0^+} (-x) = \underline{0.}$$

② $\lim_{x \to \infty} e^x \left(\tan^{-1}(x) - \frac{\pi}{2}\right)$ $(\infty \cdot 0 \text{ form})$

$$= \lim_{x \to \infty} \frac{\tan^{-1}(x) - \pi/2}{\cancel{e^{-x}}\ 1/e^x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{\tan^{-1}(x) - \frac{\pi}{2}}{e^{-x}}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{-e^{-x}} = \lim_{x\to\infty} -\frac{e^x}{1+x^2} \left(\frac{-\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{-e^x}{2x} \left(\frac{-\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{-e^x}{2} = -\frac{1}{2}\cdot\infty$$

$$= -\infty.$$

③ $\lim_{x\to\frac{\pi}{2}^-} (\tan x)(x - \frac{\pi}{2})$ $(\infty \cdot 0 \text{ form})$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{(x-\pi/2)}{1/\tan x}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{x-\pi/2}{\cot x} \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{1}{-\operatorname{cosec}^2 x}$$

$$= \frac{1}{-\operatorname{cosec}^2(\frac{\pi}{2})} = -1$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\frac{1}{(x-\pi/2)}} \left(\frac{\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{\sec^2 x}{\frac{-1}{(x-\pi/2)^2}} \left(\frac{\infty}{\infty}\right)$$

$$\overset{L'H}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{2\sec x . \sec x \tan x}{\frac{-2}{(x-\frac{\pi}{2})^3}} \left(\frac{\infty}{\infty}\right)$$

$\vdots$

Type 4 $(\infty - \infty$ type$)$

Examples ① $\lim_{x \to \frac{\pi}{2}^-} \sec x - \tan x$

$\left(\sec(\tfrac{\pi}{2}) - \tan(\tfrac{\pi}{2}) \right)$
$\infty - \infty$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{1}{\cos x} - \frac{\sin x}{\cos x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{1 - \sin x}{\cos x} \quad \left(\frac{0}{0}\right)$$

$$\stackrel{L'H}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} = \lim_{x \to \frac{\pi}{2}^-} \cot x$$

$$= \underline{\underline{0}}.$$

② $\lim_{x \to \infty} \sqrt{x^2+1} - x \quad (\infty - \infty)$

$$\lim_{x \to \infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}$$

$$= \lim_{x \to \infty} \frac{\cancel{x^2} + 1 - \cancel{x^2}}{\sqrt{x^2+1} + x}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{x^2+1} + x} = 0 \quad \left(\frac{1}{\infty}\right)$$

Type 5 0^0 $\lim_{x\to 0^+} x^x$

Type 6 ∞^0 $\lim_{x\to\infty} (\ln x)^{1/x}$

Type 7 1^∞ $\lim_{x\to 0} (1+x)^{1/x}$

NOT INDETERMINATE

$$0^\infty = 0$$
$$\infty^\infty = \infty$$
$$\infty^{-\infty} = \frac{1}{\infty^\infty} = \frac{1}{\infty} = 0$$

TRICK

B

$$y = f(x)_0^{g(x)_0} \quad (0^0)$$
$$\ln y = \ln\left(f(x)^{g(x)}\right) = \underset{0\ \cdot\ -\infty}{g(x)\ln(f(x))} = \frac{\ln(f(x))}{1/g(x)}$$

Example

① $\lim_{x\to 0^+} x^x$. Set $y = x^x$

So, $\ln(y) = x\ln(x) = \dfrac{\ln(x)}{1/x}$

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \quad \left(\frac{-\infty}{\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0^+} -x = 0.$$

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln(y)} \quad \boxed{y = e^{\ln(y)}}$$

$$= e^{\lim_{y\to 0^+} \ln(y)}$$

$$= e^0 = 1.$$

So, $\lim_{x\to 0^+} x^x = 1.$

② $\lim_{x\to\infty} (\ln x)^{1/x} \quad (\infty^0)$

Set $y = (\ln x)^{1/x}$

$$\ln(y) = \ln\left((\ln x)^{1/x}\right)$$

$$= \frac{1}{x} \ln(\ln(x))$$

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln(\ln(x))}{x} \quad \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{\ln(x)} \frac{1}{x}}{1}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{1}{x\ln(x)} \quad \left(\frac{1}{\infty}\right)$$

$$= 0.$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y}$$

$$= e^{\lim_{x\to\infty} \ln(y)} = e^0 = 1.$$

③ $\lim_{x\to 0} (1+x)^{1/x}$ $\quad (1^\infty)$

Set $y = (1+x)^{1/x}$

$$\ln y = \frac{1}{x} \ln(1+x)$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1+x)}{x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = \frac{1}{1+0} = 1$$

$$\lim_{x\to 0} y = \lim_{x\to 0} e^{\ln(y)} = e^{\lim_{x\to 0}\ln(y)}$$

$$= e^{1} = \underline{\underline{e}}$$

$$\lim_{x\to\infty} (1+x)^{1/x} = e.$$

(4) $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$ (1^{∞}).

$$y = (1+\sin 4x)^{\cot x}$$

$$\ln y = \cot x \cdot \ln(1+\sin 4x)$$

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \underset{\infty}{\cot x} \cdot \underset{0}{\ln(1+\sin 4x)}$$

$$= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} \quad \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+\sin 4x}\cos 4x \cdot 4}{\sec^2 x}$$

$$= \frac{\frac{1}{1+0}\cdot 1 \cdot 4}{1} = 4$$

$$\lim_{x\to 0^+} y = e^{\lim_{x\to 0^+}\ln(y)} = \underline{\underline{e^4}}.$$