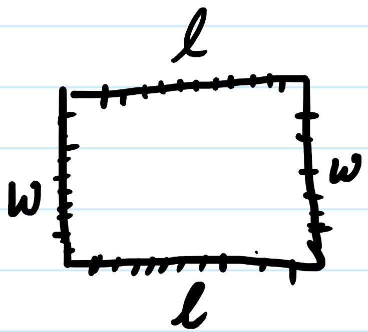
11.04.2015 4.7. OPTIMIZATION

REVISE : CLOSED INTERVAL METHOD

Ex. At a land-grabbing fest, you are given 2400 ft of fencing and allowed to keep any rectangular piece of land that you can completely surround with a fence. What is the largest area of land you can acquire at this fest?



ℓ: length of the plot
w: width of the plot

To maximize : A = area $= \ell w$ ①

Constraint: $2\ell + 2w = 2400$ ②

Isolate w in ② $w = \frac{2400 - 2\ell}{2} = 1200 - \ell$.

Substitute in ①

$$A(\ell) = \ell(1200 - \ell) = 1200\ell - \ell^2$$

To maximize $A(\ell)$ in $[0, 1200]$

CAN USE THE CLOSED INTERVAL METHOD BECAUSE A is continuous in $[0,1200]$.

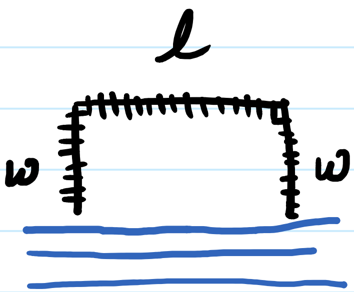
Critical numbers of $A(\ell)$ in $(0, 1200)$.

$A'(\ell) = 1200 - 2\ell$. So, $A'(\ell) = 0$, when $\ell = 1200/2 = 600$

$A(0) = 0$, $A(600) = 600(600) = 360{,}000$, $A(1200) = 0$ } So abs. maximum @ $\ell = 600$

The largest area that can be fenced by 2400 ft of fencing is 360,000 ft^2.

Ex. A farmer has 2400 ft. of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fencing along the river. What are the dimensions of the field that has the largest area?



l: length of the field
w: width " " "

To maximize:
A = area of the field
$= lw$. — ①

Constraint: $l + 2w = 2400$ — ②

Isolate l in ②, $l = 2400 - 2w$
Substitute in ①

$A(w) = (2400 - 2w)w = 2400w - 2w^2$

Maximize: $A(w)$, w in $[0, 1200]$.

① Since $A(w)$ is continuous in $[0,1200]$, we will use the C.I.M.

② Critical numbers in $(0, 1200)$.

$A'(w) = 2400 - 4w$.
So, $A'(w) = 0$ when $w = \frac{2400}{4} = 600$.

③ $A(0)$, $A(600)$, $A(1200)$

$A(0) = 0$

$A(600) = (2400 - 1200)\,600 = 1200 \cdot 600 = 720000$.

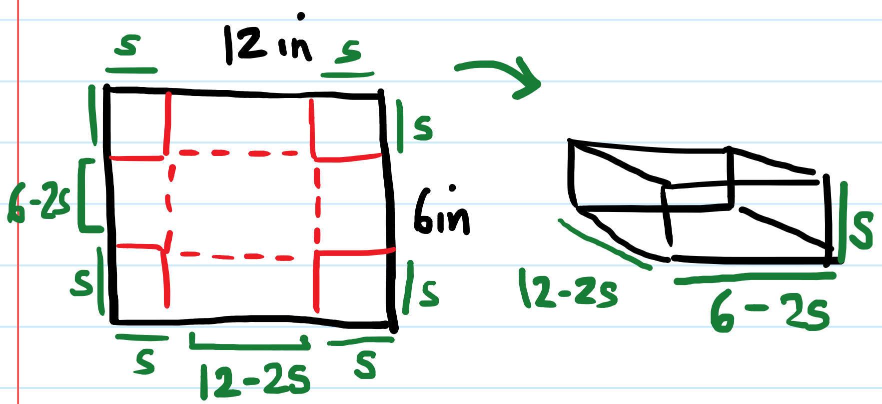
$A(1200) = 0$

So, A attains an abs. max. at $w = 600$ in $[0, 1200]$.

When $w = 600$, $l = 2400 - 2(600) = 1200$

So, the area maximizing are 600 ft by 1200 ft.

Ex. We have a piece of cardboard that is 12 in. by 6 in. We're going to cut out identical square corners and fold up the sides to make a box.
Determine the height of the box that will give the maximum volume.



s: length of the square cuts
$12-2s$: length of the base of the box
$6-2s$: width " " " " " "

To maximize V: volume of the box

$$= (s)(12-2s)(6-2s)$$
$$= 4s^3 - 36s^2 + 72s$$

Maximize $V(s)$ in $[0,3]$ $(6-2s \geq 0)$

① C.I.M: $V(s)$ is continuous in $[0,3]$.

② Critical numbers: $V'(s) = 12s^2 - 72s + 72$
$$= 12(s^2 - 6s + 6).$$
$V'(s) = 0$ when $s^2 - 6s + 6 = 0$
or when $s = \dfrac{6 \pm \sqrt{36-24}}{2}$
$$= 3 \pm \sqrt{3}.$$
But $3+\sqrt{3}$ is not in $[0,3]$

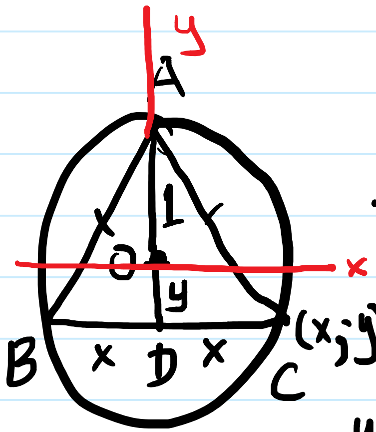
③ $\underset{0}{\underset{\shortparallel}{V(0)}},\ \underset{(3-\sqrt{3})(12-2(3-\sqrt{3}))(6-2(3-\sqrt{3}))}{\underset{\shortparallel}{V(3-\sqrt{3})}},\ \underset{0}{\underset{\shortparallel}{V(3)}}$ V attains an abs max @ $3-\sqrt{3}$ in $[0,3]$.

$$= (3-\sqrt{3})(6+2\sqrt{3})(2\sqrt{3}) > 0$$

Answer.

The height of the box with the maximum volume is $\underline{3-\sqrt{3} \text{ inches}}$.

Ex. Find the dimensions of the isosceles triangle of the largest area that can be inscribed in a circle of radius 1 unit.



$x^2+y^2=1$

length(OA) = 1

$(x,-y)$ x : $\frac{1}{2}$ base of triangle

$y+1$: height " "

$(x,-y)$ is on the circle.

So, $x^2+(-y)^2=1$

$x^2+y^2=1$ — ① Constraint.

To maximize : A : area of triangle

$= \frac{1}{\cancel{2}}(y+1)(\cancel{2}x)$

$A = x(y+1)$. — ②

Isolate x in ①

$x^2 = 1-y^2 \rightarrow$ so $x = \sqrt{1-y^2}$
$(x \geq 0)$

Substitute in ②

$A(y) = \sqrt{1-y^2}\,(y+1)$

So, we want to maximize $A(y)$ in $[0,1]$

① C.I.M. $\rightarrow$ A is continuous in $[0,1]$

② Critical numbers in $(0,1)$.

$A'(y) \overset{P.R.}{=} \frac{1}{2\sqrt{1-y^2}}\cdot(-2y)(y+1)$
$+ \sqrt{1-y^2}\cdot 1$

② Critical numbers in $(0,1)$.

$$A'(y) \overset{\text{P.R.}}{=} \frac{1}{\cancel{2}\sqrt{1-y^2}} \cdot (-\cancel{2}y)(y+1) + \sqrt{1-y^2} \cdot 1$$

$$= \frac{-(y+1)y}{\sqrt{1-y^2}} + \sqrt{1-y^2}$$

$$= \frac{-(y+1)y + 1-y^2}{\sqrt{1-y^2}}$$

$$= \frac{1-y-2y^2}{\sqrt{1-y^2}}.$$

$A'(y)=0$ when $1-y-2y^2=0$

$(0,1)$ $\qquad 1-2y+y-2y^2=0$

$A(y) = \sqrt{1-y^2}(y+1)$

OR $(1+y)(1-2y)=0$

$\to y=-1$ or $y=\frac{1}{2}$

But -1 is not in $(0,1)$, so we only consider $1/2$.

③ $A(0)$, $\quad A(1/2)$, $\quad A(1)$

$1 \qquad \sqrt{1-\frac{1}{4}}\left(\frac{1}{2}+1\right) \qquad 0$

$$= \frac{\sqrt{3}}{2} \cdot 3/2$$

So, A attains an abs. max. at $y=1/2$

So, the dimensions of the isosceles triangle with the largest area are

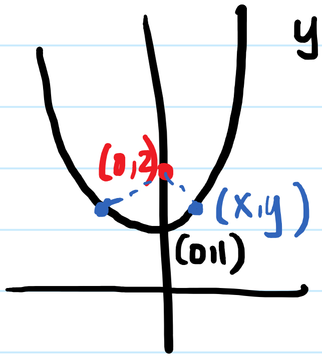
height : $y+1 = \frac{1}{2}+1 = 3/2$ units

base : $2x = 2\sqrt{1-y^2} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$ units.

Ex. Determine the points on the parabola

$$y = x^2+1$$

that are closest to $(0,2)$.



$y = x^2+1$

(x,y) : coordinates of points on the parabola closest to $(0,2)$.

To minimize : d : dist. between (x,y) and $(0,2)$

$$d = \sqrt{(x-0)^2+(y-2)^2} \quad — ①$$

Constraint : $y = x^2+1 \quad — ②$

TRICK : Minimize d^2 instead!

$$d^2 = (x)^2 + (y-2)^2 \quad — ③$$

Substitute ② in ③

$$D(x) = d^2 = x^2 + (x^2+1-2)^2$$
$$= x^2 + (x^2-1)^2 = x^2 + x^4 - 2x^2 + 1$$
$$= x^4 - x^2 + 1.$$

Minimize $D(x)$ in $(-\infty,\infty)$

CANNOT USE THE CLOSED INTERVAL METHOD!

CRITICAL NUMBERS:

$$D'(x) = 4x^3 - 2x = 2x(2x^2-1)$$

So, $D'(x) = 0$ when $x = 0$, $x = \pm\sqrt{\frac{1}{2}}$

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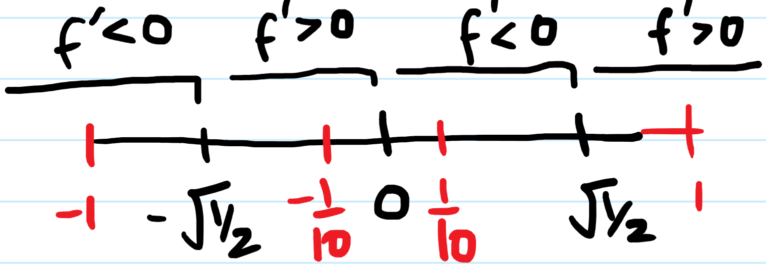
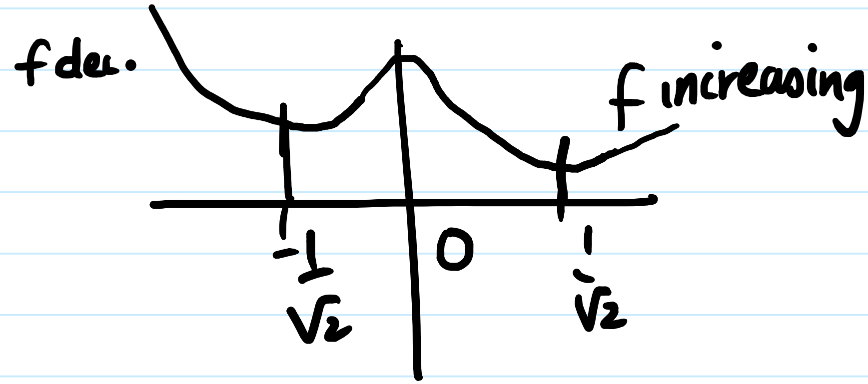


	$2x$	$2x^2-1$	f'
$x < -\frac{1}{\sqrt{2}}$	$-$	$+$	$-$
$-\frac{1}{\sqrt{2}} < x < 0$	$-$	$-$	$+$
$0 < x < \frac{1}{\sqrt{2}}$	$+$	$-$	$-$
$x > \frac{1}{\sqrt{2}}$	$+$	$+$	$+$

f has a local min at $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$

f has a local max at 0.



$D(x) = x^4 - x^2 + 1$

$D\left(-\frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\right)^4 - \left(-\frac{1}{\sqrt{2}}\right)^2 + 1$

$= \frac{1}{4} - \frac{1}{2} + 1 = \frac{1-2+4}{4}$

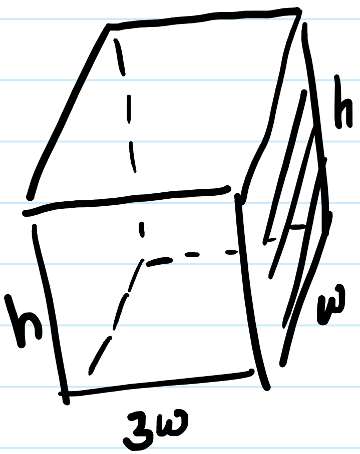
$= 3/4$

$D\left(\frac{1}{\sqrt{2}}\right) = 3/4$ | $x = \pm\frac{1}{\sqrt{2}}$, $y = x^2 + 1$
$= 1/2 + 1 = 3/2$

$D(0) = 1$

The points on $y = x^2+1$ closest to $(0,2)$ are $\left(-\frac{1}{\sqrt{2}}, 3/2\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$.

Ex. We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom costs \$ $10/ft^2$ and the material used to build the sides costs \$ $6/ft^2$. If the box must have a volume of $50\ ft^3$, determine the dimensions that will minimize the cost to build the box.



h: height of the box.

w: width of the box.

3w: length of the box.

To minimize : C : cost to build the box

$$C = \underbrace{2 \cdot 3w^2 \cdot 10}_{\text{base and lid}} + (2hw + 6hw)6$$

$$= 60w^2 + 48hw. \quad - (1)$$

Constraint

$$V = 50 = h \cdot 3w \cdot w$$

$$= 3hw^2 \quad \rightarrow (2)$$

Isolate h in (2) $\quad h = \dfrac{50}{3w^2}$

Substitute this in (1)

$$C(w) = 60w^2 + 48 \cdot \frac{50}{3w^2} \cdot w$$

$$= 60w^2 + \frac{2400}{3w} = 60w^2 + \frac{800}{w}.$$

Minimize $C(w)$ in $(0, \infty)$.

CANNOT USE THE CLOSED INTERVAL M.!

$$C(w) = 60w^2 + \frac{800}{w}$$

$$C'(w) = 120w - \frac{800}{w^2}.$$

$C'(w) = 0$ when $120w = \frac{800}{w^2}$

or $w^3 = \frac{\cancel{800}^{20}}{3 \times \cancel{120}}$

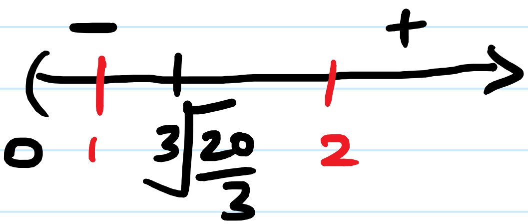
or $w = \sqrt[3]{\frac{20}{3}}.$

FIRST DERIVATIVE TEST FOR <u>GLOBAL</u> EXTREME VALUES.

Suppose c is a critical number of a continuous function on some interval then

① If $f'<0$ for all $x<c$ and $f'>0$ for all $x>c$, then f attains a global minimum value at c.

② If $f'>0$ for all $x<c$ and $f'<0$ for all $x>c$, then f has a global maximum value at $x=c$.



$\frac{20}{3} \sim 6.6$

$1 < \sqrt[3]{\frac{20}{3}} < 2$

$\left(0, \sqrt[3]{\frac{20}{3}}\right)$: $120(1) - \frac{800}{1} < 0$

$\left(\sqrt[3]{\frac{20}{3}}, \infty\right)$: $120(2) - \frac{\cancel{800}^{200}}{\cancel{4}} = 40 > 0$

f attains a global minimum at $w = \sqrt[3]{20/3}$. (By FDT) when $w = \sqrt[3]{\frac{20}{3}}$

$$l = 3w = 3 \cdot \sqrt[3]{\frac{20}{3}}$$

$$h = \frac{50}{3w^2} = \frac{50}{3\left(\frac{20}{3}\right)^{2/3}}$$

$$= \frac{50}{3} \cdot \left(\frac{3}{20}\right)^{2/3}.$$

So, the dimensions of the box that costs the least amount of money to build at

$\sqrt[3]{\frac{20}{3}}$ ft $\times$ $3 \cdot \sqrt[3]{\frac{20}{3}}$ ft $\times$ $\frac{50}{3} \cdot \left(\frac{3}{20}\right)^{2/3}$ ft.