10.28.2015 4.3 Continued.

Ex. Find and classify the local extreme values of

$$f(x) = x - 2\cos x$$

in $0 \le x \le 2\pi$.

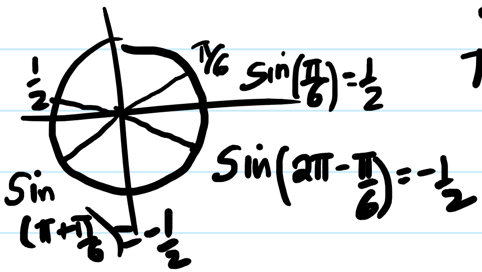
Solution: Critical points of f

$$f'(x) = 1 - 2(-\sin x) = 1 + 2\sin x.$$

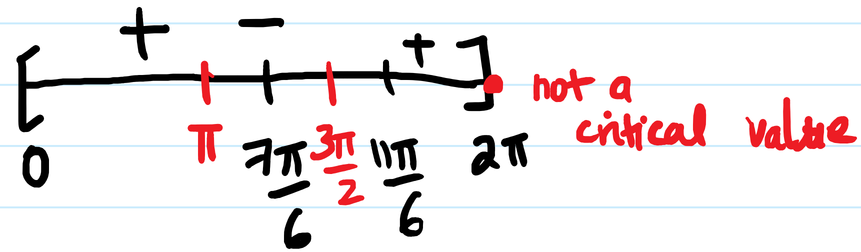
So, the critical points of x satisfy $f'(x) = 0$

OR $1 + 2\sin x = 0$ $\quad 0 \le x \le 2\pi$

OR $\sin x = -\frac{1}{2}$.



The only critical values of $f(x)$ are at $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$



interval	sign of $f'(x)$	behaviour of f
$0 < x < \frac{7\pi}{6}$	$f'(\pi) = 1 + 2\sin(\pi) = 1 + 0 = 1$ $+$	increasing
$\frac{7\pi}{6} < x < \frac{11\pi}{6}$	$f'(\frac{3\pi}{2}) = 1 + 2\sin(\frac{3\pi}{2}) = 1 + 2(-1) = -1$ $-$	decreasing
$\frac{11\pi}{6} < x < 2\pi$	$f'(2\pi) = 1 + 2\sin(2\pi) = 1$ $+$	increasing

By the first derivative test,

since f' changes sign from +ve to −ve at $\frac{7\pi}{6}$, it attains a <u>local max</u>. value of $f\left(\frac{7\pi}{6}\right) = \frac{7\pi}{6} - 2\cos\left(\frac{7\pi}{6}\right)$

$$= \frac{7\pi}{6} - 2\left(-\frac{\sqrt{3}}{2}\right)$$

$$= \frac{7\pi}{6} + \sqrt{3}$$

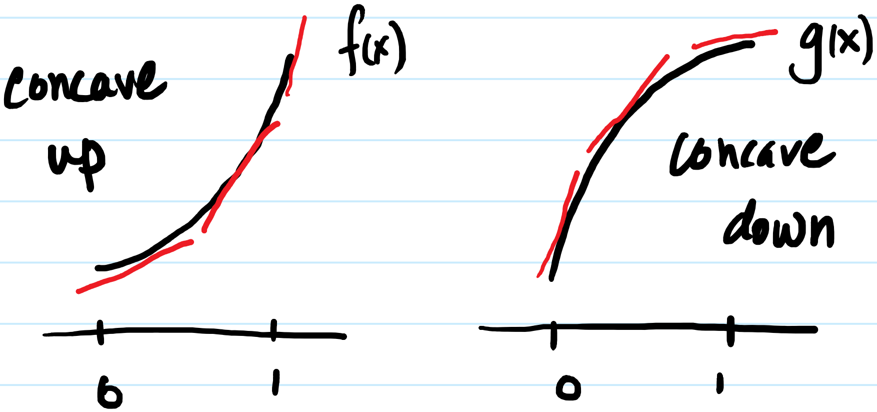
at $\frac{7\pi}{6}$.

Since f' changes sign from −ve to +ve at $\frac{11\pi}{6}$, it attains a <u>local min</u>. of $f\left(\frac{11\pi}{6}\right) = \frac{11\pi}{6} - 2\cos\left(\frac{11\pi}{6}\right)$

$$= \frac{11\pi}{6} - 2\left(\frac{\sqrt{3}}{2}\right) = \frac{11\pi}{6} - \sqrt{3}$$

at $\frac{11\pi}{6}$.

Effect of f'' on the shape of $y=f(x)$



Both f and g are increasing
Both f' and g' are positive.

f: the tangents are below the graph
as you go towards 1, the tangents are becoming more vertical — i.e. f' is becoming larger (or increasing)
so
$$(f')' = f'' > 0$$

g: all the tangent lines are above the graph.

As you move towards 1, the tangents are becoming more horizontal — i.e. g' is becoming smaller (decreasing)
so
$$(g')' = g'' < 0$$

<u>Concavity test</u>: f, f', f'' all exist.

① If $f'' > 0$ on I, f is <u>concave up</u> there

② If $f'' < 0$ on I, f is <u>concave down</u> there

A point on the graph of f where the concavity changes from either up to down or down to up is called an <u>inflection point</u> of f. (both coordinates)

Ex. Draw a sketch of the graph of

$$f(x) = 3x^5 - 10x^3 - 120x.$$

Solution

$$f(x) = 3x^5 - 10x^3 - 120x$$
$$= x(3x^4 - 10x^2 - 120)$$

$$f'(x) = 15x^4 - 30x^2 - 120$$
$$= 15(x^4 - 2x^2 - 8)$$
$$= 15(x^4 - 4x^2 + 2x^2 - 8)$$
$$= 15(x^2+2)(x^2-4).$$

$$f''(x) = 60x^3 - 60x = 60x(x^2-1)$$

CRITICAL POINTS: when $f'(x) = 0$

OR $15(x^2+2)(x^2-4) = 0$

OR $\cancel{x^2+2=0}$ OR $x^2-4=0$

$x^2 = 4 \rightarrow x = \pm 2$

+ local max − local min +

−3 −2 0 2 3

interval	(x^2+2)	(x^2-4)	$f'(x)$	f
$(-\infty, -2)$	+	+	+	inc.
$(-2, 2)$	+	−	−	dec.
$(2, \infty)$	+	+	+	inc.

$$f(-2) = 3(-2)^5 - 10(-2)^3 - 120(-2) = 224$$

$$f(2) = 3(2)^5 - 10(2)^3 - 120(2) = -224$$

Concavity:

$f''(x) = 0$ when $60x(x^2-1) = 0$

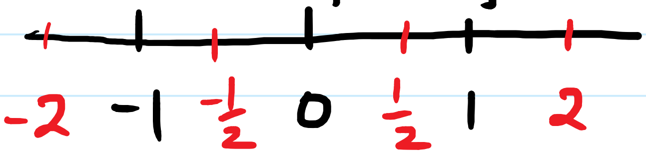
OR $x = 0$ and $x = \pm 1$.

Concavity:

$f''(x)=0$ when $60x(x^2-1)=0$

OR $x=0$ and $x=\pm 1$.

$-$ inf. $+$ inf. $-$ inf. $+$



intervals	x	x^2-1	f''	f
$(-\infty,-1)$	$-$	$+$	$-$	concave down
$(-1,0)$	$-$	$-$	$+$	c. up.
$(0,1)$	$+$	$-$	$-$	c. down
$(1,\infty)$	$+$	$+$	$+$	c. up.

$-1, 0, 1$ are all the inflection pts.

zeros: $f(x) = 3x^5 - 10x^3 - 120x$

$f(x) = x(3x^4 - 10x^2 - 120)$

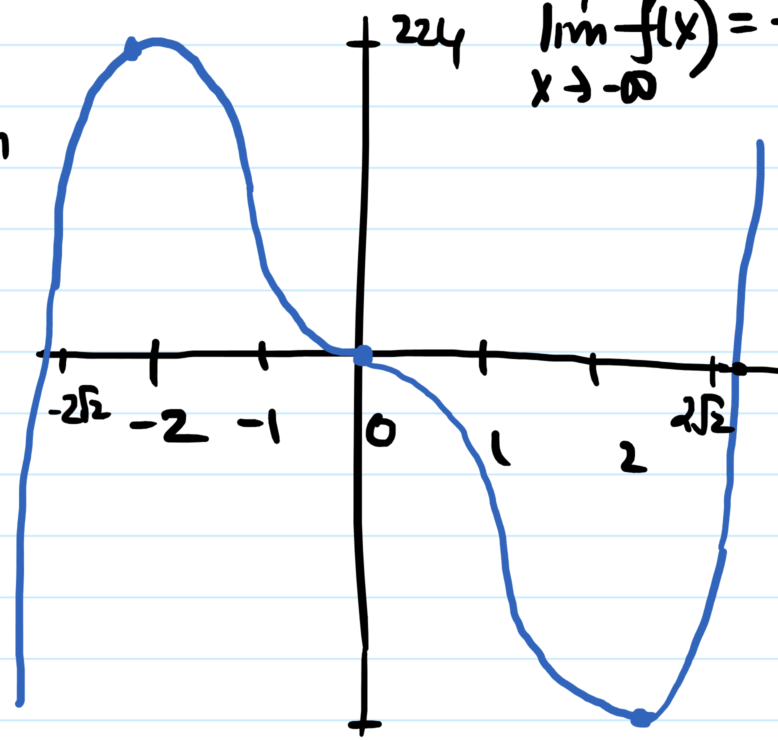
$x(3(x^2)^2 - 10x^2 - 120)$

Quadratic formula to find values for x^2

zeros at x, approx. at $\pm 2\sqrt{2}$

$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = -\infty$



Ex. Draw a sketch of the graph of

$$y = e^{\sqrt[3]{x}}.$$

Solution $f(x) = e^{\sqrt[3]{x}}$ (>0)

$$= e^{x^{1/3}}$$

$$= \underbrace{e^{\underbrace{x^{1/3}}_{\text{ins.}}}}_{\text{out.}}$$

$$f'(x) = e^{x^{1/3}} \cdot \frac{1}{3} x^{1/3 - 1}$$

$$= e^{x^{1/3}} \cdot \frac{1}{3} x^{-2/3}. = \frac{e^{x^{1/3}}}{3x^{2/3}}$$

$$= \frac{1}{9} e^{x^{1/3}} \left[\frac{1}{x^{2/3}} - \frac{2}{x^{5/3}} \right]$$

Critical points:

$f'(0)$ not defined $\rightarrow$ critical pt at $x=0$.

$f'(x) = 0$ OR $\dfrac{e^{x^{1/3}}}{3x^{2/3}} = 0 \Rightarrow e^{x^{1/3}} = 0$ never happens

at $x = 0$

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \frac{e^{x^{1/3}}}{3x^{2/3}} = \infty$$

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} \frac{e^{x^{1/3}}}{3x^{2/3}} = \infty$$