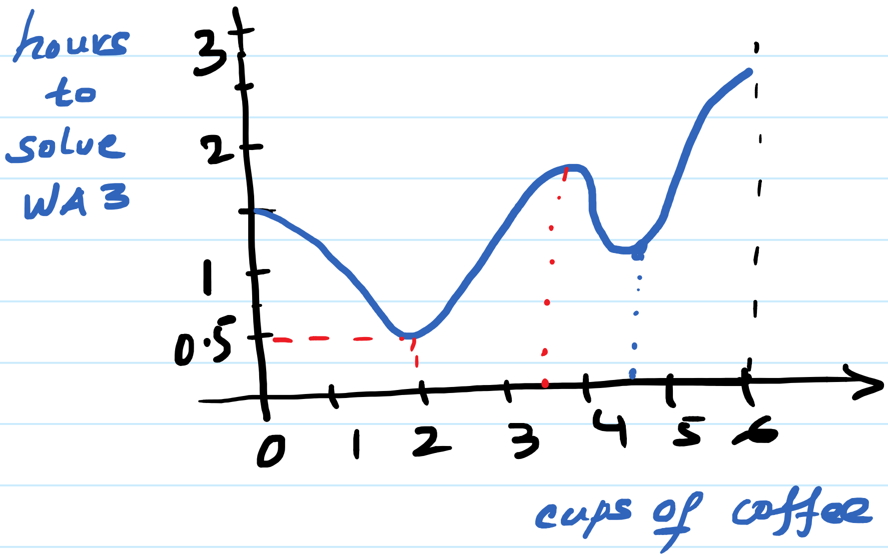
10.26.2015 Chapter 4.

- 4.1, 4.3, 4.4, 4.7, 4.9
- Class ends at 7:50pm on Wed.
- WA 3 is up!

4.1 Maximum and minimum values



$f(2) = 0.5$ is the global minimum value of f

DEFINITIONS: Let c be a number in the domain D of $f(x)$.

* $f(c)$ is called the absolute maximum value of $f(x)$ in D if $f(c) \geq f(x)$ for any x in D

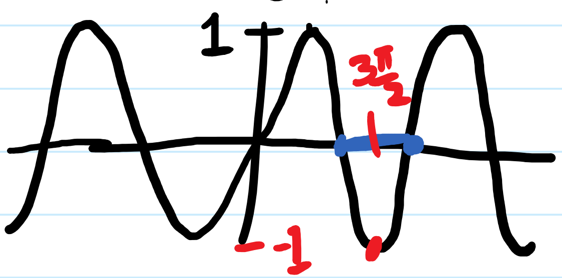
* $f(c)$ is called the absolute minimum value of $f(x)$ in D if $f(c) \leq f(x)$ for any x in D.

ABSOLUTE = GLOBAL

* $f(c)$ is called a local maximum value of $f(x)$ if $f(c) \geq f(x)$ for x near c.

* $f(c)$ is called a local minimum value if $f(c) \leq f(x)$ for x near c.

Examples: ① $y = \sin x$



$\sin(x) \le 1$

1 is a global maximum value of $\sin x$ — it is attained at infinitely many x-values.

1 is the only local maximum value of $\sin x$.

$-1 \le \sin(x)$ for any x in $(-\infty, \infty)$

-1 is the global minimum value of $\sin x$ and it is attained at infinitely many x-values.

-1 is the only local minimum value of $\sin x$.

$f(x) = \sin x \quad , \quad \pi \le x \le 2\pi$

Maximum/minimum = Extreme

$\sin x \le 0$ for x in $[\pi, 2\pi]$

0 the global maximum value of $f(x)$ in $[\pi, 2\pi]$ — it is attained at π and 2π

0 is not a local maximum value for $f(x)$ — $\sin(x) \le 0 \quad x \ge \pi$; $\sin(x) \ge 0 \quad x \le \pi$ } near π

$f(x)$ does not have a local max. value!

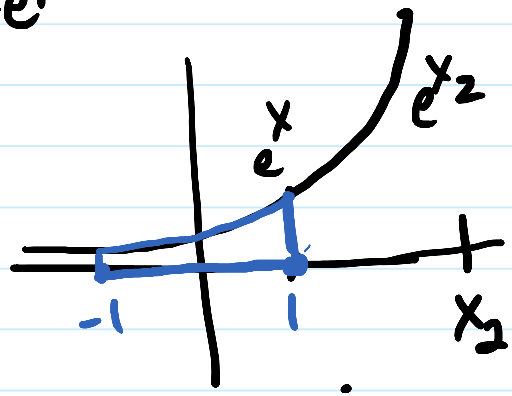
$\sin x \ge -1$

-1 is a global (and local) minimum value.

-1 is the only local minimum value

-1 is a global (and local) max value.

② $y = e^x$



On its entire domain

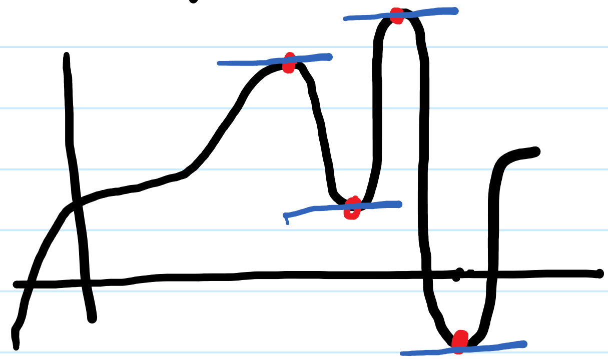
e^x does not have any global/local maximum or minimum values.

Ex. Find all the extreme (global/local) values

$$f(x) = e^x \quad -1 \le x \le 1.$$

SPOTTING EXTREME VALUES.

1



$y = f(x)$

Fermat's theorem: If f has a local extreme point at $x = c$, then If $f'(c)$ exists then

$$f'(c) = 0.$$

Back to Example: $y = \sin x \quad \pi \le x \le 2\pi$

$f(\pi)$ is not a local max. value.

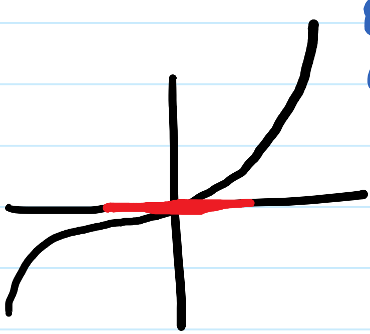
$\frac{dy}{dx} = \cos x$ and $\cos(\pi) = -1 \ne 0$.

WARNINGS: ① Just because $f'(c)=0$, that does not mean that there is a local extreme point over there.

$y=x^3$

$y'=3x^2$

$y'(0)=0$



0 is a critical point of x^3.

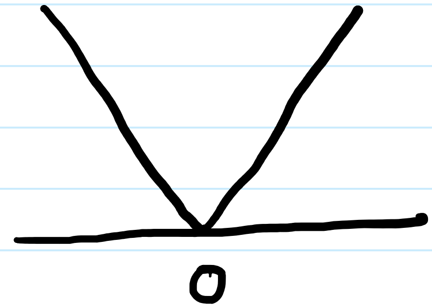
But 0 is not a local max. or min. !

$x \geq 0$, $\quad x^3 \geq 0$

$x \leq 0$, $\quad x^3 \leq 0$.

② $f(x)=|x|$

0 is a critical point of $|x|$



$|x| \geq 0 \rightarrow 0$ is a local and global min. value.

but $f'(0)$ DNE!

CRITICAL POINT (NUMBER, VALUE)

c is called a critical point of $f(x)$ if

① $f'(c)$ does not exist

OR

② $f'(c)=0$.

CRITICAL POINT (NUMBER, VALUE)

c is called a critical point of $f(x)$ if

① $f'(c)$ does not exist

OR

② $f'(c) = 0$.

FERMAT's THEOREM (RESTATEMENT)

If f has a local max./min. at c, the c is a critical value of $f(x)$.

WARNING: NOT EVERY CRITICAL VALUE GIVES RISE TO A LOCAL MAX./MIN. VALUE!

The critical numbers of $f(x)$ are $0, \frac{1}{5}, 1$ ← Answer

Example:

Let $f(x) = x^{1/2}(x-1)^2$. Find all the critical values of $f(x)$.

Solution: $f(x) = x^{1/2}(x-1)^2$

$$f'(x) \overset{P.R.}{=} \frac{1}{2}x^{-1/2}(x-1)^2 + x^{1/2}\,2(x-1)$$

$$= \frac{(x-1)^2}{2\sqrt{x}} + 2\sqrt{x}(x-1).$$

at $\boxed{x=0}$, $f'(0)$ DNE

Set $f'(x) = 0$ and solve for x.

$$(x-1)\left[\frac{x-1}{2\sqrt{x}} + 2\sqrt{x}\right] = 0$$

$x - 1 = 0$ OR $\boxed{x=1}$

$$\frac{x-1}{2\sqrt{x}} = -2\sqrt{x}$$

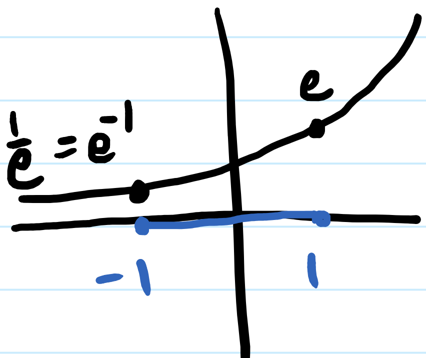
$$x - 1 = -4x \quad \boxed{x = \frac{1}{5}}$$

The critical numbers of $f(x)$ are $\underline{0, \frac{1}{5}, 1}$ $\quad x-1=0$ OR $\boxed{x=1}$ $\quad x-1=-4x+ \boxed{x=\frac{1}{5}}$

$\tan^{-1}(e^x)$ → Composition NOT a product

$\tan^{-1}$ this has no meaning without (x)

differentiate this using the chain rule — not the product rule!



$e^x \quad -1 \le x \le 1$

$\frac{1}{e} \le e^x \le e$

$-1 \le x \le 1$

e is a global max. value and
1/e is " " min. " of f(x).

* EXTREME VALUE THEOREM:

Let f be a continuous function on [a, b]. Then, f attains a global max. value f(c) and a global min. value f(d) for some c and d in [a, b].

* CLOSED INTERVAL METHOD — TECHNIQUE FOR FINDING GLOBAL EXT. VAL.

1. Confirm that f is continuous in [a, b].
2. Find the critical values of f in (a, b).
3. Compute the function at these critical values.
4. Compute f(a) and f(b).
5. Largest (Lowest) of the values from step 3. and 4. is the global max. (min.) of f(x) in [a, b]

e is a global max. value and
$1/e$ is " " min. " of $f(x)$.

$f(x)$ in $[a,b]$ (min)

WEBWORK:

$$\frac{d}{dx}\Big(\underbrace{f}_{\text{out.}}\big(\underbrace{2x^2}_{\text{ins.}}\big)\Big) = 3x^3$$

$f'(2x^2)$

Find $f'(x)$.

By the chain rule

$$f'(2x^2)\cdot 4x = 3x^{\cancel{3}}$$

$$f'(\underbrace{2x^2}_{x}) = \frac{3}{4}x^2.$$

Rewrite this using t instead of x

$$f'(\underbrace{2t^2}_{x}) = \frac{3}{4}t^2$$

$$x = 2t^2 \rightarrow t = \sqrt{\frac{x}{2}}$$

$$f'(x) = \frac{3}{4}\left(\sqrt{\frac{x}{2}}\right)^2 = \frac{3}{4}\cdot\frac{x}{2}$$

$$= \frac{3x}{8}.$$

$$\boxed{f'(x) = \frac{3x}{8}}$$

Back to section 4.1.

Examples: Find the absolute maximum and minimum values of

$$f(x) = x^3 - 3x^2 + 1$$

on $\left[-\frac{1}{2}, 4\right]$.

Solution: f is continuous in

① $[-1/2, 4]$.

② Critical numbers:

$$f'(x) = 3x^2 - 6x$$

Set $f'(x) = 0 \Rightarrow 3x^2 - 6x = 0$

OR $3x(x-2) = 0$

$x = 0, \quad x = 2$

③ Compute f at critical numbers

$f(0) = 0^3 - 3 \cdot 0^2 + 1 = 1.$

$f(2) = 2^3 - 3 \cdot 2^2 + 1$

$= 8 - 12 + 1 = -3.$

④ Compute f at the end points:

$$f(-1/2) = \left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 1$$

$$= -\frac{1}{8} - \frac{3}{4} + 1 = -\frac{1}{8} + \frac{1}{4}$$

$$= \frac{1}{8}$$

$f(4) = 4^3 - 3 \cdot 4^2 + 1 = 64 - 48 + 1$

$= 16 + 1$

$= 17.$

@ $x = 4$

17 is the global max. value

@ $x = 2$

-3 is the global min value

② Find the absolute maximum and minimum values of

$$f(x) = x - 2\tan^{-1}(x)$$

on $[0, 4]$.

Solution: ① f is continuous in $[0,4]$.

Closed Interval Method:

②. $f'(x) = 1 - \dfrac{2}{1+x^2}$.

Set $f'(x) = 0$. Solve for x.

$$1 - \frac{2}{1+x^2} = 0$$

$$\frac{1+x^2-2}{1+x^2} = 0$$

$x^2 - 1 = 0 \to x = \pm 1 \to$ but -1 is not in $[0,4]$

Critical numbers of f in $[0,4]$: $x = 1$.

③ $f(1) = 1 - 2\tan^{-1}(1)$
$= 1 - 2\frac{\pi}{4} = \underline{1 - \frac{\pi}{2}} < 0$

④ $f(0) = 0 - 2\tan^{-1}(0) = \underline{0}$

$f(4) = \underline{4 - \underbrace{2\tan^{-1}(4)}_{3.1}} > 0$

$\tan^{-1}(x)$ can be at most $\frac{\pi}{2}$

$2\tan^{-1}(4)$ can be at most π

Answer ⑤ $4 - 2\tan^{-1}(4)$ is the global max. value.
$1 - \pi/2$ is the global min. value.

4.3.

What about local max/min?

How f' affects the shape of the graph of f.

INCREASING/DECREASING TEST (I/D)

FACT: $f'(x) > 0$ in some interval, then f is increasing in that interval. (e^x)

$(x_1 < x_2,\ f(x_1) < f(x_2))$

$f'(x) < 0$ in some interval, then $f(x)$ is decreasing in that interval

$x_1 < x_2 \rightarrow f(x_1) > f(x_2)$. (e^{-x})

Example: In what intervals is the fn. $f(x) = \frac{1}{3}x^3 - 2x^2 - 5x + 20$ increasing and decreasing.

Solution: $f'(x) = \frac{1}{\cancel{3}} \cdot \cancel{3}x^2 - 2 \cdot 2x - 5$

$= x^2 - 4x - 5$

First set $f'(x) = 0$

$= x^2 - 5x + x - 5$

$= (x+1)(x-5)$

$f'(x) = 0$ when $x = -1$, $x = 5$

$+ \quad 0 \quad - \quad 0 \quad +$

$-1 \qquad 5$

Interval	$(x+1)$	$(x-5)$	$f'(x)$
$(-\infty, -1)$	$-$	$-$	$+$
$(-1, 5)$	$+$	$-$	$-$
$(5, \infty)$	$+$	$+$	$+$

Interval	$(x+1)$	$(x-5)$	$f'(x)$
$(-\infty,-1)$	$-$	$-$	$+$
$(-1,5)$	$+$	$-$	$-$
$(5,\infty)$	$+$	$+$	$+$

So

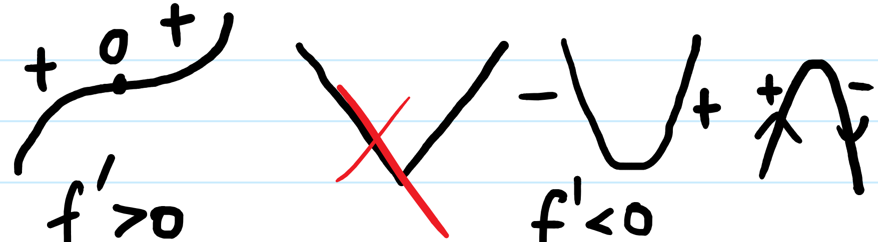
$$\underset{}{+}\quad \underset{-1}{0}\quad - \quad \underset{5}{0}\quad +$$

By the I/D test,

So, $f(x)$ is increasing in

$(-\infty,-1)$ and $(5,\infty)$

and decreasing in $(-1,5)$.



$f'>0$ for all x

$f'<0$ before the critical pt and $f'>0$ after the critical pt

First Derivative Test:

If $f'(c)=0$. then,

$f(c)$ is a local min. if f' changes from −ve to +ve as you pass through

$f(c)$ is a local max. then f' changes sign from >0 to <0.

$f(c)$ is neither – if it does not change sign.