10.21.2015 Review - contd.

Example: Let $f(x) = \begin{cases} \frac{\pi}{2}e^{x}, & x \leq 0 \\ \cos^{-1}x, & 0 < x \leq 1 \\ x, & x > 1 \end{cases}$

(i) Is f continuous at 0? Justify.

HW (ii) Is f continuous at 1? Justify.

Solution (ii): * $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \cos^{-1}(x)$

$= \cos^{-1}(1)$

$= 0.$

* $f(1) = \cos^{-1}(1) = 0$

* $\lim_{x \to 1^+} f(x)$

$= \lim_{x \to 1^+} x = 1.$

No, $f(x)$ is not continuous at 1, because

$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

OR

$\lim_{x \to 1} f(x)$ DNE.

Example: Let

$$f(x) = \begin{cases} \frac{(x-1)^2}{x^2+x-2}, & x<1 \\ a, & x=1 \\ \cos^{-1}\left(b\tan^{-1}\left(\frac{1}{x-1}\right)\right), & x>1. \end{cases}$$

f is continuous at 1. Find a and b.

Solution: Since f is cont. at 1

$$\lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$$

$$\lim_{x\to 1^-} \frac{(x-1)^2}{x^2+x-2} = a.$$

$$\lim_{x\to 1^-} \frac{(x-1)^2}{x^2+2x-x-2} = a$$

$$\Rightarrow \lim_{x\to 1^-} \frac{(x-1)^{\cancel{2}}}{\cancel{(x-1)}(x+2)} = a.$$

$$\boxed{0 = a}$$

For b,

$$0 = \lim_{x\to 1^+} \cos^{-1}\left(b\tan^{-1}\left(\frac{1}{x-1}\right)\right)$$

$$0 = \cos^{-1}\left(\lim_{x\to 1^+} b\tan^{-1}\left(\frac{1}{x-1}\right)\right)$$

$$0 = \cos^{-1}\left(b\lim_{x\to 1^+} \tan^{-1}\left(\underbrace{\frac{1}{x-1}}_{t}\right)\right)$$

$t = \frac{1}{x-1}$, $x \to 1^+$

$t = \frac{1}{x-1} \to \infty$

$$0 = \cos^{-1}\left(b \lim_{t\to\infty} \tan^{-1}(t)\right)$$

$$0 = \cos^{-1}\left(b\frac{\pi}{2}\right) \Rightarrow \cos(0) = \frac{b\pi}{2}$$

$$\Rightarrow 1 = b\frac{\pi}{2}$$

So $b = 2/\pi$

Intermediate Value Theorem:

$f(x)$ is continuous in $[a,b]$ and N is a number between $f(a)$ and $f(b)$. Then, there exists a number c in (a,b) such that

$$f(c) = N.$$

Given: $f(x)$, $[a,b]$, N
$\downarrow$
desired output.

Example: (WA2,(i)) f is continuous in $[-1,1]$. $f(-1/2)=2$, $f(0)=0$, $f(1/2)=1$. Is f one-to-one in $[-1,1]$?

ONE-TO-ONE: If does not take the same output values for two different values.

Solution: $f(-1/2)=2$, $f(0)=0$, $f(1/2)=1$.

f is continuous in $[-1,1]$ so, f is continuous in $[-1/2,0]$

$$\underset{f(a)}{f(-1/2)} > \underset{N}{1} > \underset{f(b)}{f(0)}$$

By the I.V.T. there is a c in $(-1/2,0)$ so that $f(c)=1$.

$f(c)=1$ and $f(1/2)=1$.

So, f cannot be one-to-one!

Example: Show that there is a solution (or root) of the equation

$$\cos x = x^2 - 1.$$

in the interval $(0, \pi)$, using the Intermediate Value Theorem.

Solution: $\cos x = x^2 - 1$

To show: $\cos x - x^2 + 1 = 0$

holds for some number in $(0, \pi)$.

Set

$f(x) = \cos x - x^2 + 1$

$[a, b] = [0, \pi]$

f is continuous in $[0, \pi]$

$f(0) = \cos(0) - 0^2 + 1 = 1 - 0 + 1$
$= 2$

$f(\pi) = \cos(\pi) - \pi^2 + 1 = -1 - \pi^2 + 1$
$= -\pi^2$

$$-\pi^2 < \underset{(N)}{0} < 2$$

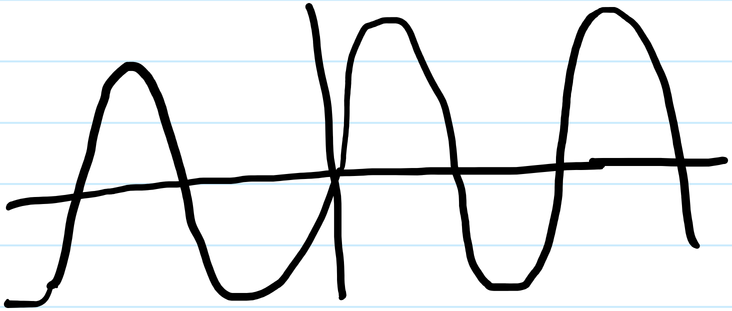
So, by the I.V.T. there is a c in $(0, \pi)$ such that

$f(c) = 0$ – i.e.

c is a solution of $\cos x = x^2 - 1$.

D. TRIGONOMETRY

$y = \sin(x)$

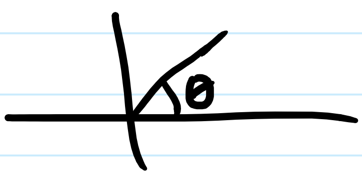


Domain : $(-\infty, \infty)$

Range: $[-1, 1]$

$$-1 \leq \sin(x) \leq 1$$

Think of $\theta = x$ as an angle



Then, $\sin\theta \geq 0$ when θ is in the I and II quadrants

$$0 \leq \theta \leq \pi.$$

$\sin(\pi - x) = \sin x$
$\sin(\pi + x) = -\sin x$
$\sin(2\pi - x) = -\sin x$
$\sin(2\pi + x) = \sin x$ (periodic w/ periodic 2π)

$\sin(-x) = -\sin(x)$ (odd)

$\lim_{x \to 0} \sin x = \sin(0) = 0$

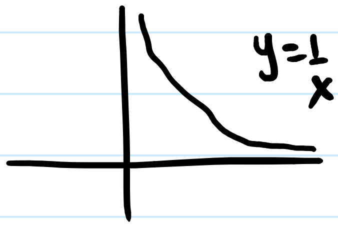
$\lim_{x \to 0} \frac{\sin x}{x} = 1$ (FACT)

$\lim_{x \to \infty} \sin x = \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ DNE.

Compute $\boxed{\lim_{x\to\infty} \sin\left(\frac{1}{x}\right). = 0}$

$t = 1/x \qquad x \to \infty$

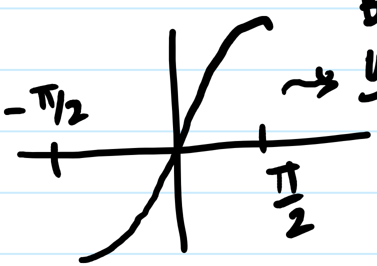
$t = \frac{1}{x} \to 0$



$\lim_{t\to 0} \sin(t) = 0$

$\boxed{\lim_{x\to\infty} \sin(x) \text{ DNE}}$ $\boxed{\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \text{ DNE}}$

$\sin(x)$ is NOT ONE-TO-ONE



But, $y = \sin(x)$

$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

is one-to-one.

$\arcsin(x)$ OR $\sin^{-1}(x)$ is the inverse of 'restricted' $\sin(x)$

domain of $\sin^{-1}(x) = [-1, 1]$

range of $\sin^{-1}(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \sin^{-1}(x) \iff \sin(y) = x$ &

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\sin(\sin^{-1}(x)) = x \quad , \quad -1 \le x \le 1$

$\sin^{-1}(\sin(x)) = x \quad , \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$

$-\quad \frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.$

$y=\sin^{-1}(x) \iff \sin(y)=x$ &

$-\frac{\pi}{2}\le y\le \frac{\pi}{2}$

$\sin^{-1}(0) = 0$ $\quad$ $\sin^{-1}\left(\sqrt{3}/2\right)=\pi/3$

$\sin^{-1}(1/2)=\frac{\pi}{6}$ $\quad$ $\sin^{-1}(1) = \pi/2$

$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$ $\quad$ $\sin^{-1}(-1/2) = -\frac{\pi}{6}$

$\sin(0) = 0$

$\sin\left(\frac{\pi}{6}\right) = 1/2$

Compute: $\lim\limits_{x\to\infty}\frac{\sin x}{x}$

$$= \lim_{x\to\infty} \underbrace{\frac{1}{x}}_{\to 0}\cdot \underbrace{\sin x}_{\to \text{DNE}}$$

$-1\le \sin x \le 1$

$$\underbrace{-\frac{1}{x}}_{f} \le \underbrace{\frac{\sin x}{x}}_{g} \le \underbrace{\frac{1}{x}}_{h}$$

SQUEEZE THM: $f(x)\le g(x)\le h(x)$ x near a. If

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

the $\lim\limits_{x\to a} g(x) = L$.

IF THE LIMITS DON'T MATCH, USE ANOTHER METHOD!

$$\lim_{x\to\infty} -\frac{1}{x} = 0 = \lim_{x\to\infty}\frac{1}{x}$$

By the squeeze theorem $\boxed{\lim\limits_{x\to\infty}\frac{\sin x}{x}=0}$

Example:

$$\lim_{x\to 0} \frac{x^2+x}{\sin(x)} = ?$$

Substitution : $\frac{0}{0}$ form

$\Downarrow$

think of $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1 \rightsquigarrow \lim_{x\to 0} \frac{x}{\sin x} = 1$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = -\lim_{x\to 0} \frac{1-\cos x}{x}$$

$$= -0 = 0$$

$$\lim_{x\to 0} \frac{x^2+x}{\sin x} = \lim_{x\to 0} \frac{x(x+1)}{\sin x}.$$

$$= \lim_{x\to 0} \frac{x}{\sin x} \cdot (x+1)$$

$$= \lim_{x\to 0} \frac{x}{\sin x} \lim_{x\to 0} x+1 = 1 \times (0+1) = \underline{\underline{1}}$$