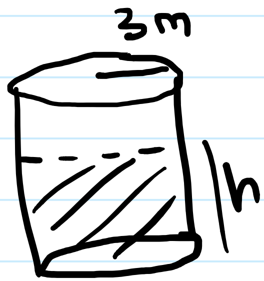
10.19.2015

3.9 Related Rates

To compute the rate of change of one quantity in terms of the rate of change of another.

Example: A cylindrical tank is being filled with water at the rate of $2 m^3/min$. The radius of the tank is 3m. How fast is the height of the water increasing?



$V(t)$: volume of water

$h(t)$: height of water

r : 3m.

Given $\frac{dV}{dt} = 2 \frac{m^3}{min}$

Find : $\frac{dh}{dt}$.

$$V = \pi r^2 h$$

$$V(t) = 9\pi h(t)$$

Differentiate w.r.t. t

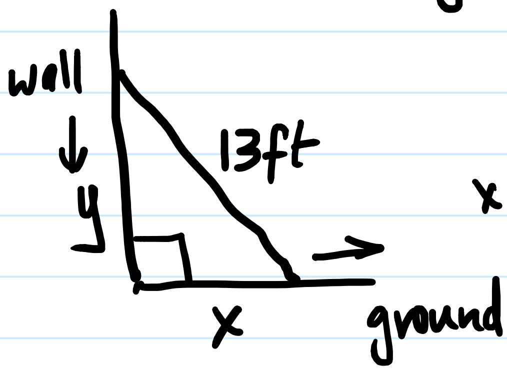
$$\frac{dV}{dt} = 9\pi \frac{dh}{dt}$$

$$2 = 9\pi \frac{dh}{dt}$$

So, $\frac{dh}{dt} = \boxed{\frac{2}{9\pi} \frac{m}{min}}$

+ve → so h is increasing.

Example: A 13 ft Ladder is resting against a vertical wall. If the bottom of the ladder is slipping away from the wall at the rate of 1 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom is 5 ft away from the wall?



y: height of the top of the ladder.

x: distance of the bottom of the ladder from the wall.

Given: $\frac{dx}{dt} = 1 \frac{ft}{sec}$

Find $\frac{dy}{dt}$ when $x = 5$ ft.

Solution: We know

$$x^2(t) + y^2(t) = (13)^2$$

Differentiate both sides w.r.t. t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{2x}{2y} \cdot \frac{dx}{dt} = -\frac{x}{y}.$$

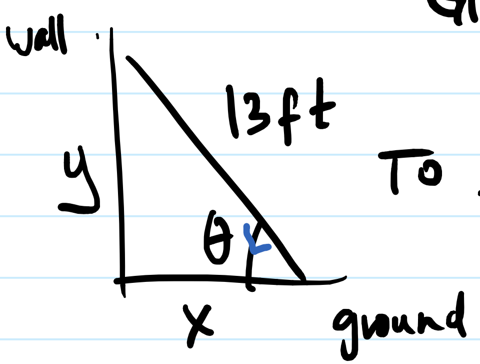
When $x=5$, $5^2 + y^2 = (13)^2$

So, $y^2 = \underline{(13)^2 - 5^2}$

So, $y = \sqrt{144} = 12$

So, $\frac{dy}{dt} = -\frac{5}{12}$ ft/sec negative means y is decreasing

Example: A 13 ft Ladder is resting against a vertical wall. If the bottom of the ladder is slipping away from the wall at the rate of 1 ft/sec, how fast is the angle between the ladder and the ground changing when the bottom of the ladder is 5ft away from the wall?



Given: $\frac{dx}{dt} = 1 \frac{ft}{sec}$

To find: $\frac{d\theta}{dt}$ when $x = 5$ ft.

We know that

$$\cos\theta(t) = \frac{x(t)}{13}$$

Differentiate both sides w.r.t t

$$-\sin\theta \cdot \frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt} \quad \left(\frac{dx}{dt} = 1\right)$$

$$= \frac{1}{13}$$

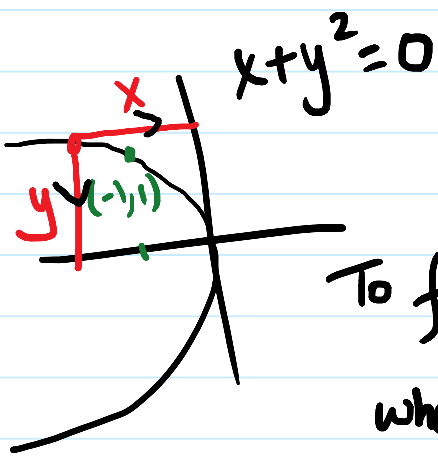
So, $\frac{d\theta}{dt} = \frac{-1}{13\sin\theta}$

@ $x = 5$, we saw that $y = 12$

So, $\sin\theta = \frac{12}{13}$

$$\frac{d\theta}{dt} = \frac{-1}{13 \cdot 12/13} = -\frac{1}{12} \text{ radians/sec}$$

Example: A point is moving towards the origin on the parabola $x+y^2=0$ with its x-coordinate increasing at the rate of 2 units/s. What is the rate of change of the y-coordinate as it passes through $(-1,1)$?



Given:

$$\frac{dx}{dt} = 2\,\frac{\text{units}}{s}$$

To find: $\frac{dy}{dt}$ when $x=-1$ & $y=1$.

Solution We know

$$x(t) + y^2(t) = 0$$

Differentiate both sides w.r.t t

$$\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

So,

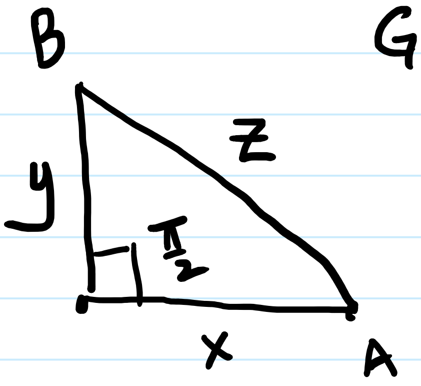
$$\frac{dy}{dt} = -\frac{1}{2y}\cdot\frac{dx}{dt}$$

$$= \frac{-2}{2y} = \frac{-1}{y}.$$

When the particle passes through $(-1,1)$,

$$\frac{dy}{dt} = -1\ \frac{\text{unit}}{s}.$$

Example: Two people, A & B, start from the same point. One walks east at 3 miles/hour and other walks north at 2 miles/hour. How fast is the distance between them changing after 15 minutes?



Given: $\frac{dx}{dt} = 3 \frac{\text{miles}}{\text{hour}}$

$\frac{dy}{dt} = 2 \frac{\text{miles}}{\text{hour}}$

Find: $\frac{dz}{dt}$ when $t = 15$ min $= \frac{1}{4}$ hrs.

Solution: We know

$$x^2(t) + y^2(t) = z^2(t)$$

Differentiate both sides w.r.t. t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$6x + 4y = 2z\frac{dz}{dt}$$

When $t = \frac{1}{4}$ hours

$$x = 3 \cdot \frac{1}{4} = 3/4 \text{ miles}$$

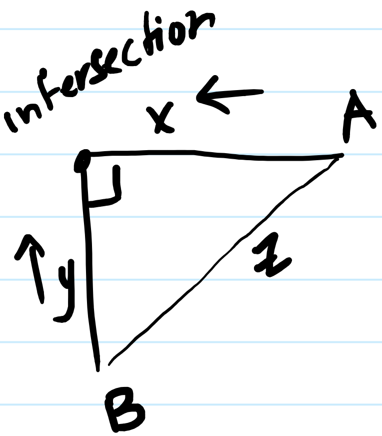
$$y = 2 \cdot \frac{1}{4} = \frac{1}{2} \text{ miles}$$

$$z = \sqrt{x^2+y^2} = \frac{\sqrt{13}}{4} \text{ miles.}$$

$$6 \cdot \frac{3}{4} + 4 \cdot \frac{1}{2} = 2\frac{\sqrt{13}}{4} \, dz/dt$$

$$dz/dt = \sqrt{13} \text{ miles/hour.}$$

Example: Car A is travelling west at 50 mi/hr and Car B is travelling north at 60 mi/hr. They are headed towards the same intersection. At what rate are the cars approaching one another when car A is 0.3 mi away from the intersection and car B is 0.4 mi away from the intersection?



Given:

$\frac{dx}{dt} = -50 \frac{mi}{hour}$

$\frac{dy}{dt} = -60 \frac{mi}{hour}$

To find: $\frac{dz}{dt}$ when $x = 0.3$ mi, $y = 0.4$ mi.

Solution: $x^2_{(t)} + y^2_{(t)} = z^2_{(t)}$

Diff. both sides w.r.t. t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$2x(-50) + 2y(-60) = 2z\frac{dz}{dt}$$

when $x = 0.3$ & $y = 0.4$

$z = \sqrt{(0.3)^2 + (0.4)^2} = 0.5$ mi

$$-100(0.3) - 120(0.4) = 2(0.5)\frac{dz}{dt}$$

$$\frac{dz}{dt} = -78 \text{ mi/hour.}$$

Review:

A. TANGENT LINES

Given a curve

Example (i) $y = x^3 + x^2$

(ii) $\tan(x+y) = 2x$

To find the equation of any tangent line, you need

(i) slope
(ii) a point on the line.

(i) Slope: If the line is tangent to the given curve at (a,b), then the slope of line is $\frac{dy}{dx}$ at $x=a$, $y=b$.

(ii) Point is given in the problem.

Example: Find an equation of the tangent line to the graph

$$x\cos(xy) = \frac{\sqrt{3}}{2}$$

at $(1, \pi/6)$.

Solution:

Continued on next page.

Example: Find an equation of the tangent line to the graph

$$x\cos(xy) = \frac{\sqrt{3}}{2}$$

at $(1, \pi/6)$.

Solution: Differentiate both sides w.r.t. x

$$1\cdot\cos(xy) + x\frac{d}{dx}\cos(xy) = 0$$

$$\cos(xy) + x\left[-\sin(xy)\frac{d}{dx}(xy)\right] = 0$$

$$\cos(xy) - x\sin(xy)\left[1\cdot y + xy'\right] = 0$$

$$\cos(xy) - xy\sin(xy) - x^2\sin(xy)y' = 0$$

$$\cos(xy) - xy\sin(xy) = x^2\sin(xy)y'$$

$$\frac{\cos(xy) - xy\sin(xy)}{x^2\sin(xy)} = y'.$$

when $x=1$, $y=\pi/6$

$$y' = \frac{\cos(\pi/6) - \pi/6\sin(\frac{\pi}{6})}{(1)^2\sin(\pi/6)}$$

$$= \frac{\sqrt{3}/2 - \pi/6\cdot 1/2}{1/2}$$

$$= \sqrt{3} - \pi/6$$

By the slope-point formula

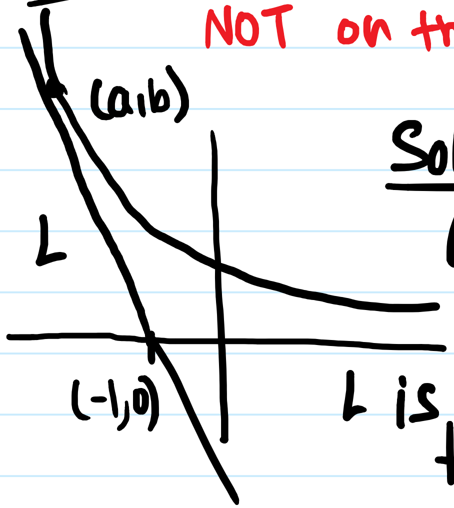
$$y - \frac{\pi}{6} = \left(\sqrt{3} - \frac{\pi}{6}\right)(x-1).$$

Example: Find an equation of the tangent line to

$$y = e^{-2x}$$

which passes through $(-1, 0)$.

Solution: CAUTION: $(-1,0)$ is NOT on the graph!



Solution

(a,b): the point where L is tangent to the curve.

Slope of L: $\dfrac{b-0}{a-(-1)} = \dfrac{b}{a+1}$

Slope of L: $\dfrac{dy}{dx}$ @ $x=a$, $y=b$

OR $\dfrac{d e^{-2x}}{dx}$ at $x=a$, $y=b$.

$f(x) = e^{x}$ $\quad g(x) = -2x$

$e^{-2x} = f(g(x))$

$$\frac{d}{dx} e^{-2x} = f'(g(x)) \cdot g'(x).$$

$$= e^{g(x)} \cdot g'(x)$$

$$= e^{-2x} (-2).$$

So, slope of L: $-2e^{-2a}$

So, $\boxed{\dfrac{b}{a+1} = -2e^{-2a}}$ (1)

Since (a,b) is on the graph of $y = e^{-2x}$

$$\boxed{b = e^{-2a}} \text{ — (ii)}$$

Substitute (ii) in (i)

$$\frac{\cancel{e^{-2a}}}{a+1} = -2\cancel{e^{-2a}}$$

$$\frac{1}{a+1} = -2 \quad \text{OR} \quad -\frac{1}{2} = a+1$$

OR $\boxed{a = -3/2}$

Slope of L $= -2 \cdot e^{-\cancel{2}(-3/\cancel{2})}$

$= -2e^{3}$

Line passes through $(-1, 0)$

Equation : $y = -2e^{3}(x-(-1))$

$$y = -2e^{3}(x+1) \quad //$$

B. CONTINUITY

$f(x)$ is said to be continuous at the input value a if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a).$$

Example: Let $f(x) = \begin{cases} \frac{\pi}{2}e^{x}, & x \le 0 \\ \cos^{-1}x, & 0 < x \le 1 \\ x, & x > 1 \end{cases}$

(i) Is f continuous at 0? Justify.

HW (ii) Is f continuous at 1? Justify.

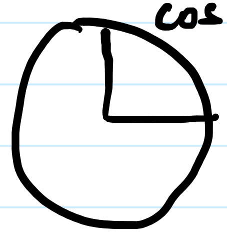
Solution (i) Compute 3 things

$*$ $\lim\limits_{\substack{x\to 0^- \\ (x\le 0)}} f(x) = \lim\limits_{x\to 0^-} \frac{\pi}{2}e^{x}$

$$= \frac{\pi}{2}e^{0} = \pi/2.$$

$*$ $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \cos^{-1}x$

$$= \cos^{-1}(0) = \frac{\pi}{2}$$



$*$ $f(0) = \frac{\pi}{2}e^{0} = \pi/2$

Yes, f is continuous at 0 because $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = \underline{\underline{f(0)}}.$