10.14.2015

MIDTERM LOCATIONS ARE POSTED ON THE SECTION WEBSITE.

3.6 Logarithmic Differentiation.

Ex. $\frac{d}{dx}\log_b x$.

Solution: Set $y = \log_b x$.

OR

$b^y = x$ (Definition)

Implicit differentiation (keep in mind that y is a function of x).

$$b^y \ln b \cdot y' = 1$$

$$y' = \frac{1}{b^y \ln b}.$$

$$\boxed{\frac{d}{dx}\log_b x = \frac{1}{x\ln b}}$$

Example: Compute y', when

$$y = \underbrace{\log_3}_{\text{outside}}\underbrace{(\cos^{-1}(x))}_{\text{inside}}.$$

Solution. By the chain rule,

$$y' = \frac{1}{(\cos^{-1}(x))\ln 3} \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{-1}{\cos^{-1}(x)\cdot\ln 3\cdot\sqrt{1-x^2}} \;//$$

CALCULUS 1000A – PROBLEM WORKSHEET I

Circle the best answer. You do not need to show your work.

1. The domain of the function $\ln(\ln(\ln(x)))$ is

(A) $(0,\infty)$ (B) $(1,\infty)$ (C) (e,∞) (D) $(1,e)$ (E) $(\frac{1}{e},e)$.

2. $\lim_{x\to -3^+} \log_3\left(\frac{1}{6-t}\right) =$

(A) ∞ (B) $-\infty$ (C) 2 (D) -2 (E) None of (A) to (D).

3. If $f(x) = \sqrt{2-e^x}$, then the domain of $f^{-1}(x)$ is

(A) $(-\infty,\infty)$ (B) $(0,\infty)$ (C) $(-\sqrt{2},\sqrt{2})$ (D) $[0,\sqrt{2})$ (E) $(-\infty,0)$.

4. $\lim_{t\to -\infty} \frac{\sqrt{9t^2+t-2}}{t-3} =$

(A) $-\infty$ (B) ∞ (C) 0 (D) 3 (E) -3.

5. $\frac{d}{dx} e^{\sqrt{1-x^2}} =$

(A) $2xe^{\sqrt{1-x^2}}$ (B) $\frac{xe^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}$ (C) $\frac{-xe^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}$

(D) $\frac{x^2e^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}$ (E) None of (A) to (D).

6. $\log_3 \frac{1}{27} + \log_3 30 - \log_3 10 =$

(A) -3 (B) 3 (C) -2 (D) 2 (E) None of the above.

7. $\lim_{x \to \infty} \arctan\left(e^{2x}\right)$

(A) $\frac{\pi}{2}$ (B) 0 (C) ∞ (D) $-\infty$ (E) None of the above.

8. If $f(x) = 5^x$, its 10^{th} derivative $f^{(10)}(x)$ is

(A) 5^x (B) $\frac{5^x}{(\ln 5)^{10}}$ (C) $5^x(\ln 5)^9$ (D) $10 \cdot 5^x$ (E) $5^x(\ln 5)^{10}$.

9. If $f(x) = \ln(x^3 - 5)$, then $f^{-1}(x) =$

(A) e^{x^3-5} (B) $e^{x^3} + 5$ (C) $\sqrt[3]{e^x + 5}$ (D) $e^{\frac{x}{3}+5}$ (E) $\frac{1}{\ln(x^3-5)}$.

10. The slope of the tangent line to the curve $\sin^2(x) + y^2 = 1$ at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ is

(A) $\frac{\pi}{4}$ (B) $\frac{1}{\sqrt{2}}$ (C) 0 (D) $\frac{-1}{\sqrt{2}}$ (E) $\sqrt{2}$.

11. If $f(x) = g(e^x)$, $f'(\ln(2)) = 2e$, then $g'(2) =$

(A) e (B) $\ln(2)$ (C) $\frac{e}{2}$ (D) 2 (E) Cannot be determined.

12. Let $g(x) = (f(x) - 2)\tan x$. Suppose $g(x)$ is continuous at $\frac{\pi}{2}$. Then, $\lim_{x \to \frac{\pi}{2}} f(x) =$

(A) ∞ (B) $-\infty$ (C) Does not exist (D) 2 (E) 0.

WORKSHEET 1 SOLUTIONS

1. The domain of the function $\ln(\ln(\ln(x)))$ is

(A) $(0, \infty)$ (B) $(1, \infty)$ (C) (e, ∞) (D) $(1, e)$ (E) $\left(\frac{1}{e}, e\right)$.

Ans. $\ln(x)$ is defined for $x > 0$.

So, $\ln(\ln(\ln(x)))$ is defined for

$$\ln(\ln(x)) > 0$$

OR $$\cancel{e}^{\cancel{\ln}(\ln(x))} > e^0 \quad (e^x \text{ is increasing})$$

OR $$\ln(x) > 1$$

OR $$\cancel{e}^{\cancel{\ln}(x)} > e^1 \quad (\quad " \quad)$$

OR $$x > e.$$

2. $\lim_{x \to -3^+} \log_3 \left(\frac{1}{6-t}\right) =$

(A) ∞ (B) $-\infty$ (C) 2 (D) -2 (E) None of (A) to (D).

Ans. $$\lim_{x \to -3^+} \log_3\left(\frac{1}{6-t}\right) = \log_3\left(\frac{1}{6-(-3)}\right)$$

$$= \log_3\left(\frac{1}{9}\right)$$

$$= \log_3(3^{-2})$$

$$= -2.$$

3. If $f(x) = \sqrt{2 - e^x}$, then the domain of $f^{-1}(x)$ is

(A) $(-\infty, \infty)$ (B) $(0, \infty)$ (C) $(-\sqrt{2}, \sqrt{2})$ (D) $[0, \sqrt{2})$ (E) $(-\infty, 0)$.

Ans. We first find $f^{-1}(x)$.

$$y = \sqrt{2 - e^x} \quad (\text{then } y \geq 0)$$

$$y^2 = 2 - e^x$$

$$e^x = 2 - y^2$$

$$x = \ln(2 - y^2).$$

So, $2 - y^2 > 0$

OR $y^2 < 2$

OR $-\sqrt{2} < y < \sqrt{2}$

But $y \geq 0$ as it is a square root.

So $0 \leq y < \sqrt{2}$.

4. $\lim_{t\to-\infty} \dfrac{\sqrt{9t^2+t-2}}{t-3} =$

(A) $-\infty$ (B) ∞ (C) 0 (D) 3 (E) -3.

Ans. $\lim_{t\to-\infty} \dfrac{\sqrt{9t^2+t-2}}{t-3}$

$$= \lim_{t\to-\infty} \frac{\frac{\sqrt{9t^2+t-2}}{t}}{\frac{t-3}{t}}$$

$$= \lim_{t\to-\infty} \frac{\frac{\sqrt{9t^2+t-2}}{-\sqrt{t^2}}}{1-\frac{3}{t}}$$

If $t<0$, $t = -\sqrt{t^2}$.

Example $-1 = -\sqrt{(-1)^2}$

$$= \lim_{t\to-\infty} - \frac{\sqrt{9+\underbrace{1/t}_{\to 0}-\underbrace{2/t^2}_{\to 0}}}{1-\underbrace{3/t}_{\to 0}}$$

$$= -\sqrt{9} = -3.$$

5. $\frac{d}{dx} e^{\sqrt{1-x^2}} =$

(A) $2x e^{\sqrt{1-x^2}}$ (B) $\frac{x e^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}$ (C) $\frac{-x e^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}$

(D) $\frac{x^2 e^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}$ (E) None of (A) to (D).

Ans. By the chain rule

$$\frac{d}{dx} e^{\sqrt{1-x^2}} = e^{\sqrt{1-x^2}} \frac{d}{dx}\left(1-x^2\right)^{1/2} \left[\frac{d}{dx}e^x = e^x\right]$$

$$= e^{\sqrt{1-x^2}} \cdot \frac{1}{2}\left(1-x^2\right)^{-1/2} \frac{d}{dx}\left(1-x^2\right)$$

[power rule]

$$= \frac{e^{\sqrt{1-x^2}}}{\cancel{2}\sqrt{1-x^2}}\left(-\cancel{2}x\right)$$

$$= \frac{-x\, e^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}$$

6. $\log_3 \frac{1}{27} + \log_3 30 - \log_3 10 =$

(A) -3 (B) 3 (C) -2 (D) 2 (E) None of the above.

Ans. $\log_3 \frac{1}{27} + \underline{\log_3 30} - \underline{\log_3 10}$

$= \log_3 \frac{1}{27} + \log_3\left(\frac{30}{10}\right)$

$= \log_3\left(\frac{1}{27}\cdot 3\right) = \log_3\left(\frac{1}{9}\right) = \log_3\left(3^{-2}\right)$

$= -2\log_3(3)$

$= -2.$

7. $\lim\limits_{x\to\infty} \arctan\left(e^{2x}\right)$

(A) $\frac{\pi}{2}$ (B) 0 (C) ∞ (D) $-\infty$ (E) None of the above.

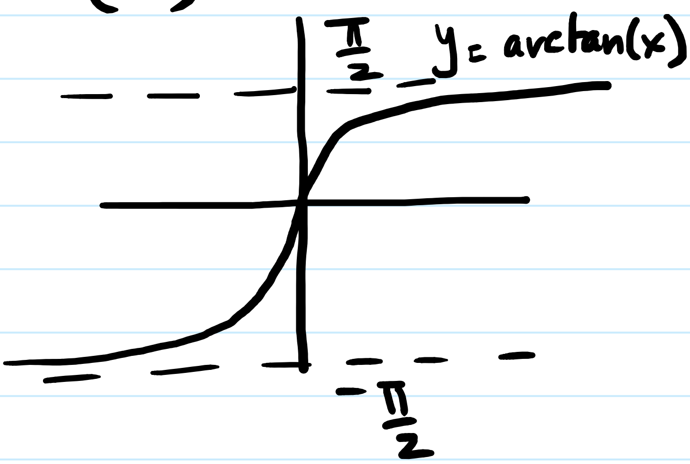
Ans. Let $t = e^{2x}$. As $x \to \infty$

$t = e^{2x} \to \infty$

So, $\lim\limits_{x\to\infty} \arctan\left(e^{2x}\right)$

$= \lim\limits_{t\to\infty} \arctan(t)$

$= \frac{\pi}{2}.$



8. If $f(x) = 5^x$, its 10^{th} derivative $f^{(10)}(x)$ is

(A) 5^x (B) $\dfrac{5^x}{(\ln 5)^{10}}$ (C) $5^x(\ln 5)^9$ (D) $10 \cdot 5^x$ (E) $5^x(\ln 5)^{10}$.

Ans. $f(x) = 5^x$

$f'(x) = 5^x \ln 5$

(ln5 is a constant)

$f''(x) = 5^x \ln 5 \cdot \ln 5 = 5^x (\ln 5)^2$

$f^{(3)}(x) = 5^x \ln 5 (\ln 5)^2 = 5^x (\ln 5)^3$

$f^{(4)}(x) = 5^x (\ln 5)^4$

$\vdots$

$f^{(10)}(x) = 5^x (\ln 5)^{10}$.

9. If $f(x) = \ln(x^3 - 5)$, then $f^{-1}(x) =$

(A) e^{x^3-5} (B) $e^{x^3}+5$ (C) $\sqrt[3]{e^x+5}$ (D) $e^{\frac{x}{3}}+5$ (E) $\frac{1}{\ln(x^3-5)}$.

Ans. $y = \ln(x^3-5)$. Solve for x.

$$e^y = e^{\ln(x^3-5)}$$

$$e^y = x^3-5$$

$$e^y+5 = x^3$$

$$(e^y+5)^{1/3} = (x^3)^{1/3}$$

$$x = (e^y+5)^{1/3} = \sqrt[3]{e^y+5}$$

So, $f^{-1}(x) = \sqrt[3]{e^x+5}$

10. The slope of the tangent line to the curve $\sin^2(x) + y^2 = 1$ at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ is

(A) $\frac{\pi}{4}$ (B) $\frac{1}{\sqrt{2}}$ (C) 0 (D) $\frac{-1}{\sqrt{2}}$ (E) $\sqrt{2}$.

Ans. $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$, $y = \frac{1}{\sqrt{2}}$.

Implicit differentiation.

$$2\sin x \cos x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2\sin x \cos x$$

OR $2y\frac{dy}{dx} = -\sin(2x)$ [double angle]

OR $\frac{dy}{dx} = \frac{-\sin(2x)}{2y}$

So, $\frac{dy}{dx}$ at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) = \frac{-\sin\left(2\frac{\pi}{4}\right)}{2(1/\sqrt{2})}$

$$= \frac{-\sin(\pi/2)}{\sqrt{2}}$$

$$= \frac{-1}{\sqrt{2}}.$$

11. If $f(x) = g(e^x)$, $f'(\ln(2)) = 2e$, then $g'(2) =$

(A) e (B) $\ln(2)$ (C) $\frac{e}{2}$ (D) 2 (E) Cannot be determined.

Ans. By the chain rule

$$f'(x) = g'(e^x) \cdot \frac{de^x}{dx}$$

$$= g'(e^x) \cdot e^x$$

So, $f'(\ln(2)) = g'(e^{\ln(2)})\, e^{\ln(2)}$

so, $f'(\ln(2)) = g'(2)\, 2$

So, $\cancel{2}e = g'(2)\,\cancel{2}$

$$g'(2) = e.$$

12. Let $g(x) = (f(x) - 2)\tan x$. Suppose $g(x)$ is continuous at $\frac{\pi}{2}$. Then, $\lim_{x \to \frac{\pi}{2}} f(x) =$

(A) ∞ (B) $-\infty$ (C) Does not exist (D) 2 (E) 0.

Ans. $g(x) = (f(x) - 2)\tan x$ OR

$\frac{g(x)}{\tan x} = f(x) - 2$ OR

$f(x) = 2 + g(x)\cot x$ $\left[\frac{1}{\tan x} = \cot x\right]$

So, $\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} (2 + g(x)\cot x)$

$= 2 + g\left(\frac{\pi}{2}\right)\cot\left(\frac{\pi}{2}\right) = 2.$ $\left[\cot\frac{\pi}{2} = 0\right]$