10.07.2015

3.4 CHAIN RULE

With the rules seen so far we cannot differentiate

$$f(x) = \sqrt[3]{\sin x}. \quad \text{— (1)}$$

Note that $f(x) = g(h(x))$

$$g(x) = \sqrt[3]{x}, \quad h(x) = \sin(x).$$

outside

CHAIN RULE: $F(x) = f(g(x))$.

inside

Suppose $g'(x)$ and $f'(g(x))$ exists, then

$$F'(x) = f'(g(x)) \cdot g'(x).$$

The derivative of F at x is the product of the derivative of the outside function evaluated at the inside function, and the derivative of the inside function.

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

OR $y = f(u)$, $u = g(x)$

$$y = f \circ g(x).$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Going back to (1).

$$g(x) = \sqrt[3]{x}, \quad h(x) = \sin(x)$$

$$(g \circ h)'(x) = g'(h(x)) \cdot h'(x)$$

$$= \frac{1}{3}(\sin(x))^{-2/3} \cdot \cos x.$$

We know that $\frac{d}{dx}e^x = e^x$.

But what about $\frac{d}{dx}a^x = ?$

Recall that if

$$y = a^x$$

then $\log_a y = x$

then $\frac{\ln y}{\ln a} = x$

Change of basis

So, $\ln y = (\ln a)x$

So $\cancel{e}^{\cancel{\ln} y} = e^{(\ln a)x}$

OR $y = e^{(\ln a)x}$

OR $\underline{a^x = e^{(\ln a)x}}$

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{(\ln a)x}$$

$f(x) = e^x \quad g(x) = (\ln a)x.$

$$= f'(g(x))\,g'(x)$$

$$= e^{(\ln a)x} \cdot (\ln a)$$

$$= a^x \ln a$$

$\frac{d}{dx}x^a = a x^{a-1}$

So, $\boxed{\frac{d}{dx}a^x = a^x \ln a}$

Slope of the tangent to $y = a^x$ at $(0,1)$ is $a^0 \ln a = \ln(a)$.

So, the only value of a for which this slope is 1, is $a = e$.

WARNING $x^a \neq a^x$

Ex. $F(x) = \sqrt[5]{\tan x}$. What is $F'(x)$?

$f(x) = \sqrt[5]{x} = x^{1/5}$

$g(x) = \tan x$.

$f'(x) = \frac{1}{5} x^{-4/5}$

$g'(x) = \sec^2 x$.

By the chain rule,

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{5}(\tan x)^{-4/5} \sec^2 x.$$

In general,

$$\frac{d}{dx}(f(x))^n = n(f(x))^{n-1} \cdot f'(x).$$

Ex. $h(x) = \sqrt{4 + e^{f(x)}}$

$f(1) = 1$, $h'(1) = 1$. What is $f'(1)$?

Ans. $h'(x) = \frac{1}{2}\left(4 + e^{f(x)}\right)^{-1/2} \cdot \left(\frac{d}{dx} e^{f(x)}\right)$

$$= \frac{1}{2\sqrt{4 + e^{f(x)}}} \left[e^{f(x)} \cdot f'(x)\right]$$

So, $h'(x) = \frac{e^{f(x)}}{2\sqrt{4 + e^{f(x)}}} \cdot f'(x)$.

$x = 1$

$$h'(1) = \frac{e^{f(1)}}{2\sqrt{4 + e^{f(1)}}} f'(1)$$

$1 = \frac{e}{2\sqrt{4+e}} f'(1)$. So, $\boxed{f'(1) = \frac{2}{e}\sqrt{4+e}}$.

Ex. At what point of the curve $y = \sqrt{1+2x}$ is the tangent line perpendicular to the line $6x + 2y = 1$?

Ans Slope of the given line:

$$6x + 2y = 1$$
$$2y = -6x + 1$$
OR
$$y = \underline{-3}\,x + \frac{1}{2}$$

Slope of the given line $= -3$.

Slope of the required tangent

$$= \frac{-1}{-3} = \frac{1}{3}.$$

$$y' = \frac{1}{\cancel{2}}(1+2x)^{-1/2} \cdot \cancel{2}$$

$$y' = \frac{1}{\sqrt{1+2x}}$$

Solve for x

$$\frac{1}{\sqrt{1+2x}} = \frac{1}{3}$$

OR $\quad 3 = \sqrt{1+2x}$

OR $\quad 9 = 1 + 2x$

$$\boxed{x = 4}$$

And $f(4) = \sqrt{1+2\cdot 4} = \sqrt{9} = 3$

So, the point is $\boxed{(4, 3)}$

WARNING

composition $(f \circ g)'(x) = f'(g(x))\,g'(x)$

product $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

$y' = \cancel{2}(1+2x)\cdot\cancel{2}$

product $(fg)'(x) = f'(x)g(x)+f(x)g'(x).$

Ex. Compute $\frac{d}{dx} 5^{\sin x}$.
(a)

Solution $F(x) = 5^{\sin x}$

$f(x) = 5^x$; $f'(x) = 5^x \cdot \ln 5$

$g(x) = \sin x$; $g'(x) = \cos x$.

By the chain rule

$\frac{d}{dx} 5^{\sin x} = F'(x) = 5^{\sin x} \ln 5 \cdot \cos x$

(b) Compute $\frac{d^2}{dx^2} 5^{\sin x}$.

$$F''(x) = \ln 5 \left(\left(\frac{d}{dx} 5^{\sin x}\right) \cos x + 5^{\sin x} (-\sin x) \right)$$

$$= \ln 5 \left(5^{\sin x} \ln 5 \cos x \cdot \cos x - \sin x \, 5^{\sin x} \right)$$

$$= (\ln 5) 5^{\sin x} \underline{(\ln 5 \cos^2 x - \sin x)}$$

$$\boxed{\frac{d}{dx} a^x = a^x \ln a \qquad (a=5)}$$

3.5 Implicit Differentiation

Suppose y is a function of x

$$\sqrt{y+1} = x^2-2.$$

Find y'.

Isolate y

$$y+1 = (x^2-2)^2$$

OR

$$y = (x^2-2)^2 - 1.$$

$$y' = 2(x^2-2)^1(2x)$$

— But we can't always isolate y

For. eg. $y^3+x^3 = xy$ —— (i)

Hard to isolate y and we say that y is implicitly defined as a function of x.

Think of y as $y(x)$

$$y(x)^3 + x^3 = x\cdot y(x)$$

Differentiate w.r.t. x,

chain product

$$3\left(y(x)\right)^2\cdot y'(x) + 3x^2 = \left[1\cdot y(x) + xy'(x)\right]$$

$$3y(x)^2y'(x) + 3x^2 = y(x) + xy'(x)$$

$$3y(x)^2y'(x) - xy'(x) = y(x) - 3x^2$$

$$y'(x)\left[3y(x)^2 - x\right] = y(x) - 3x^2$$

OR $$y'(x) = \frac{y(x)-3x^2}{3y(x)^2-x}$$

DROP (x)

$$\boxed{y' = \frac{y-3x^2}{3y^2-x}}$$

hard to isolate y and we say that

y is implicitly defined as a function

$$\boxed{y' = \frac{y - 3x^2}{3y^2 - x}}$$

Find the slope of the tangent to the curve

$$y^3+x^3 = xy$$

at the point $\left(\frac{1}{2},\frac{1}{2}\right)$.

Answer

$$\boxed{y' = \frac{y-3x^2}{3y^2-x}}$$

So, we want to compute y' at $x=\frac{1}{2}$, $y=\frac{1}{2}$

Slope: $$\frac{1/2-3(1/2)^2}{3(\frac{1}{2})^2-1/2} = -1.$$

Ex. Find y'' if

$$\cos(x+y) = y$$

Answer: Differentiate w.r.t. x

$$-\sin(x+y)(1+y') = y'$$

$$-\sin(x+y)-\sin(x+y)y' = y'$$

$$-\sin(x+y) = y'+y'\sin(x+y)$$

$$= y'(1+\sin(x+y))$$

So, $$y' = \frac{-\sin(x+y)}{1+\sin(x+y)}.$$

Ex. Find y'' if

$$\cos(x+y) = y$$

Answer: Differentiate w.r.t. x

$$-\sin(x+y)(1+y') = y'$$

$$-\sin(x+y) - \sin(x+y)y' = y'$$

$$-\sin(x+y) = y' + y'\sin(x+y)$$

$$-\sin(x+y) = y'(1+\sin(x+y)) \quad \text{—(i).}$$

So, $$y' = \frac{-\sin(x+y)}{1+\sin(x+y)} \quad \text{—(ii)}$$

Differentiate (i) w.r.t. x

$$-\cos(x+y)(1+y') = y''(1+\sin(x+y)) + y'(\cos(x+y)(1+y'))$$

$$-\cos(x+y)(1+y') - y'\cos(x+y)(1+y') = y''(1+\sin(x+y))$$

$$\frac{-\cos(x+y)(1+y')(1+y')}{1+\sin(x+y)} = y''$$

$$y'' = \frac{-\cos(x+y)(1+y')^2}{1+\sin(x+y)}$$

$$= \frac{-\cos(x+y)}{1+\sin(x+y)}\left(1 + \frac{-\sin(x+y)}{1+\sin(x+y)}\right)^2$$

$$= \frac{-\cos(x+y)}{1+\sin(x+y)}\left(\frac{1}{1+\sin(x+y)}\right)^2$$

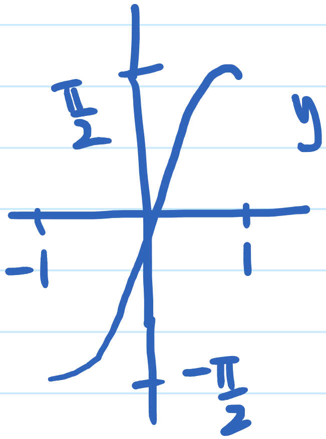
$$= \frac{-\cos(x+y)}{(1+\sin(x+y))^3}.$$

$$+\,y'\left(\cos(x+y)(1+y')\right) = \frac{-\cos(x+y)}{(1+\sin(x+y))^3}.$$

APPLICATION: DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS.

(A) $\sin^{-1}(x)$ or $\arcsin(x)$

Recap $y = \sin^{-1}(x)$ MEANS $\sin(y) = x$ $\quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$



Dom: $[-1,1]$
Ran: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\frac{d}{dx}\sin^{-1}(x)$?

Set $y = \sin^{-1}(x)$
$\sin(y) = x \qquad y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Diff. w.r.t. x,

$$\cos(y) \cdot y' = 1$$

So, $y' = \frac{1}{\cos(y)}$ —— Φ

Recall $\sin^2(y) + \cos^2(y) = 1$

So $x^2 + \cos^2 y = 1$

OR $\cos^2 y = 1 - x^2$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\cos y > 0$

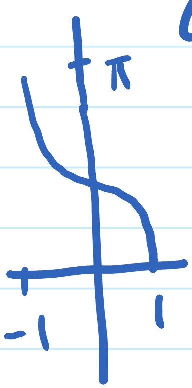
So $\cos y = \sqrt{1-x^2}$

So, in Φ $\boxed{\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}}$

(B) $\cos^{-1}(x)$ or $\arccos(x)$.

Recap $y=\cos^{-1}(x)$ MEANS

$\cos(y)=x \qquad 0\le y\le \pi$



$y=\cos^{-1}(x)$ Dom: $[-1,1]$
Ran.: $[0,\pi]$

$$\frac{d}{dx}\cos^{-1}(x)$$

Set $y=\cos^{-1}(x)$

$\cos(y)=x \qquad y\in[0,\pi]$.

Diff. both sides w.r.t. x

$$-\sin(y)y'=1$$

So, $y'=-\dfrac{1}{\sin(y)}$.

But, $\sin^2 y+\cos^2 y=1$

$\sin^2 y+x^2=1$

So $\sin^2 y=1-x^2$

$y\in[0,\pi]$ so $\sin(y)>0$

Ist & IInd Quadrant.

$\sin(y)=\sqrt{1-x^2}$

$$\boxed{\frac{d}{dx}\cos^{-1}(x)=\frac{-1}{\sqrt{1-x^2}}}$$

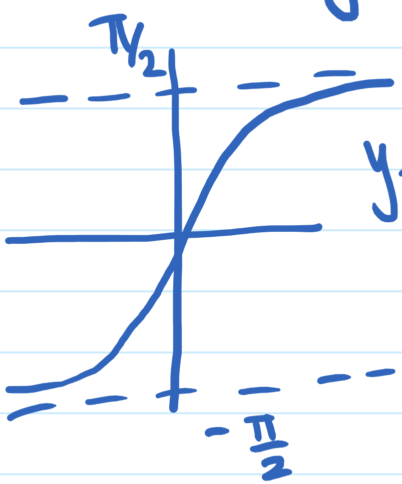
$$\tan x = \frac{\sin x}{\cos x}$$

$$\cancel{\tan^{-1} x = \frac{\sin^{-1} x}{\cos^{-1} x}}$$

(c) $y = \tan^{-1}(x)$ or $y = \arctan(x)$.

Recap. $y = \tan^{-1}(x)$ MEANS

$$\tan(y) = x \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$



$$\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$$

$\frac{d}{dx} \tan^{-1}(x)$. Set $y = \tan^{-1}(x)$

OR $\tan(y) = x \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$.

Diff. both sides w.r.t. x

$$\sec^2(y) \cdot y' = 1$$

$$y' = \frac{1}{\sec^2(y)}.$$

$$\tan^2 y + 1 = \sec^2(y)$$

So, $x^2 + 1 = \sec^2(y)$

$$\boxed{\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}}$$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}.$$

<u>Example</u> Find y' if

$$\sin^{-1}(\cos(y)) + x = \tan(y)$$

<u>Solution</u>: Diff. both sides w.r.t x

$$\frac{1}{\sqrt{1-(\cos(y))^2}} \cdot \left[-\sin(y)\cdot y'\right] + 1 = \sec^2(y)\cdot y'$$

$$1 = +\frac{\sin(y)}{\sqrt{1-\cos^2 y}}\,y' + \sec^2(y)\,y'$$

$$1 = y'\left(\frac{\sin(y) + \sqrt{1-\cos^2 y}\,\sec^2(y)}{\sqrt{1-\cos^2 y}}\right)$$

$$y' = \frac{\sqrt{1-\cos^2 y}}{\sin(y) + \sqrt{1-\cos^2 y}\,\sec^2 y}.$$

$$\sin(y) + \sqrt{1-\cos^2 y}\,\sec^2 y$$

Ex. The line

$$ax+by=1$$

is tangent to the ellipse

$$x^2-xy+y^2=3 \text{ at } (-1,1).$$

What are a and b?

Solution $x^2-xy+y^2=3$

Differentiate w.r.t. x.

$$2x-(1\cdot y+xy')+2y\cdot y'=0$$

$$2x-y+(-x+2y)y'=0$$

So, $(-x+2y)y'=y-2x$

So, $y'=\dfrac{y-2x}{-x+2y}$

So, y' at $(-1,1)$

$$\frac{1-2(-1)}{-(-1)+2(1)}=\frac{1+2}{3}$$

$$=1$$

Eqn. of the tangent

$$(y-1)=1(x-(-1))$$

$$y-1=x+1$$

OR $x-y=-2$

OR $\underbrace{-\tfrac{1}{2}}_{a}x+\underbrace{\tfrac{1}{2}}_{b}y=1$

$$\boxed{\begin{aligned}a&=-\tfrac{1}{2}\\ b&=\tfrac{1}{2}\end{aligned}}$$