10.05.2015 3.1 and 3.2 (Review)

FACTS YOU MUST ALREADY KNOW:

$$\frac{d}{dx}c = 0 \qquad c = \text{constant}$$

$$\frac{d}{dx}x^n = nx^{n-1} \qquad n \text{ is a real number}$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}(cf) = c\frac{df}{dx} \qquad c = \text{constant}$$

$$\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}.$$

Ex. Compute $f'(x)$ if $f(x) = e^x\left(\frac{x^{5/3} + \sqrt[3]{x}}{2x}\right)$.

Ans $f(x) = e^x\left(\frac{x^{5/3} + x^{1/3}}{2x}\right)$

$$= e^x\left(\frac{1}{2}x^{2/3} + \frac{1}{2}x^{-2/3}\right)$$

$$f'(x) = e^x\left(\frac{1}{2}x^{2/3} + \frac{1}{2}x^{-2/3}\right)$$

$$+ e^x\left(\frac{1}{\cancel{2}}\frac{\cancel{2}}{3}x^{-1/3} + \frac{1}{\cancel{2}}\left(-\frac{\cancel{2}}{3}\right)x^{-5/3}\right)$$

$$= e^x\left[\frac{x^{2/3}}{2} + \frac{x^{-2/3}}{2} + \frac{x^{-1/3}}{3} - \frac{x^{-5/3}}{3}\right]$$

Ex. Find the tangent to the given
(i) curve at $(0, \frac{1}{2})$

$$y = \frac{1+x}{1+e^x} = f(x)$$

Ans. By the slope-point formula

tangent : $\left(y - \frac{1}{2}\right) = y'\Big|_{(x=0)} (x-0)$

$$y' = \frac{1+e^x(1) - (1+x)(e^x)}{(1+e^x)^2}$$

$$= \frac{1+\cancel{e^x} - \cancel{e^x} - xe^x}{(1+e^x)^2}$$

at $x=0$

$$y' = \frac{1 - 0 \cdot e^0}{(1+e^0)^2} = \frac{1}{4}$$

tangent : $\boxed{y - \frac{1}{2} = \frac{1}{4}x}$

(ii) Find the normal line at $(0, \frac{1}{2})$.

Normal line = perpendicular to the tangent line.

Slope of the normal at $(0, \frac{1}{2})$ $= \dfrac{-1}{\text{slope of the tangent at } (0, \frac{1}{2})}$

$$= \frac{-1}{f'(0)}$$

$$= \frac{-1}{1/4} = -4.$$

Normal line : $y - \frac{1}{2} = -4x$.

OR $y + 4x = \frac{1}{2}$

Ex. Show that tangent line to

$$y = \frac{e^x}{1+x^2} \text{ at } \left(1, \frac{e}{2}\right)$$

is parallel to the x-axis (horizontal)

Ans $f(x) = \dfrac{e^x}{1+x^2}$.

Find $f'(1)$.

$$f'(x) = \frac{(1+x^2)(e^x) - e^x(2x)}{(1+x^2)^2}$$

$$= \frac{e^x(1-2x+x^2)}{(1+x^2)^2}$$

$$f'(1) = \frac{e(1-2+1)}{4} = 0$$

Ex. Suppose $f(4)=2$, $f'(4)=6$
$g(4)=5$, $g'(4)=-3$.

If $h(x) = \dfrac{f(x)}{f(x)+g(x)}$.

What is $h'(4)$?

Ans. $h'(x) = \dfrac{(f(x)+g(x))f'(x) - f(x)(f'(x)+g'(x))}{(f(x)+g(x))^2}$

$$= \frac{\cancel{f(x)f'(x)} + g(x)f'(x) - \cancel{f(x)f'(x)} - f(x)g'(x)}{(f(x)+g(x))^2}$$

$$h'(4) = \frac{g(4)f'(4) - f(4)g'(4)}{(f(4)+g(4))^2}$$

$$= \frac{5 \cdot 6 - 2(-3)}{(2+5)^2} = \frac{30+6}{7^2} = \frac{36}{49}$$

Ex. Suppose g is differentiable

(i) $f(x) = \dfrac{1+xg(x)}{\sqrt{x}}$.

$f'(1)=2$ and $g(1)=1$, then what is $g'(1)$?

<u>Ans.</u> $f(x) = x^{-1/2} + x^{1/2} g(x)$.

$$f'(x) = -\frac{1}{2}x^{-3/2} + \frac{1}{2}x^{-1/2}g(x) + x^{1/2}g'(x).$$

$x=1$

$$f'(1) = -\frac{1}{2}(1)^{-3/2} + \frac{1}{2}(1)^{-1/2}g(1) + 1^{1/2}g'(1)$$

$$2 = \cancel{-\frac{1}{2}} + \cancel{\frac{1}{2}\cdot 1} + g'(1)$$

$$\boxed{g'(1)=2}$$

(ii) $f''(1)=2$, what is $g''(1)$?

second derivative

differentiate $f'(x)$.

differentiate!

$$f''(x) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} + \frac{1}{2}\left[-\frac{1}{2}x^{-3/2}g(x) + x^{-1/2}g'(x)\right] + \frac{1}{2}x^{-1/2}g'(x) + x^{1/2}g''(x).$$

$x=1$

$$f''(1) = \frac{3}{4} - \frac{1}{4}g(1) + \frac{1}{2}g'(1) + \frac{1}{2}g'(1) + g''(1)$$

$$2 = \frac{3}{4} - \frac{1}{4} + \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 2 + g''(1)$$

$$\cancel{2} = \frac{1}{2} + \cancel{2} + g''(1) \qquad \boxed{g''(1) = -\tfrac{1}{2}}$$

3.3 Derivatives of trigonometric functions

$$\frac{d}{dx}\sin x = \cos x.$$

$$\parallel$$

definition!

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

addition for sin!

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\sin(x)\left(\frac{\cos(h)-1}{h}\right)+\lim_{h\to 0}\cos(x)\frac{\sin(h)}{h}.$$

$$= \sin(x)\underbrace{\cancel{\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right)}}_{0}+\cos(x)\underbrace{\cancel{\lim_{h\to 0}\frac{\sin h}{h}}}_{1}$$

$$\boxed{\lim_{h\to 0}\frac{\sin h}{h}=1} \quad \& \quad \boxed{\lim_{h\to 0}\frac{\cosh -1}{h}=0}$$

$$\lim_{h\to 0}\frac{\cosh-1}{h}\cdot\frac{\cosh+1}{\cosh+1}$$

$$\boxed{\begin{aligned}&\cos^2 h-1\\&=-\sin^2 h\end{aligned}}$$

$$= \lim_{h\to 0}\frac{\cos^2 h-1}{h\cdot(\cosh+1)}$$

$$= \lim_{h\to 0}\frac{-\sin^2 h}{h(\cosh+1)}$$

$$= \lim_{h\to 0}\frac{\overset{0}{\sin h}}{h}\cdot -\sin h\cdot\frac{1}{\cosh+1}$$

$$= \lim_{h\to 0}\frac{\sin h}{h}\ \lim_{h\to 0}-\overset{0}{\sin}h\ \lim_{h\to 0}\frac{1}{\cosh+1}$$

$$= 1\cdot 0\cdot\frac{1}{2} = 0$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

need these two

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x.$$

Ex/

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x}$$

$$= \frac{\cos x\left(\frac{d}{dx}1\right) - 1\,\frac{d}{dx}\cos x}{\cos^2(x)}$$

$$= \frac{0 - 1\cdot(-\sin x)}{\cos^2(x)}$$

$$= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}$$

$$= \sec x \cdot \tan x.$$

NOTATION: $f^{(k)}(x)$ — this is the function obtained from $f(x)$ by differentiating k-times.

Ex. $f(x) = \cos x$. What is $f^{(21)}(x)$?

ANS. FIND A PATTERN!

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f^{(3)}(x) = -(-\sin x)$
$= \sin x$

$f^{(4)}(x) = \cos x$

$f^{(5)}(x) = -\sin x$

$f^{(6)}(x) = -\cos x$

$f^{(7)}(x) = \sin x$

cycle of period of four

Every fourth multiple is going to be $\cos x$.

$f^{(20)}(x) = \cos x$

$$\boxed{f^{(21)}(x) = -\sin x}$$

Ex. Find the equation of the tangent line to the curve

$$y = x + \cot x$$

at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

Ex. Find the equation of the tangent line to the curve

$$y = x + \cot x$$

at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

Ans. $y' = 1 + \frac{d}{dx}\cot x$

$$= 1 + \frac{d}{dx}\frac{\cos x}{\sin x}$$

$$= 1 + \frac{\sin x \frac{d}{dx}\cos x - \cos x \frac{d}{dx}\sin x}{\sin^2 x}$$

$$= 1 + \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x}$$

$$= 1 - \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= 1 - \frac{1}{\sin^2 x}$$

$$= 1 - \operatorname{cosec}^2 x$$

Slope at $x = \frac{\pi}{2}$ is $1 - \operatorname{cosec}^2\left(\frac{\pi}{2}\right)$

$$= 1 - 1 = \boxed{0.}$$

Horizontal tangent.

$$y - \frac{\pi}{2} = 0\left(x - \frac{\pi}{2}\right)$$

$$\boxed{y = \frac{\pi}{2}}$$

Chapter 2 revisited.

Ex. $\lim_{x\to\infty} \frac{x+3}{\sqrt{x^2-2}}$

$= \lim_{x\to\infty} \dfrac{\frac{x+3}{\sqrt{x^2}}}{\frac{\sqrt{x^2-2}}{\sqrt{x^2}}}$

$= \lim_{x\to\infty} \dfrac{\frac{x+3}{\sqrt{x^2}}}{\sqrt{1-\frac{2}{x^2}}}$

$x>0, \sqrt{x^2} = x$

$= \lim_{x\to\infty} \dfrac{\frac{x+3}{x}}{\sqrt{1-2/x^2}} = \lim_{x\to\infty} \dfrac{1+\overset{0}{3/x}}{\sqrt{1-\overset{0}{\frac{2}{x^2}}}}$

$= \frac{1}{\sqrt{1}} = 1$

$\lim_{x\to-\infty} \frac{x+3}{\sqrt{x^2-2}}$

$= \lim_{x\to-\infty} \dfrac{\frac{x+3}{\sqrt{x^2}}}{\frac{\sqrt{x^2-2}}{\sqrt{x^2}}}$

$= \lim_{x\to-\infty} \dfrac{\frac{x+3}{\sqrt{x^2}}}{\sqrt{1-2/x^2}}$

$x<0, \sqrt{x^2} = |x| = -x$

$= \lim_{x\to-\infty} \dfrac{\frac{x+3}{-x}}{\sqrt{1-2/x^2}} = \lim_{x\to-\infty} \dfrac{-1-\overset{0}{\frac{3}{x}}}{\sqrt{1-\overset{0}{2/x^2}}}$

$= \frac{-1}{\sqrt{1}} = -1.$