9-30-2015

① Written Assignment 2 is due next Wednesday. NO LATE SUBMISSIONS.

② MIDTERM : OCTOBER 23rd. 7:00-9:00pm
E-MAIL ME IF YOU HAVE A CONFLICT

Syllabus: 8th ed. Appendix D, 1.4, 1.5, 2.2, 2.3, 2.5, 2.6, 2.7, 2.8, 3.1, 3.2, 3.3, 3.4 & 3.5.

2.7. Derivatives and rates of change

Definition: The derivative of the function $f(x)$ at $x=a$ is defined as

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \left[\begin{array}{l} h = x-a \\ x\to a \text{ then } \\ h\to 0 \end{array}\right].$$

Ex. The following limit represents the derivative of a function f at the point a

$$\lim_{x\to \frac{\pi}{2}} \frac{\cos(x)}{x-\frac{\pi}{2}}.$$

What are f and a?

Solution $\cos\left(\frac{\pi}{2}\right) = 0$

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos(x)}{x-\frac{\pi}{2}} = \lim_{x\to\frac{\pi}{2}} \frac{\cos(x)-\cos\left(\frac{\pi}{2}\right)}{x-\frac{\pi}{2}}$$

$f(x) = \cos(x)$
$a = \frac{\pi}{2}.$

If the question says 'first principles', 'using the definition' or 'using limits', then compute using the definition.

Ex. Compute, using first principles, the derivative of $f(x) = \sqrt[3]{x}$ at $x = 1$.

Solution : $f'(1) = \lim_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$

$$= \lim_{h \to 0} \frac{(1+h)^{1/3} - (1)^{1/3}}{h}$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

set $a = (1+h)^{1/3}$, $b = (1)^{1/3}$

$$f'(1) = \lim_{h \to 0} \frac{\left((1+h)^{1/3} - (1)^{1/3}\right)}{h} \frac{\left((1+h)^{2/3} + (1+h)^{1/3}(1)^{1/3} + (1)^{2/3}\right)}{(1+h)^{2/3} + (1+h)^{1/3} + 1}$$

$$= \lim_{h \to 0} \frac{\left((1+h)^{1/3}\right)^3 - (1^{1/3})^3}{h\left((1+h)^{2/3} + (1+h)^{1/3} + 1\right)}$$

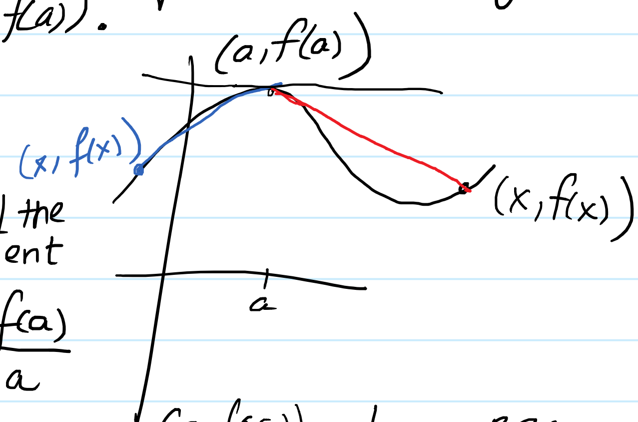
$$= \lim_{h \to 0} \frac{\cancel{1} + \cancel{h} - \cancel{1}}{\cancel{h}\left((1+h)^{2/3} + (1+h)^{1/3} + 1\right)}$$

$$= \frac{1}{1^{2/3} + 1^{1/3} + 1} = \frac{1}{3} \quad //$$

Interpretations

① Slope of tangents.

The tangent line to the curve $y = f(x)$ at $(a, f(a))$ is the line passing through this point and whose slope is the limit of the slope of secants passing through $(a, f(a))$.



slope of the tangent $= \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

the tangent at $(a, f(a))$ has eqn.

$$y - f(a) = f'(a)(x - a)$$

If l is a line with slope m and passing through (x_0, y_0), l has equation

slope-point formula

$$y - y_0 = m(x - x_0).$$

Example Find the tangent to the curve $y = x^4$ at $(1,1)$.

Solution: $f(x) = x^4$.

Then the slope of the tangent at $(1,1)$ is $f'(1)$.

$$f'(1) = \lim_{h\to 0} \frac{(1+h)^4 - (1)^4}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{1} + 4h + 6h^2 + 4h^3 + h^4 - \cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(4 + 6h + 4h^2 + h^3)}{\cancel{h}}$$

$$= 4.$$

Eqn: $(y - f(1)) = f'(1)(x-1)$

$$(y-1) = 4(x-1).$$

(1) Slope of the tangent at $x = a$ is also called the slope of the curve at $x = a$.

(2) $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \pm\infty$ then this means that the function $f(x)$ has a vertical tangent line at $(a, f(a))$.

(II) Velocity: $s = f(t)$: describes the motion of a moving particle. t = time and s is displacement.

$f'(a)$ = velocity of the particle at time a.

(III) Instantaneous rate of change

2.8 Derivative as a function.

The derivative function of $f(x)$ is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(i) A function f is said to be differentiable at $x = a$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

(ii) f is said to be differentiable in (a,b) if it is differentiable at every point in (a,b).

Ex. Where is $f(x) = |x|$ differentiable?

Solution: $|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$

Case 1 $x > 0$.

I can choose h so small that $x+h>0$

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h} = 1.$$

Case 2 $x<0$

I can choose h so small that $x+h<0$

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

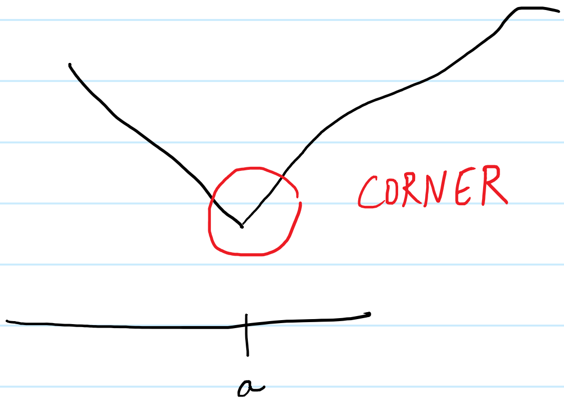
$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = -1.$$

Case 3. $f'(0) = \lim_{h \to 0} \frac{|h|}{h} = \text{DNE}.$

$|x|$ is differentiable everywhere except at $x = 0$.

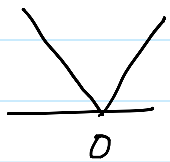
The following functions are not differentiable at $x=a$.

① 

CORNER

Example

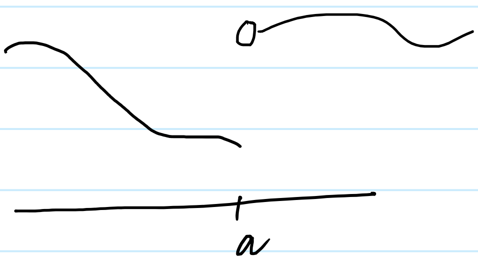
$f(x)=|x|$



② Theorem: If f is differentiable at $x=a$, then it is continuous at $x=a$.

OR

If f is not continuous at a, it is not differentiable at a.



LACK OF CONTINUITY.

③ 

VERTICAL TANGENTS

Example

$f(x) = \sqrt[3]{x}$ at $x=0$.

Ex: Compute the derivative of $\sqrt[3]{x}$ at 0.

Second derivative: The derivative of $y=f'(x)$ is called the second derivative of $f(x)$ and is written as $y=f''(x)$.

$y=f(x)$

NOTATION: $y' = f'(x) = \dfrac{df(x)}{dx} = Df(x)$

$= D_x f(x)$.

QUIZ 1 - SOLUTIONS

① Reflect the graph of $y=3^x$ about the x-axis and shift it upwards by 3 units.

$y=3^x$

$\downarrow$ reflect about the x-axis

$y=-3^x$

$\downarrow$ shift

$y=3-3^x$

Solution C

② $$3e^{3(\ln(p)-\ln(q))}$$

$$=3e^{3\left(\ln\left(\frac{p}{q}\right)\right)}$$

$$=3e^{\ln\left(\left(\frac{p}{q}\right)^3\right)}$$

$$=3\left(\frac{p}{q}\right)^3.$$

$$\boxed{e^{\ln(x)}=x}$$

B

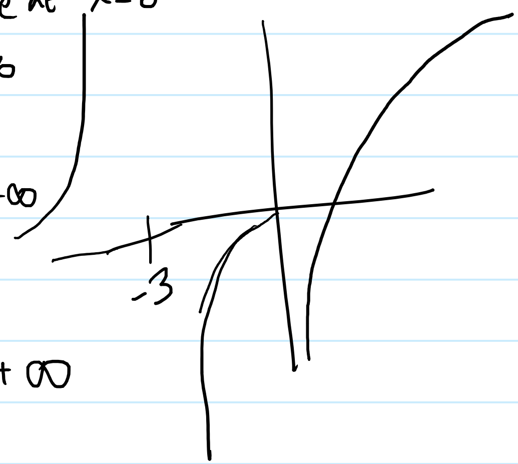
③ $$f(x)=\begin{cases}\ln(x) & x>0\\ \dfrac{x}{x+3} & x\le 0\end{cases}$$

✓ vertical asymptote at $x=0$

$\lim_{x\to 0^+} f(x) = -\infty$

✓ $\lim_{x\to -3^+} f(x) = -\infty$

✓ $\lim_{x\to 3^-} f(x) = +\infty$



$\lim_{x\to 0} f(x)$ exists — NOT TRUE

Right Hand Limit does not exist

D

④ $$\lim_{x\to -2} \frac{2-|x|}{2+x}.$$

x is near -2.

x is negative.

So $|x| = -x$ when x is close to -2.

$$\lim_{x\to -2} \frac{2-|x|}{2+x} = \lim_{x\to -2} \frac{2-(-x)}{2+x}$$

$$= \lim_{x\to -2} \frac{\cancel{2+x}}{\cancel{2+x}}$$

$$= 1.$$

(A).

Review Problems

① What are the horizontal and vertical asymptotes of

$$f(x) = \frac{-2x^{2/3}}{x^2+2x-3}.$$

Solution: Horizontal asymptotes $\longleftrightarrow$ Limits at infinity.

$$\lim_{x\to\infty} \frac{-2x^{2/3}}{x^2+2x-3} = \frac{\frac{-2x^{2/3}}{x^2}}{\frac{x^2+2x-3}{x^2}} \qquad \frac{-x^{2/3}}{x^2} = \frac{-1}{x^{4/3}}$$

$$= \lim_{x\to\infty} \frac{\frac{-2}{x^{4/3}} \to 0}{1+\frac{2}{x}\to 0 - \frac{3}{x^2} \to 0} = 0$$

$y=0$ is a horizontal asymptote.

$\boxed{Ex}$ $$\lim_{x\to -\infty} \frac{-2x^{2/3}}{x^2+2x-3} = 0$$

① What are the horizontal and vertical asymptotes of

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$\boxed{y=0}$ is a horizontal asymptote.

$\boxed{Ex}$ $$\lim_{x\to-\infty} \frac{-2x^{2/3}}{x^2+2x-3} = 0$$

<u>Vertical asymptotes</u>

Look for values of x where the denominator vanishes but the numerator does not.

$$x^2+2x-3=0$$
$$x^2+3x-x-3=0$$
$$(x-1)(x+3)=0.$$
$$x=1, \quad x=-3$$

$\rightarrow$ Numerator does not vanish at these points.

$$f(x) = \frac{-2x^{2/3}}{(x-1)(x+3)}$$

$$\lim_{x\to1^+} f(x) = -\infty$$ Vertical asymptote $\boxed{x=1}$.

$$\lim_{x\to-3^+} f(x) = \lim_{x\to-3^+} \frac{-2x^{2/3} \ \text{-ve}}{\underset{\text{-ve}}{(x-1)}\underset{\text{+ve}}{(x+3)}} = +\infty.$$

Vertical asymptote at $\boxed{x=-3}$

(2) Use the Intermediate Value Theorem to show that the graphs of $y = \tan(2x)$ and $y = \sin(x) + 1$ intersect for x in $\left(0, \frac{\pi}{6}\right)$.

Solution: Show that

$$\tan(2x) = \sin(x) + 1$$

has at least one solution in $\left(0, \frac{\pi}{6}\right)$.

OR Show that

$$\tan(2x) - \sin(x) - 1 = 0$$

for some x in $\left(0, \frac{\pi}{6}\right)$.

Set $f(x) = \tan(2x) - \sin(x) - 1$

We need to show that $f(x)$ has a zero in $\left(0, \frac{\pi}{6}\right)$.

$f(x)$ is continuous in $\left[0, \frac{\pi}{6}\right]$

PROOF:

$\sin(x)$ is continuous everywhere

$\tan(x)$ is continuous for all real numbers except numbers of the form $\frac{\pi}{2} + n\pi$, n is an integer

In particular

$\tan(x)$ is continuous in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan(2x)$ is continuous when

IMPORTANT
$$\left\{\begin{array}{l} -\frac{\pi}{2} < 2x < \frac{\pi}{2} \quad \text{isolate } x \\ -\frac{\pi}{4} < x < \frac{\pi}{4} \end{array}\right.$$

$\tan(2x)$ is continuous in $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

$\tan(2x) - \sin(x) - 1$ is continuous in $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

But, $\left[0, \frac{\pi}{6}\right]$ is a smaller interval.

f is continuous in $\left[0, \frac{\pi}{6}\right]$

$$f(0) = \tan(2 \cdot 0) - \sin(0) - 1 = -1$$

$$f\left(\frac{\pi}{6}\right) = \tan\left(2 \cdot \frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right) - 1 = \sqrt{3} - \frac{1}{2} - 1 = \sqrt{3} - \frac{3}{2} > 0$$

$$\boxed{\begin{array}{l} \sqrt{3} > \frac{3}{2} \\ \text{square} \\ 3 > \frac{9}{4} \end{array}}$$

$$f(0) < 0 < f\left(\frac{\pi}{6}\right)$$

So, by the I.V.T. there must be an x in $\left(0, \frac{\pi}{6}\right)$ such that $f(x) = 0$.

③ Compute $\lim\limits_{x \to \infty} e^{x} \sin(e^{-x})$.

$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

$$= \lim_{x \to \infty} \frac{\sin(e^{-x})}{e^{-x}}$$

$t = e^{-x}$ | $x \to \infty$, $t = e^{-x} \to 0$

$$= \lim_{t \to 0} \frac{\sin(t)}{t} = 1. \quad //$$