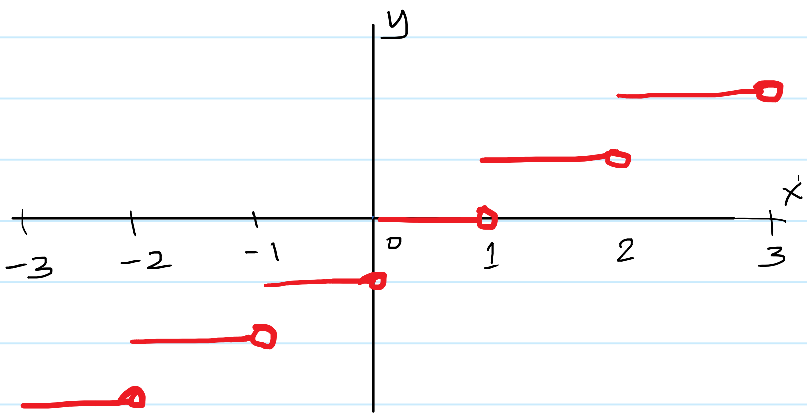
9.28.2015

Sections 2.1 and 2.4 are not on your syllabus.

The floor function: $f(x) = \lfloor x \rfloor$ OR $[\![x]\!]$

Greatest integer less than or equal to x.



For positive numbers, "drop the decimals"!

In general, $[\![x]\!]$ is the first integer you meet if you walk left from x.

2.5. Continuity

Theorem: Suppose $\lim_{x\to a} g(x) = b$ and $f(x)$ is continuous at b. Then,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b).$$

Example: Compute $\lim_{x\to 1} \sin\left(\frac{x^2-2x+1}{x-2}\right)$.

Solution $\lim_{x\to 1} \frac{x^2-2x+1}{x-2} = \frac{1-2+1}{1-2} = 0.$

$\sin(x)$ is continuous at 0.

So,

$$\lim_{x\to 1} \sin\left(\frac{x^2-2x+1}{x-2}\right) = \sin(0) = 0.$$

Example: (where this rule cannot be applied)

Compute $\lim_{x\to \frac{1}{\sqrt{2}}} \tan\left(\sin^{-1}(x) + \frac{\pi}{4}\right)$.

Example: (where this rule cannot be applied)

Compute $\lim_{x \to \frac{1}{\sqrt{2}}} \tan\left(\sin^{-1}(x) + \frac{\pi}{4}\right)$.

Solution: $\lim_{x \to \frac{1}{\sqrt{2}}} \sin^{-1}(x) + \frac{\pi}{4}$

$\sin^{-1}$ is cont. in $[-1,1]$ so

$$\lim_{x \to \frac{1}{\sqrt{2}}} \sin^{-1}(x) + \frac{\pi}{4} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

tan is not continuous at $\frac{\pi}{2}$!

We will compute this in section 2.6.!

Example: The rule also applies to one-sided limits

Compute $\lim_{t \to 1^-} \tan\left(\sqrt{1-t^2}\right)$.

Solution $\lim_{t \to 1^-} \sqrt{1-t^2} = 0$ and tan is cont. at 0.

$$\lim_{t \to 1^-} \tan\left(\sqrt{1-t^2}\right) = \tan(0) = 0.$$

Theorem: If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Example: Where is the following function continuous

$$f(x) = \frac{\ln(1-x)}{\cos^{-1}(x) - \frac{\pi}{2}}.$$

Solution: Numerator: $1-x$ and $\ln(x)$ are continuous in their domains. $\ln(x)$ is defined for $x > 0$.

So, $\ln(1-x)$ is defined when $1-x > 0$. $\to x < 1$

So, the numerator is continuous in $(-\infty, 1)$.

Denominator: $\cos^{-1}$ is continuous in $[-1,1]$.

The denominator " " " $[-1,1]$

So, both the numerator & the denominator are continuous in $[-1,1)$.

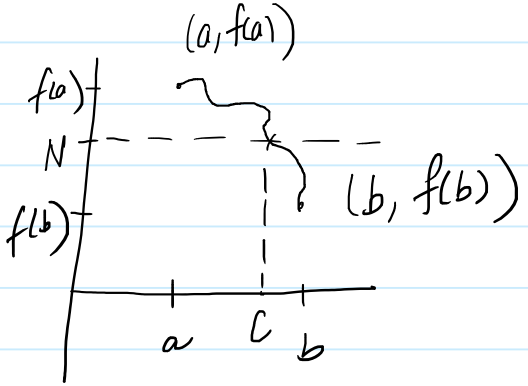
But, $\cos^{-1}(x) - \frac{\pi}{2}$ is zero when $x = 0$.

So, $f(x)$ is continuous for all x in $[-1,1)$ except for $x = 0$.

Intermediate Value Theorem

Suppose f is continuous in $[a,b]$ and N is a number between $f(a)$ and $f(b)$. Then, there is a number c in (a,b) so that

$$f(c) = N.$$



Example: Show that there is a solution to the equation

$$x^5 - 4x - 2 = 0$$

in the interval $(-1, 0)$.

Solution: $f(x) = x^5 - 4x - 2$

We want: there is some c in $(-1,0)$ so that

$$f(c) = 0.$$

f is continuous in $[-1, 0]$

$$f(-1) = (-1)^5 - 4(-1) - 2 = -1 + 4 - 2 = 1$$

$$f(0) = 0^5 - 4 \cdot 0 - 2 = -2.$$

$$-2 < \overset{N}{0} < 1$$

So, by the I.V.T., there is a c in $(-1, 0)$ so that

$$f(c) = 0.$$

Alternate phrasing:

Example: Show that there is a solution to the equation

$$x^5 - 4x = 2$$

in the interval $(-1, 0)$.

2.6. Limits at infinity

NOTE: Ignore precise definitions.

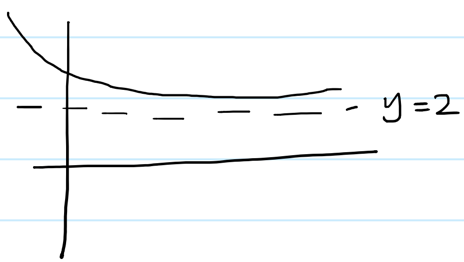
We study what happens to the values of $f(x)$ as x becomes very large (in positive or negative values)

Definition: Suppose f is defined on (a, ∞), $a>0$.

$$\lim_{x \to \infty} f(x) = L$$

means that $f(x)$ can be made arbitrarily close to L by choosing x sufficiently large.

Ex. $f(x) = 10^{-x} + 2$ $\qquad \lim_{x \to \infty} f(x) = 2.$



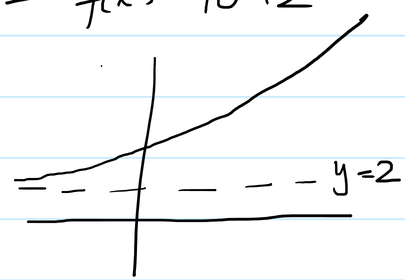
x	$f(x)$
10	$\frac{1}{10^{10}} + 2$
100	$\frac{1}{10^{100}} + 2$
1000	$\frac{1}{10^{1000}} + 2$
$\downarrow$	$\downarrow$
∞	2

Definition: f is defined on $(-\infty, a)$, $a<0$.

$$\lim_{x \to -\infty} f(x) = L$$

means that $f(x)$ can be made arbitrarily close to L by choosing x sufficiently negative.

Ex. $f(x) = 10^{x} + 2$ $\qquad \lim_{x \to -\infty} 10^{x} + 2 = 2.$

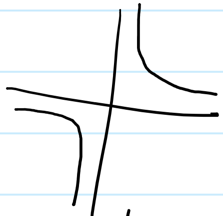


Definition: The line $y = L$ is a horizontal asymptote for the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

Ex. Both $y = 10^{-x}+2$ and $y = 10^{x}+2$ have a horizontal asymptote $y=2$.

FACTS:



$$\lim_{x\to\infty}\frac{1}{x^r}=0$$

$r>0$ rational number.

$$\lim_{x\to-\infty}\frac{1}{x^r}=0$$

[$r>0$ rational number and x^r makes sense for negative x.]

Example: Compute $\lim_{x\to\infty}\dfrac{x^3-2x+1}{3x^3+3}$.

$$\frac{x^3-2x+1}{3x^3+3} = \frac{\dfrac{x^3-2x+1}{x^3}}{\dfrac{3x^3+3}{x^3}} = \frac{1-\underbrace{\dfrac{2}{x^2}}_{\to 0}+\underbrace{\dfrac{1}{x^3}}_{\to 0}}{3+\underbrace{3/x^3}_{\to 0}}$$

Divide out by the 'dominant' term in the denominator.

highest degree

$$\lim_{x\to\infty}\frac{x^3-2x+1}{3x^3+3} = \frac{1}{3}.$$

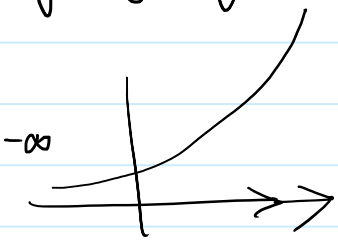
Ex. Compute $\lim_{x\to\infty}\dfrac{x^{1/3}+x^{1/5}}{x^{1/2}+x^{1/4}}$.

$$= \lim_{x\to\infty}\frac{\dfrac{x^{1/3}+x^{1/5}}{x^{1/2}}}{\dfrac{x^{1/2}+x^{1/4}}{x^{1/2}}} = \lim_{x\to\infty}\frac{\underbrace{\dfrac{1}{x^{1/6}}}_{\to 0}+\underbrace{\dfrac{1}{x^{3/10}}}_{\to 0}}{1+\underbrace{\dfrac{1}{x^{1/4}}}_{\to 0}} = 0.$$

Definition: $\lim_{x\to\infty} f(x)=\infty$ means that $f(x)$ can be made arbitrarily large by choosing x sufficiently large.

Ex. What do the foll. mean.

$\lim_{x\to\infty} f(x) = -\infty$



$\lim_{x\to-\infty} f(x)=\infty$

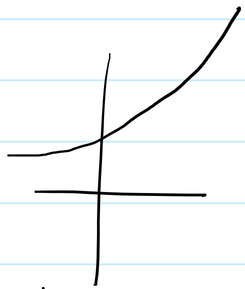
$\lim_{x\to-\infty} f(x) = -\infty.$

Important limits

$\lim_{x\to\infty} a^x = \infty$

$\lim_{x\to-\infty} a^x = 0$

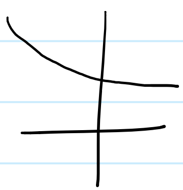
$\Big\}\ a>1$



$\lim_{x\to\infty} a^x = 0$

$\lim_{x\to-\infty} a^x = \infty$

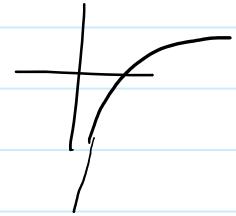
$\Big\}\ 0<a<1$



$\lim_{x\to\infty} \log_a x = \infty$

$\lim_{x\to 0} \log_a x = -\infty$

$\Big\}\ a>1$



$\lim_{x\to\infty} \log_a x = -\infty$

$\lim_{x\to 0} \log_a x = \infty$

$\Big\}\ 0<a<1$

Ex. Compute $\lim_{x\to\infty} \dfrac{e^x}{e^x + e^{-x}}$ $\quad \infty \quad 0 \quad \infty$

$$\lim_{x\to\infty} \frac{\frac{e^x}{e^x}}{\frac{e^x+e^{-x}}{e^x}} = \lim_{x\to\infty} \frac{1}{1+e^{-2x}}$$

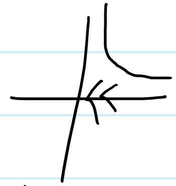
$$= 1.$$

Ex. Try $\dfrac{e^x}{e^{2x}} = \dfrac{1}{e^x} \to 0$

Substitution / change of variables trick.

Ex. Compute $\lim\limits_{x\to 0^+} \dfrac{1}{1+e^{1/x}}$

$t = \frac{1}{x}$ $\quad \boxed{\begin{array}{l} x \to 0^+ \\ t = \frac{1}{x} \to \infty \end{array}}$

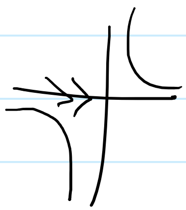


$$\lim_{x\to 0^+} \frac{1}{1+e^{1/x}} = \lim_{t\to\infty} \frac{1}{1+e^t}$$

$$= 0. \qquad \left[\lim_{t\to\infty} e^t = \infty\right]$$

Compute $\lim\limits_{x\to 0^-} \dfrac{1}{1+e^{1/x}}$

$t = \frac{1}{x}$ $\quad \boxed{\begin{array}{l} x \to 0^- \\ t = \frac{1}{x} \to -\infty \end{array}}$



$$\lim_{x\to 0^-} \frac{1}{1+e^{1/x}} = \lim_{t\to -\infty} \frac{1}{1+e^t} \qquad \left[\lim_{t\to-\infty} e^t = 0\right]$$

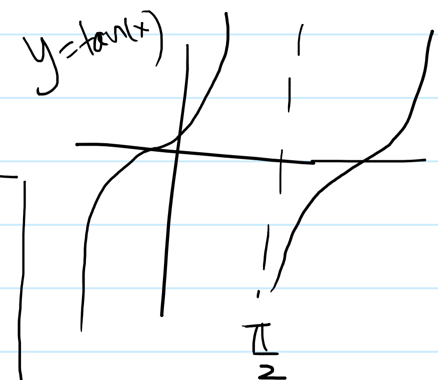
$$= 1.$$

From. 2.5 we had: compute $\lim\limits_{x\to \frac{1}{\sqrt{2}}} \tan\left(\sin^{-1}(x) + \frac{\pi}{4}\right)$.

$t = \sin^{-1}(x) + \frac{\pi}{4}$ $\quad \boxed{\begin{array}{l} x \to \frac{1}{\sqrt{2}} \\ t \to \frac{\pi}{2} \end{array}}$ $\qquad t = \sin^{-1}(x) + \frac{\pi}{4}$

$$\lim_{x\to\frac{1}{\sqrt{2}}} \tan\left(\sin^{-1}(x) + \frac{\pi}{4}\right) = \lim_{t\to\frac{\pi}{2}} \tan(t)$$

DNE.

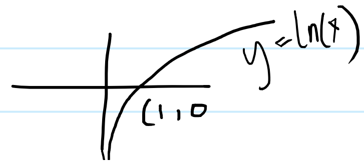


Compute: $\lim\limits_{x\to 0^+} \ln(\sin(x))$

$t = \sin(x)$ $\qquad x \to 0^+$

$\qquad\qquad t = \sin(x) \to 0^+$

$$\lim_{t\to 0^+} \ln(t) = -\infty.$$



$= 1.$