9.23.2015

WEB ASSIGNMENT 1 IS OPEN!

QUIZ ON MONDAY: Appendix D, 1.4, 1.5, 2.2, 2.3.

FOR EXAMS: ANYTHING IN A BOX IN THE TEXTBOOK!

ON INFINITE LIMITS:

Does $\lim_{x\to 0} \frac{1}{x^2}$ exist? NO

[Both left and right limits are ∞, which is not a real number]

Compute $\lim_{x\to 0} \frac{1}{x^2}$ ∞

Here $\lim_{x\to 0} \frac{1}{x^2} = \infty$ has more information than $\lim_{x\to 0} \frac{1}{x^2} = \text{DNE}$ so the former is better!

$\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$.

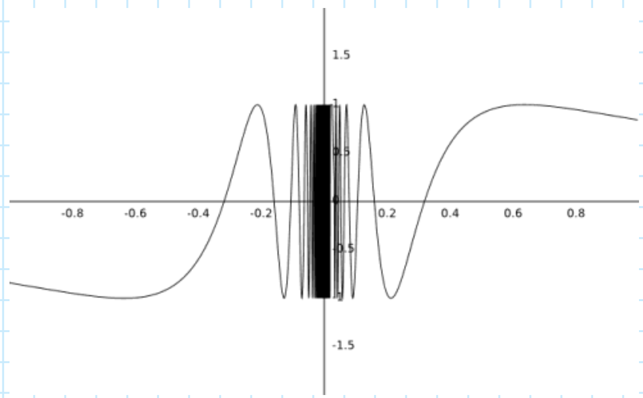
x	$f(x)$
$\frac{1}{\pi}$	0
$\frac{1}{2\pi}$	0
$\frac{1}{3\pi}$	0
$\frac{1}{4\pi}$	0

$\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi} \to 0$

$f(x) = \sin\left(\frac{1}{x}\right)$.

x	$f(x)$
$\frac{2}{\pi}$	1
$\frac{2}{5\pi}$	1
$\frac{2}{9\pi}$	1
$\frac{2}{13\pi}$	1.

$\frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi} \to 0$



$\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ DNE.

2.3. Limit Laws

Suppose $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$

L, M are real numbers.

① $\lim\limits_{x\to a} (f(x)+g(x)) = L+M$

② $\lim\limits_{x\to a} (f(x)-g(x)) = L-M$

③ $\lim\limits_{x\to a} (f(x)g(x)) = LM$

④ $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$

⑤ $\lim\limits_{x\to a} cf(x) = cL$ c = constant

$\left[\lim\limits_{x\to a} c = c\right]$

⑥ $\lim\limits_{x\to a} (f(x))^n = L^n$

⑦ $\lim\limits_{x\to a} x = a$ apply ⑥ to $f(x)=x$ $\lim\limits_{x\to a} x^n = a^n$.

Ex. $\lim\limits_{x\to a} (f(x)+g(x)) = 3$

$\lim\limits_{x\to a} g(x) = -2$

Compute $\lim\limits_{x\to a} 2 \cdot \dfrac{(f(x))^2}{g(x)}$.

Solution

$f(x) = (f(x)+g(x)) - g(x)$

rule ② $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} (f(x)+g(x)) - \lim\limits_{x\to a} g(x)$

$= 3-(-2) = 5$

rule ③ $\lim\limits_{x\to a} f(x)^2 = 5.5 = 25$

rule ⑤ and ④ $\lim\limits_{x\to a} 2\dfrac{f(x)^2}{g(x)} = 2\lim\limits_{x\to a} \dfrac{f(x)^2}{g(x)}$

$= 2 \cdot \dfrac{25}{-2} = -25$ //

⑧ $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ (if n is even, we assume $a > 0$)

⑨ $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$

[if n is even, then assume $L > 0$]

Principle: If $f(x)$ is a polynomial or a rational function and a is in the domain of $f(x)$, then $\lim_{x \to a} f(x) = f(a)$.

① $\lim_{x \to 2} \frac{x^2 - 2x}{x+2}$ well-defined at 2

$= \frac{0}{4} = 0.$

② $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ 1 is not in the domain of $\frac{x^2-1}{x-1}$

$\frac{x^2-1}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = x+1 \quad (x \neq 1)$

Principle If $f(x) = g(x)$ for all $x \neq a$ then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ (Assuming the RHS exists)

$f(x) = \frac{x^2-1}{x-1}$ Cannot use substitution 1 is not in the domain

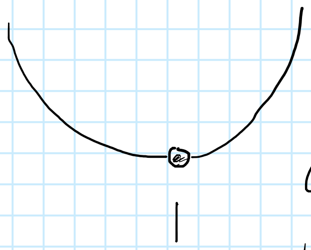
$g(x) = x+1$

$f(x) = g(x) \quad x \neq 1$

$\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = 1+1 = 2.$

$\lim_{x \to 2} 4x^2 - 2 = 4(2)^2 - 2$ [Polynomial]

$= 14.$



$f(x)$ $\quad \lim_{x\to 1} f(x) = \cancel{f(1)}$

$g(x) =$

$g(1)$ is well-defined

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① $\displaystyle \lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$ $\quad$ (substitution gives $\frac{0}{0}$)

$\displaystyle = \lim_{h\to 0} \frac{\cancel{8} + 18h^2 + 12h + h^3 - \cancel{8}}{h}$

$\displaystyle = \lim_{h\to 0} \frac{\cancel{h}(18h + 12 + h^2)}{\cancel{h}}$

$\displaystyle = \lim_{h\to 0} 18h + 12 + h^2$

$= 12 \,//.$

② $\displaystyle \lim_{t\to 0} \frac{\sqrt{1+t^2} - 1}{t^2}$ $\quad$ (Sub: $\frac{0}{0}$)

(Rationalize

$\displaystyle = \lim_{t\to 0} \frac{\sqrt{1+t^2} - 1}{t^2} \cdot \frac{\sqrt{1+t^2} + 1}{\sqrt{1+t^2} + 1}$

$\displaystyle = \lim_{t\to 0} \frac{(1+t^2) - 1}{t^2\left(\sqrt{1+t^2} + 1\right)}$ $\quad$ $[(a-b)(a+b)$ $a^2 - b^2]$

$\displaystyle = \lim_{t\to 0} \frac{\cancel{t^2}}{\cancel{t^2}\left(\sqrt{1+t^2} + 1\right)}$ $\quad$ $(t \neq 0)$

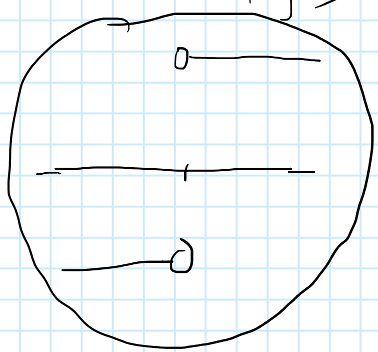
$\displaystyle = \lim_{t\to 0} \frac{1}{\sqrt{1+t^2} + 1} = \frac{1}{2} \,//$

Ex.

(1) $f(x) = \dfrac{|x|}{x}$, $\lim_{x \to 0} f(x)$.

$$\frac{|x|}{x} = \begin{cases} \dfrac{x}{x} = 1 & x > 0 \\ \dfrac{-x}{x} = -1 & x < 0 \end{cases}$$

Physics



$\lim_{x \to 0^+} f(x) = 1$

$\lim_{x \to 0^-} f(x) = -1$

$\lim_{x \to 0} f(x)$ DNE.

(2) $f(x) = \dfrac{\sqrt{(x-1)^2}}{(x-1)}$ $\lim_{x \to 1} f(x)$.

$= \dfrac{|x-1|}{(x-1)}$

$$f(x) = \begin{cases} \dfrac{x-1}{x-1} = 1 \, ; & x > 1 \\ \dfrac{-(x-1)}{x-1} = -1 \, ; & x < 1 \end{cases}$$

$\left.\begin{array}{l} \lim_{x \to 1^+} f(x) = 1 \\ \lim_{x \to 1^-} f(x) = -1 \end{array}\right\}$ — $\lim_{x \to 1} f(x)$ DNE.

Ex. $\sqrt{(-1)^2} = |-1|$ $\sqrt{(1)^2} = |1|$

$= \sqrt{1}$ $= \sqrt{1}$

$= 1$ $= 1$

$\sqrt{x^2} = |x|$.

Theorem: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

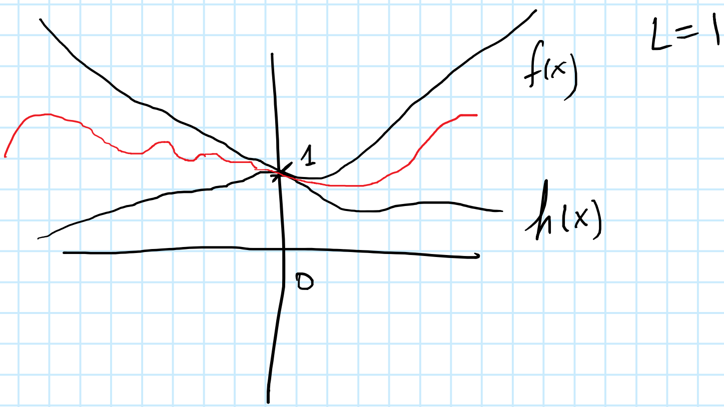
Suppose $f(x) \le g(x)$ for all x near a

then $L \le M$.

Theorem (SQUEEZE / SANDWICH) $f(x) \le g(x) \le h(x)$

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then, $\lim_{x \to a} g(x) = L$.



Ex. Compute $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

$$\cancel{\underbrace{\lim_{x \to 0} x}_{0} \cdot \underbrace{\lim_{x \to 0} \sin\left(\frac{1}{x}\right)}_{DNE}}$$

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

multiply all sides by x

$$\underset{f}{-x} \le \underset{g}{x \sin\left(\frac{1}{x}\right)} \le \underset{h}{x}$$

$\left.\begin{array}{l} \lim_{x \to 0} x = 0 \\ \lim_{x \to 0} -x = 0 \end{array}\right\}$ SQUEEZE THEOREM

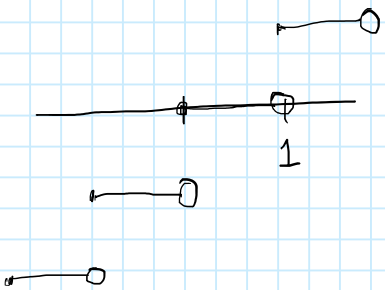
$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

//

$$\lim_{x \to 0} x^2 \lfloor x \rfloor$$

$\lfloor x \rfloor$ floor function

" greatest integer less than or equal to x.



$\lfloor 1/2 \rfloor = 0$

$\lim_{x \to 0} \lfloor x \rfloor$ DNE

$-1 \le \lfloor x \rfloor \le 1 \longrightarrow -1 \le x \le 1$

$$-x^2 \le x^2 \lfloor x \rfloor \le x^2$$

$f \quad g \quad h$

$$\lim_{x \to 0} x^2 = 0, \quad \lim_{x \to 0} -x^2 = 0$$

$$\lim_{x \to 0} x^2 \lfloor x \rfloor = 0.$$

$$\lim_{x \to 0} \sin(x) = 0$$

$$\lim_{x \to 0} \cos(x) = 1$$

$$\lim_{x \to 0} \tan(x) = 0$$

FACT.

$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

substitution trick,

$$\lim_{x \to 0} \frac{\sin(2x)}{x}$$

$t = 2x$

$$\lim_{2x \to 0} 2 \frac{\sin(2x)}{2x}$$

$$= 2 \underbrace{\lim_{2x \to 0} \frac{\sin(2x)}{2x}}_{} \sim \lim_{t \to 0} \frac{\sin(t)}{t}$$

$$= 2 \cdot 1 = 2 //$$

2.5 CONTINUITY

DNE : Does not exist

Defn : $f(x)$ is said to be continuous at a

$$\lim_{x \to a} f(x) = f(a).$$

Implications

① a is in the domain of f.

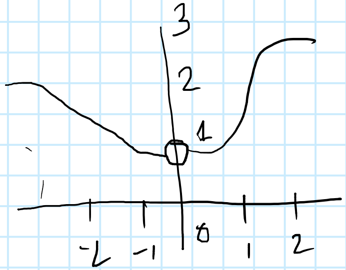
② $\lim_{x \to a} f(x)$ exists (as a real no.)

③ $\lim_{x \to a} f(x) = f(a)$.

If the above doesn't happen f is said to be discontinuous at a

All the fns below are discontinuous at $x = 0$

1.

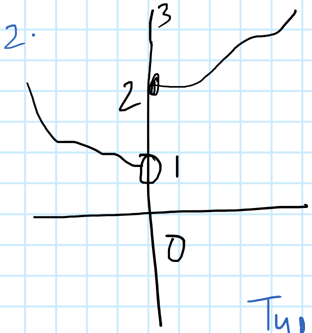


Reason: 0 is not in the domain of f.

Type of discontinuity: removable

define : $f(0) = 1$.

2.

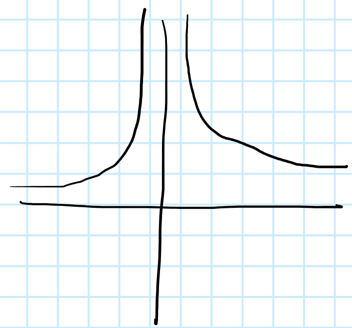


Reason: $f(0) = 2$

$\lim_{x \to 0} f(x)$ DNE

Type of discontinuity : jump

3. $f(x) = \frac{1}{x^2}$

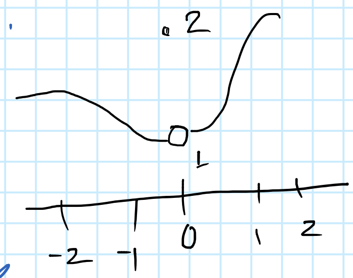


Reason : $f(0)$ is not defined

$\lim_{x \to 0} \frac{1}{x^2} = \infty$ (DNE as a real)

Reason : INFINITE

4.



Reason : $\lim_{x \to 0} f(x) = 1 \neq f(0)$

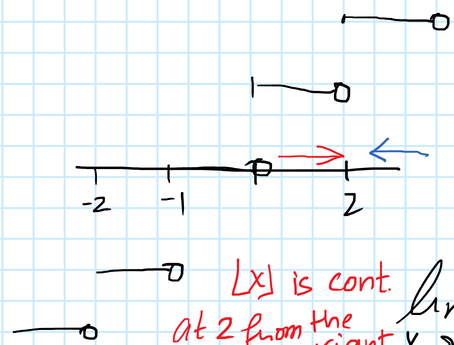
Reason: Removable because I can redefine $f(0) = 1$.

Defn: $f(x)$ is continuous at a from the left if

$$\lim_{x\to a^-} f(x) = f(a).$$

Ex: define continuity from the right.

$f(x)$



$$\lim_{x\to 2^-} f(x) = 1$$

$$\lim_{x\to 2^+} f(x) = 2$$

$$f(2) = \lfloor 2 \rfloor = 2.$$

$\lfloor x \rfloor$ is cont. at 2 from the right $\quad \lim_{x\to 2^+} f(x) = f(2)$

but discont. at 2 from the left. $\quad \lim_{x\to 2^-} f(x) \neq f(2)$

Defn ①: $f(x)$ is said to be continuous in (a,b) if $f(x)$ is continuous at every c in (a,b).

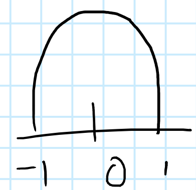
② $f(x)$ is said to be continuous in $[a,b]$ if

i) $f(x)$ is continuous in (a,b).

ii) $\lim_{x\to a^+} f(x) = f(a)$

(iii) $\lim_{x\to b^-} f(x) = f(b)$.

Example $\sqrt{1-x^2}$ is continuous in $[-1,1]$



If f and g are continuous at a, then so are:

① $f \pm g$ ② fg ③ f/g, $g(a) \neq 0$

④ cf, c is a constant

All the fns. below are continuous in their domains

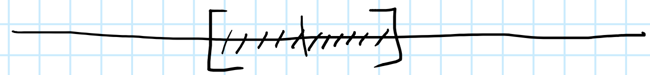
Function	Example	Domain
Polynomials	x^4+3x, c	$(-\infty,\infty)$
Rational	$\dfrac{x^2-1}{x-1}$	$(-\infty,\infty)$ but the zeros of the denominator
$\sin(x)$		$(-\infty,\infty)$
$\cos(x)$		$(-\infty,\infty)$
$\tan(x)$		$(-\infty,\infty)$ except for $x=\frac{\pi}{2}+n\pi$, n is an integer
$\sqrt[n]{x}$		n even $[0,\infty)$; n odd $(-\infty,\infty)$
a^x		$(-\infty,\infty)$
$\log_a x$		$(0,\infty)$
$\sin^{-1}(x)$		$[-1,1]$
$\cos^{-1}(x)$		$[-1,1]$
$\tan^{-1}(x)$		$(-\infty,\infty)$

Where is the following function continuous?

$$f(x)=\frac{\cos^{-1}(x)+x^2}{\ln(x)}$$

Num.: $\cos^{-1}(x)$ is continuous in $[-1,1]$

x^2 " " " $(-\infty,\infty)$

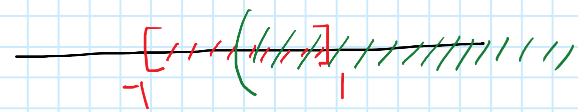


So, $\cos^{-1}(x)+x^2$ is continuous in $[-1,1]$

Den. $\ln(x)$ is continuous in $(0,\infty)$.

Num.

Den.

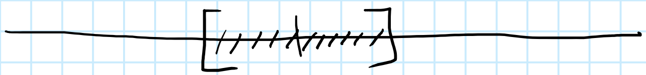


Where is the following function continuous?

$$f(x) = \frac{\cos^{-1}(x) + x^2}{\ln(x)}$$

Num.: $\cos^{-1}(x)$ is continuous in $[-1,1]$

x^2 " " " $(-\infty,\infty)$

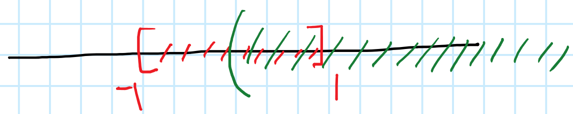


So, $\cos^{-1}(x) + x^2$ is continuous in $[-1,1]$

Den: $\ln(x)$ is continuous in $(0,\infty)$.

Num.

Den.



the overlap is $(0,1]$

also discard the zeros of the denominator!

But $\ln(1) = 0$

So, $f(x)$ is continuous in $(0,1)$.

Ex

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos^2(x)}{4+\sin^2(x)}$$

this fn. is continuous, so I can use substitution.

$\cos^2(x)$ is cont. in $(-\infty,\infty)$

$4+\sin^2(x)$ "

$4+\sin^2(x)$ is never 0

$$= \frac{\cos^2(\pi/2)}{4+\sin^2(\pi/2)} = 0.$$