9-21-2015 ANNOUNCEMENTS

Putnam Mathematics Competition

The William Lowell Putnam is a mathematics contest for undergraduates in Canada and the U.S. This year's contest will be written on Saturday, December 5th.

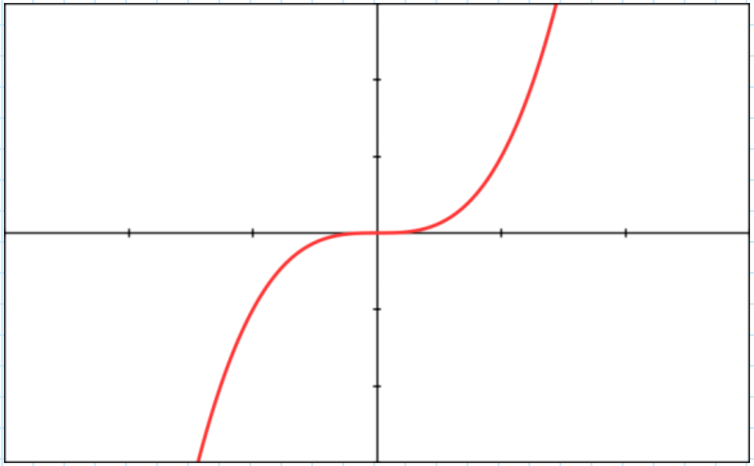
Preparation sessions for the Putnam Mathematics Competition are run by the Math department and are open to all students interested in mathematical problem solving.

There will be an organizing meeting for the Prep sessions on Friday, September 25th at 4:00PM in MC 108. Interested students who are unable to attend at this time should contact Janusz Adamus by email at jadamus@uwo.ca.

1. WeBWork 1 opens tomorrow at noon!
2. For every problem, you can 'Preview answers' & 'Submit answers'.
3. WW automatically grades your assignment.
4. Math Help Center is now open: Mon-Fri 2:30-6:30 pm MC 106
5. Lecture notes are available on section webpage.

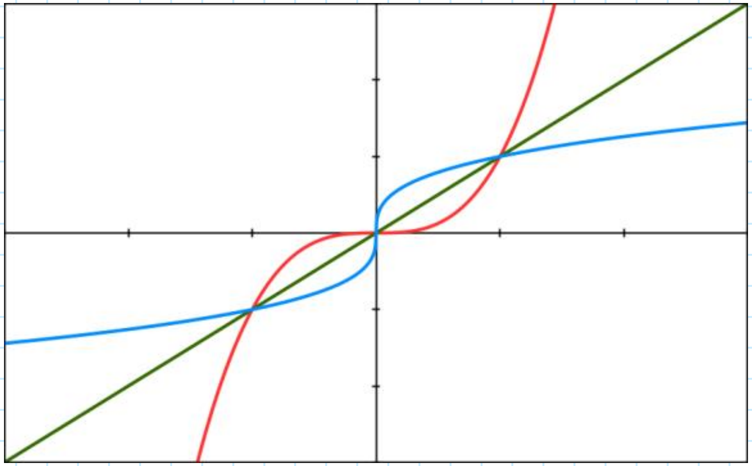
9.18.2015 1.5. Inverse functions (cont'd.)

$f(x) = x^3$



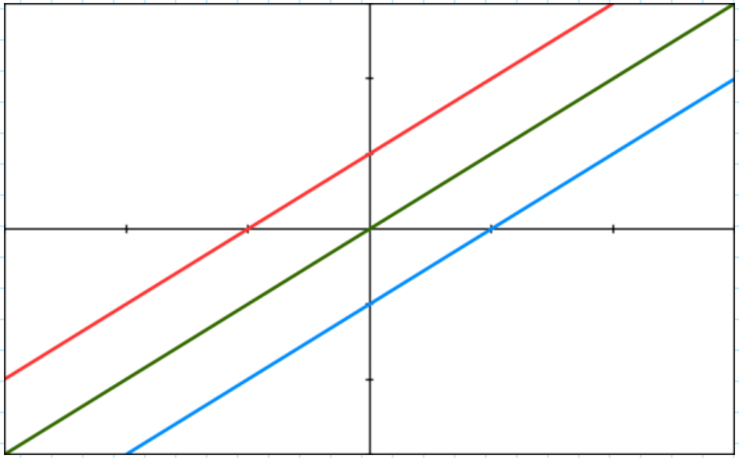
Reflect about the line $x = y$ (Green)

$f^{-1}(x) = x^{1/3}$



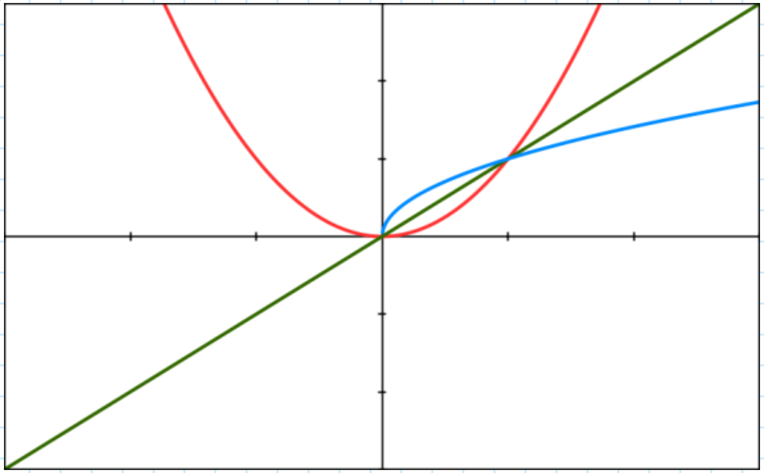
Other examples

$f(x) = x + 1$ $f^{-1}(x) = x - 1.$

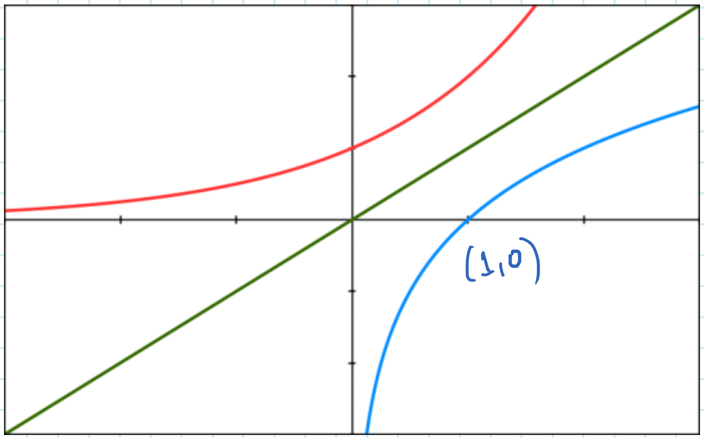


$f(x) = x^2$ (NOT ONE-TO-ONE!) (Restrict $x \geqslant 0$)

$f^{-1}(x) = \sqrt{x}$ $(x \geqslant 0)$



$f(x) = 2^x$ $\quad f^{-1}(x) = \log_2(x)$

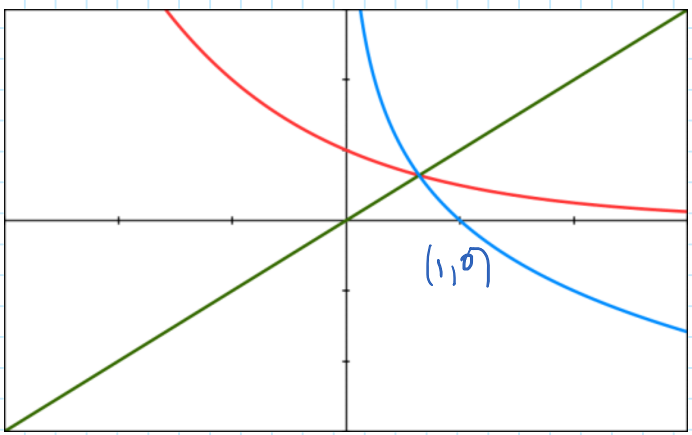


increasing

all real numbers.

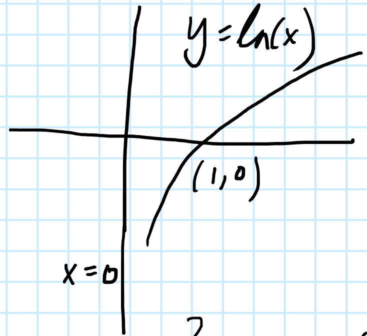
Domain of $\log_2(x)$: $(0,\infty)$, Range of $\log_2(x) = \mathbb{R}$

$f(x) = \left(\frac{1}{2}\right)^x$ $\quad f^{-1}(x) = \log_{1/2}(x)$



decreasing

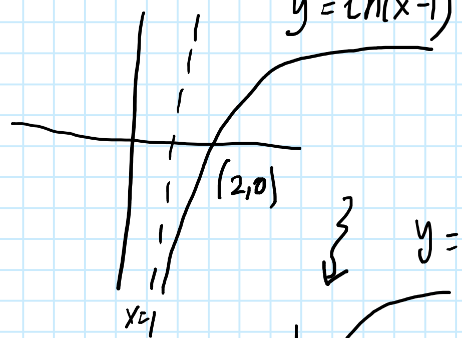
① Graph $\ln(x-1)+4$ (Domain & range).



dom: $(0,\infty)$

ran: $\mathbb{R}$.

Shift (right)

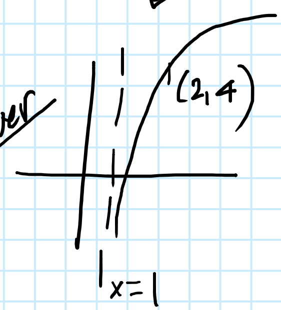


dom: $(1,\infty)$

ran: $\mathbb{R}$

$y = \ln(x-1)+4$

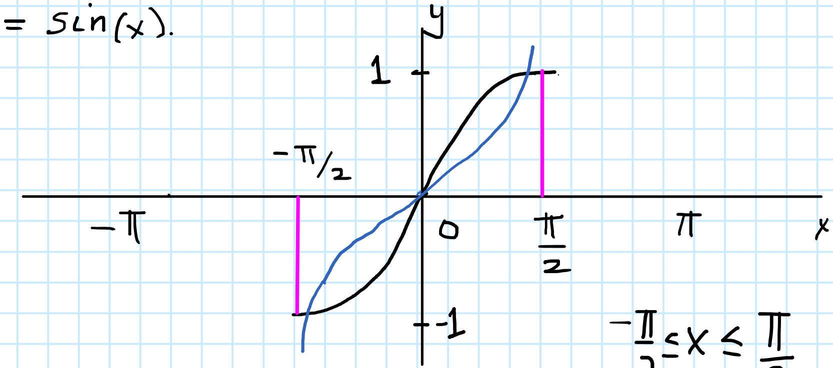
Answer



$\boxed{\text{dom. } (1,\infty) \quad \text{ran. } \mathbb{R}}$

1.5. Inverse Trigonometric Functions

$y = \sin(x)$.



$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\left.\begin{array}{c} \arcsin(x) \\ \text{OR} \\ \sin^{-1}(x) \end{array}\right\}$ is the inverse of the function

$f(x) = \sin(x)$ $\quad x$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Domain of $\arcsin(x) = [-1, 1]$

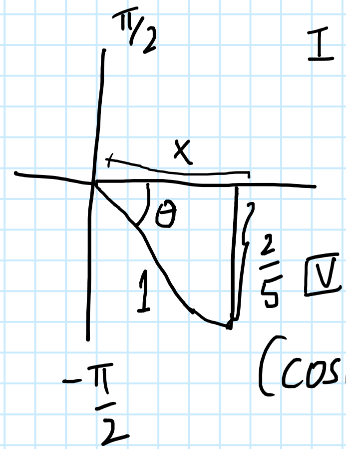
Range of $\arcsin(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Same thing for $\arccos(x) = \cos^{-1}(x)$.

Example: Evaluate $\tan(\arcsin(-2/5))$

$\theta = \arcsin(-2/5)$

WANT TO FIND: $\tan(\theta)$.



$(\cos\theta, -2/5)$

$\sin(\theta) = -2/5$.

$x^2 + \left(\frac{2}{5}\right)^2 = 1$

$x^2 = 1 - \frac{4}{25}$

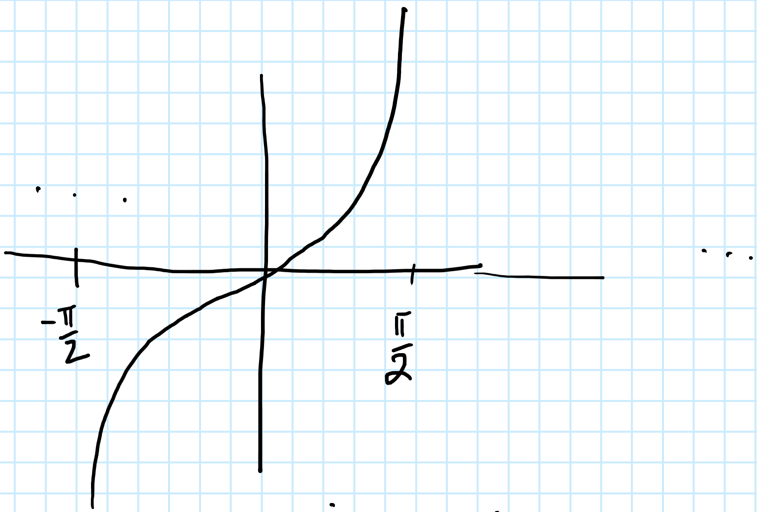
$= \frac{21}{25}$

$x = \frac{\sqrt{21}}{5}$

$\cos(\theta) = \frac{\sqrt{21}}{5}$.

$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-2/5}{\sqrt{21}/5} = \frac{-2}{\sqrt{21}}$ //.

$y = \tan(x)$



$\arctan(x)$ is the inverse of $f(x) = \tan(x)$ $\quad$ x is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\parallel$

$\tan^{-1}(x)$.

domain of $\arctan(x)$: range of $f(x) = \mathbb{R}$

range of $\arctan(x)$: domain of $f(x)$: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Cancellation laws

1. $\sin(\sin^{-1}(x)) = x \qquad -1 \le x \le 1$
2. $\sin^{-1}(\sin(x)) = x \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$
3. $\cos(\cos^{-1}(x)) = x \qquad -1 \le x \le 1$
4. $\cos^{-1}(\cos(x)) = x \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$
5. $\tan(\tan^{-1}(x)) = x \qquad$ all real numbers
6. $\tan^{-1}(\tan(x)) = x \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Cancellation laws

(1) $\sin(\sin^{-1}(x)) = x \qquad -1 \le x \le 1$

(2) $\sin^{-1}(\sin(x)) = x \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(3) $\cos(\cos^{-1}(x)) = x \qquad -1 \le x \le 1$

(4) $\cos^{-1}(\cos(x)) = x \qquad 0 \le x \le \pi$

(5) $\tan(\tan^{-1}(x)) = x \qquad$ all real numbers

(6) $\tan^{-1}(\tan(x)) = x \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Example (i) $\sin^{-1}\left(\sin\left(\frac{6\pi}{13}\right)\right)$

FIRST CHECK $\qquad -\frac{\pi}{2} \le \frac{6\pi}{13} \le \frac{\pi}{2}$ ✓

Using (2)

$$\sin^{-1}\left(\sin\left(\frac{6\pi}{13}\right)\right) = \frac{6\pi}{13}$$

(2) $\sin^{-1}\left(\sin\left(\frac{19\pi}{13}\right)\right) \overset{\text{Ans.}}{=} -\frac{6\pi}{13}$

$\frac{19\pi}{13} > \frac{\pi}{2}$

$\sin(2\pi - x) = \sin(x)$

$2\pi - x = \frac{19\pi}{13}$

$*\quad \sin(\pi + x) = -\sin(x)$

$x = \frac{6\pi}{13}$

$$\sin\left(\pi + \frac{6\pi}{13}\right) = -\sin\left(\frac{6\pi}{13}\right)$$

$$\sin\left(\frac{19\pi}{13}\right) = -\sin\left(\frac{6\pi}{13}\right)$$

$[-\sin(x) = \sin(-x)]$

$$= \sin\left(-\frac{6\pi}{13}\right)$$

$$\sin^{-1}\left(\sin\left(\frac{19\pi}{13}\right)\right) = \cancel{\sin^{-1}}\left(\cancel{\sin}\left(-\frac{6\pi}{13}\right)\right) = -\frac{6\pi}{13}$$

$-\frac{\pi}{2} \le -\frac{6\pi}{13} \le \frac{\pi}{2}$

Ex: Simplify $\cos(\underbrace{\tan^{-1}(x)}_{\theta})$ | $\cos(\theta)$

Set $\theta = \tan^{-1}(x)$ — ①

take tan on both sides — ①

$$\tan(\theta) = \tan(\tan^{-1}(x))$$ Using law ⑤

$$= x$$

$$\tan^2\theta + 1 = \sec^2\theta \quad \text{[trig. identity]}$$

$$x^2 + 1 = \frac{1}{\cos^2\theta}$$

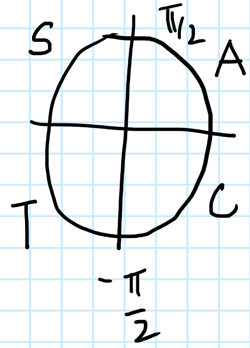
$$\cos^2\theta = \frac{1}{x^2+1}$$

$$\cos(\theta) = \pm\sqrt{\frac{1}{x^2+1}}$$

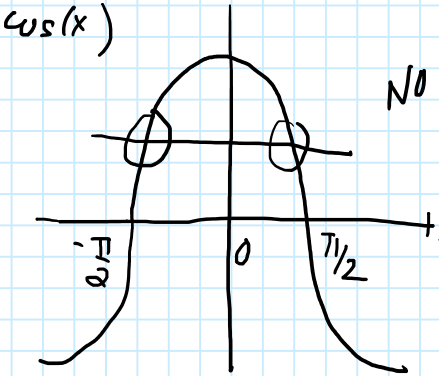
$$\cos(\theta) = \sqrt{\frac{1}{x^2+1}}$$

θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

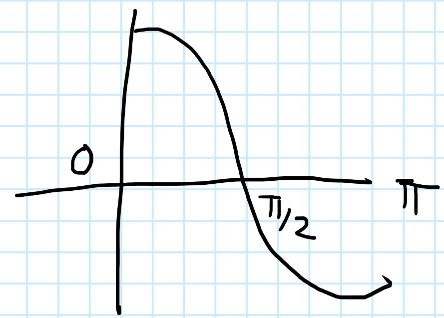
$\cos(\theta)$ is always positive if θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$y = \cos(x)$



NOT ONE-TO-ONE!



$f(x) = \cos(x)$

$0 \le x \le \pi$

arccos(x) is the inverse of f(x)

domain of arccos(x): $[-1, 1]$

range of arccos(x): $[0, \pi]$

Ex: $\cos\left(2\sin^{-1}(5/13)\right)$

Set $\theta = \sin^{-1}(5/13)$ —— 1

WANT TO FIND: $\cos(2\theta)$

Eqn 1,

$$\sin(\theta) = \sin\left(\sin^{-1}(5/13)\right)$$

$$= \frac{5}{13} \qquad -1 \le \frac{5}{13} \le 1$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$ [DOUBLE ANGLE FORMULA]

$$= 1 - \sin^2\theta - \sin^2\theta$$ [$\sin^2\theta + \cos^2\theta = 1$]

$$= 1 - 2\sin^2\theta$$

$$= 1 - 2\left(\frac{5}{13}\right)^2 = 1 - \frac{50}{169}$$

$$= \frac{119}{169} \;//.$$

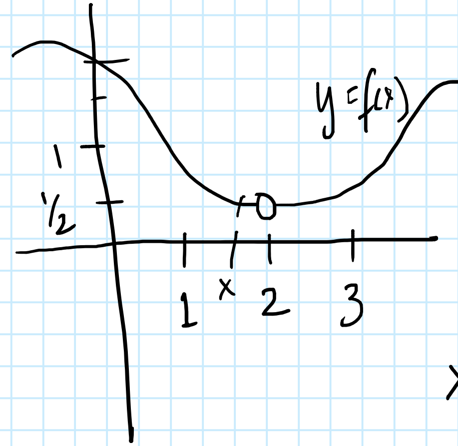
2.2. Limits of functions

Example $f(x) = x^3$. What happens to the values of $f(x)$ as x comes very close to 0?

x	$f(x)$
$1/10$	10^{-3}
$-1/10$	-10^{-3}
$1/100$	10^{-6}
$-\frac{1}{1000}$	-10^{-9}

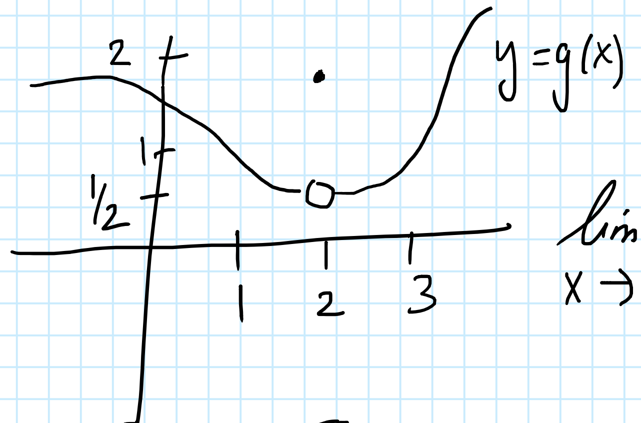
$\lim_{x\to 0} f(x) = 0$.

DEFINITION: Suppose $f(x)$ is defined for all numbers x 'close to' a (possibly not at $x=a$). If we can make $f(x)$ as close to L as we want by choosing x close to a, then we say that $\lim_{x\to a} f(x) = L$.



$f(2)$ is not defined

$\lim_{x\to 2} f(x) = 1/2$



$g(2) = 2$

$\lim_{x\to 2} f(x) = 1/2$

ONE-SIDED LIMITS

Def^n: Suppose $f(x)$ is defined for x 'close to' a and $x<a$. If $f(x)$ can be made as close to L as we want by choosing x near a and $x<a$,

$$\lim_{\substack{x\to a^- \\ (x<a)}} f(x) = L.$$

limit from the left

Left-sided limit.

$\lim_{x \to a} f(x) = L.$

$x \to a$
$(x < a)$

Left-sided limit.

$$\lim_{x \to a^-} f(x)$$

Left-sided limit

$\lim_{x \to a^+} f(x)$: right-sided limit

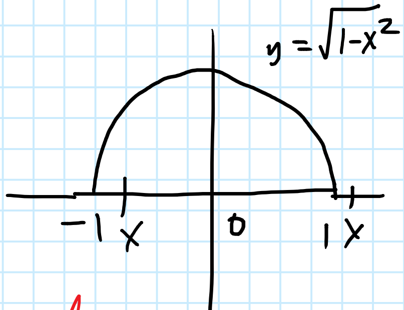
you approach a from the right $(x > a)$.

Example $f(x) = \sqrt{1-x^2}$ $\quad -1 \le x \le 1$

$\lim_{x \to 1^-} \sqrt{1-x^2} = 0$

$\lim_{x \to 1^+} \sqrt{1-x^2}$ = not defined

$\lim_{x \to -1^+} \sqrt{1-x^2} = 0$



$\lim_{x \to 1} \sqrt{1-x^2}$ DNE

$\lim_{x \to -1} \sqrt{1-x^2}$ DNE

$\lim_{x \to -1^-} \sqrt{1-x^2}$ is not defined.

FACT: If $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$,

then $\lim_{x \to a} f(x) = L$

But if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

then we say that $\lim_{x \to a} f(x)$ DNE (Does not exist).

① $\lim_{x \to 2} \frac{x^2-2x}{x+2}$ $\xrightarrow[x=2]{\text{Substitute}}$ $\frac{2^2-2 \cdot 2}{2+2} = \frac{0}{4} = 0$

② $\lim_{x \to 1} \frac{x^2-1}{x-1}$ $\xrightarrow[x=1]{\text{sub.}}$ $\frac{0}{0}$ $\quad \lim_{x \to 1} \left(\frac{x^2-1}{x-1}\right) = \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$

$x = 1$

$1 + 1 = 2$

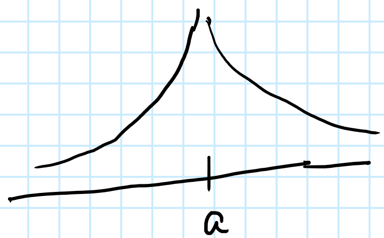
③ $\lim_{x \to 3} \frac{1}{x-3}$.

③ $\lim\limits_{x \to 3} \dfrac{1}{x-3} \overset{\text{sub.}}{\underset{x=3}{\rightsquigarrow}} \dfrac{1}{0}$

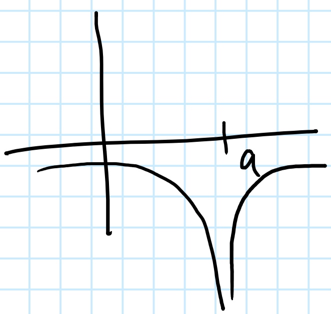
Answer

$\lim\limits_{x \to 3} \dfrac{1}{x-3}$ DNE.

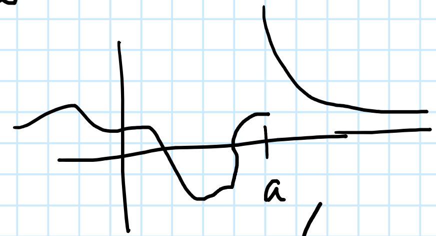
$\lim\limits_{x \to a} f(x) = \infty$ if you can make $f(x)$ as large as you want by choosing x very close to a.



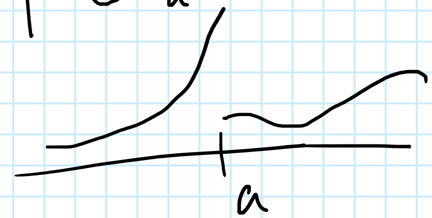
$\lim\limits_{x \to a} f(x) = -\infty$ if $f(x)$ becomes very large in negative value as x approaches a.



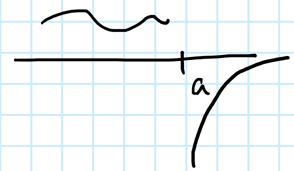
$\lim\limits_{x \to a^+} f(x) = \infty$



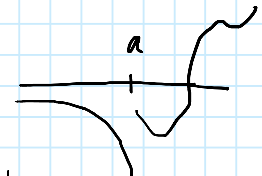
$\lim\limits_{x \to a^-} f(x) = \infty$



$\lim\limits_{x \to a^+} f(x) = -\infty$



$\lim\limits_{x \to a^-} f(x) = -\infty.$



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③ $\lim\limits_{x \to 3^+} \dfrac{1}{x-3}$ $(x > 3)$ $= +\infty.$

Denominator: positive + small

so, $\dfrac{1}{x-3}$ is positive + large

$\lim\limits_{x \to 3^-} \dfrac{1}{x-3}$ $(x<3)$ $= -\infty.$

Denominator negative, very close to 0

$1/x-3$: negative, large magnitude.

④ $\lim_{x\to 3} \frac{1}{|x-3|}$

$$\frac{1}{|x-3|} = \begin{cases} \frac{1}{x-3} & x>3 \\ \frac{1}{3-x} & x<3 \end{cases}$$

$$\lim_{x\to 3^+} f(x) = \infty$$

$$\lim_{x\to 3^-} f(x) = \infty$$

$$\lim_{x\to 3} f(x) = \infty$$

VERTICAL ASYMPTOTES

The curve $y = f(x)$ is said to have a vertical asymptote at $x=a$ if any of the following 6 occur

① $\lim_{x\to a^+} f(x) = \infty$

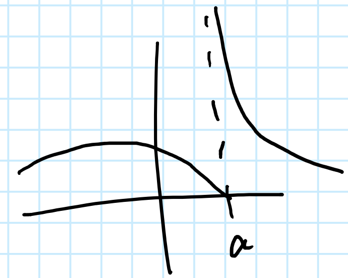
$\lim_{x\to a^-} f(x) = \infty$

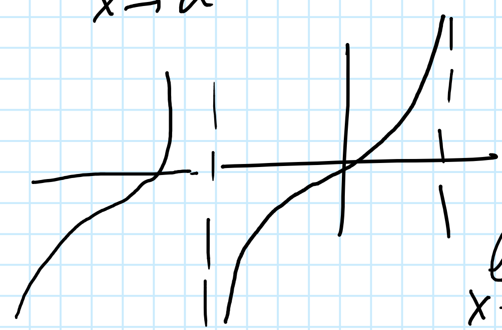
$\lim_{x\to a} f(x) = \infty$

$\lim_{x\to a^-} f(x) = -\infty$

$\lim_{x\to a^+} f(x) = -\infty$

$\lim_{x\to a} f(x) = -\infty.$





$y = \tan(x)$

vert. asymptotes at $x = \frac{\pi}{2} + n\pi$

$\lim_{x\to \frac{\pi}{2}} \tan(x)$ DNE.