9/16/2015

1.4 Exponential Functions

Functions of the form

$$f(x) = a^x \,, \quad a \text{ is a fixed positive constant}$$

Example: ① $f(x) = 2^x$

(NOT THE SAME AS x^2)

② $f(x) = \left(\frac{1}{3}\right)^x$

③ $f(x) = (1)^x = 1$

GRAPHS OF EXPONENTIAL FNS.

$f(x) = 2^x$

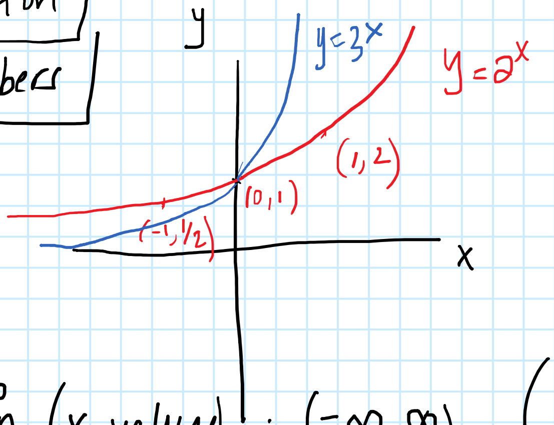
x	0	1	-1
$y=2^x$	1	2	$1/2$

$f(x) = (1/3)^x$

x	0	1	-1
$(1/3)^x$	1	$1/3$	3

fn. function

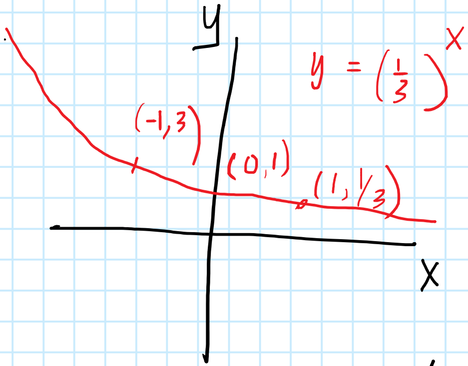
nos. numbers



domain (x-values) : $(-\infty, \infty)$ (all real nos.)

range (y-values) : $(0, \infty)$

(2^x is never 0)



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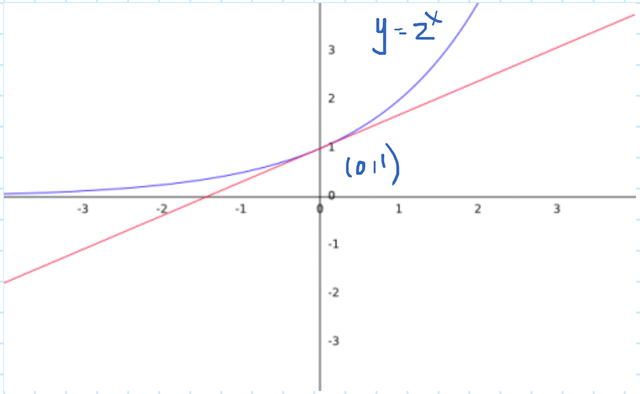
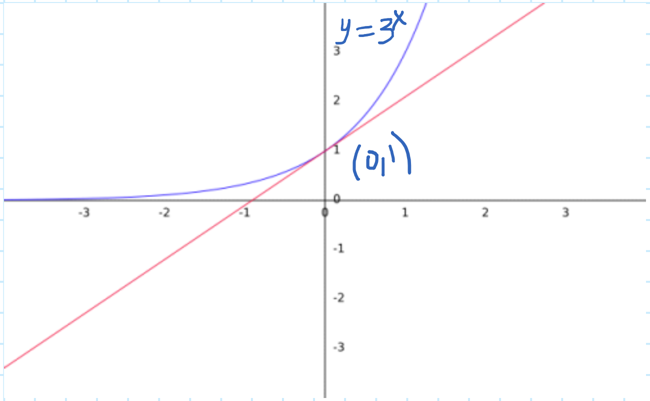


slope of the tangent ≈ 0.7



slope of the tangent ≈ 1.1.

For applications, it is useful to have the slope of the tangent at (0,1) to be exactly 1.

This happens for a special value of a call e

$$2 < e < 3$$

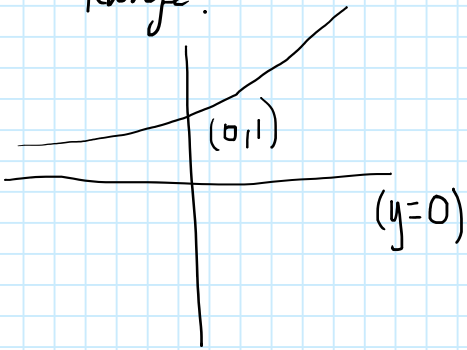
$y=e^x$: tangent to its graph at (0,1) has slope 1.

domain: $(-\infty, \infty)$
range: $(0, \infty)$

Ex. Draw the graph of $\frac{1}{2}e^{-x}-1$

Domain & Range?

$f(x) = e^{x}$

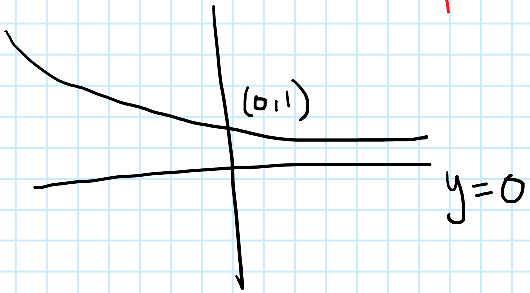


dom: $(-\infty,\infty)$

range: $(0,\infty)$

step 1 $f(x) = e^{-x}$

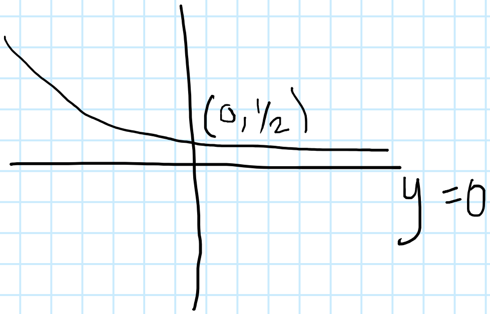
(flip abt the y-axis)



dom: $(-\infty,\infty)$

range: $(0,\infty)$

$\rightsquigarrow f(x) = \frac{1}{2}e^{-x}$ (contracting vertically by 1/2)

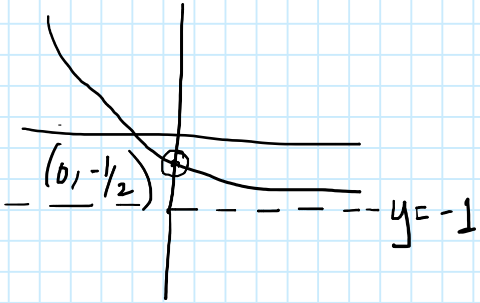


dom: $(-\infty,\infty)$

range: $(0,\infty)$

$f(x) = \frac{1}{2}e^{-x}-1$ (shifts it down by 1 unit)

Answer



dom: $(-\infty,\infty)$

range: $(-1,\infty)$

Rules of exponents

① $a^x \cdot a^y = a^{x+y}$

② $\frac{a^x}{a^y} = a^{x-y}$

③ $(a^x)^y = a^{x \cdot y}$

④ $a^x b^x = (ab)^x$

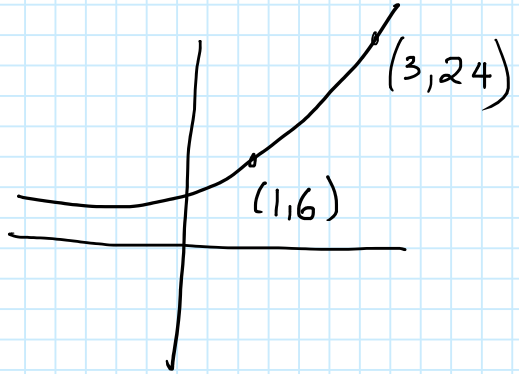
Simplify:

$$\frac{3y^r \cdot y^{3r-1}}{\sqrt[3]{y}} \overset{①}{=} \frac{3y^{r+3r-1}}{y^{1/3}}$$

$$= \frac{3y^{4r-1}}{y^{1/3}} \overset{②}{=} 3y^{4r-1-\frac{1}{3}}$$

$$= \boxed{3y^{4r-\frac{4}{3}}}$$

Ex. $y = Ca^x$



determine the function (what are C & a?)

$6 = Ca^1$ —— ①

$24 = Ca^3$ —— ②

$\frac{②}{①}$ $\frac{24}{6} = \frac{\cancel{C}a^3}{\cancel{C}a^1}$ (Know that $C \neq 0$)

$4 = a^2$

$a = \pm 2$ ($a > 0$ for exp. fns)

$a = 2$, $C = 3$ $\boxed{y = 3 \cdot 2^x}$

Ex: A population of bacteria that doubles every 3 hours. Suppose there were 100 bacteria in the beginning, what is the population after 15 hours?

$$\left(y = Ca^t\right)$$

t	0	3	15
y	100	200	2_0

$$100 = C \cdot a^0 \quad — ①$$

$$200 = C \cdot a^3 \quad — ②$$

① — $\boxed{C = 100}$

② — $200 = 100a^3$

$$2 = a^3$$

$$\boxed{2^{1/3} = a}$$

$$y = 100\left(2^{1/3}\right)^t$$

$$y = 100 \cdot 2^{t/3}.$$

$t = 15$: $y = 100 \cdot 2^{\frac{15}{3}}$

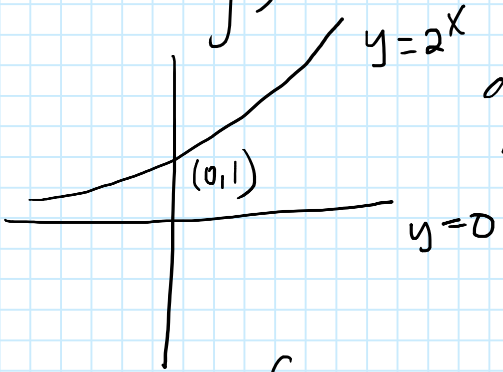
$$= 100 \cdot 2^5$$

$$= \boxed{3200 \text{ bacteria.}}$$

Ex. What is the fn. you get when you flip the graph of $y=2^x$ about the x-axis and shift to the left by 1 unit?

(domain & range)

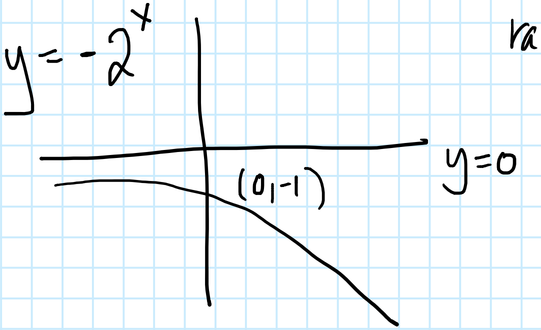
Answer



dom: $(-\infty,\infty)$

ran: $(0,\infty)$

flip abt. x-axis

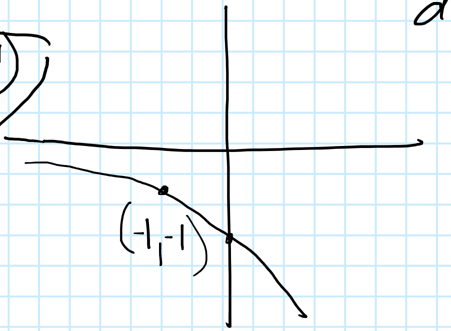


dom: $(-\infty,\infty)$

ran: $(-\infty,0)$

Shift to the left

A. $y=-2^{(x+1)}$



domain: $(-\infty,\infty)$

range: $(-\infty,0)$

1.5 : Inverse functions

Population of bacteria

time t	0	1	2	3	4
pop^n $P(t)$	100	169	224	291	332

} one-to-one!

Can also ask: If you know the population, Can you tell me how many hours it took for the population to get there?

224 → 2 hours.

↑ input ↓ output

pop^n x	100	169	224	291	332
time $g(x)$	0	1	2	3	4

g is the inverse p.

Population of bacteria + contamination

time t	0	1	2	3	4
pop^n $r(t)$	100	95	98	100	169

NOT one-to-one

Same question.

100 — 0

3

You cannot define an inverse

Problem: $r(t)$ takes the same value twice!

Definition: A function $f(x)$ is called one-to-one if it never takes the same value twice.

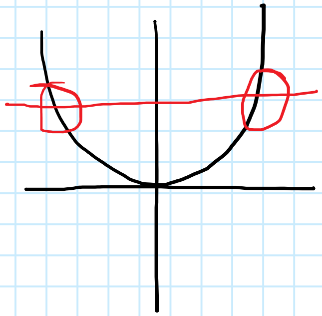
If $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Graphically, every horizontal intersects the graph at at most one point.

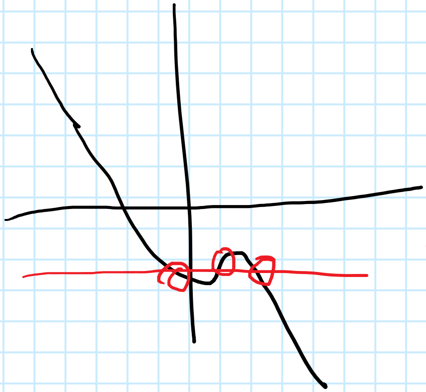
HORIZONTAL LINE TEST.

Examples ① $f(x) = x^2$

$f(1) = 1$
$f(-1) = 1$ } NOT one-to-one



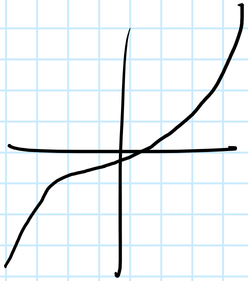
②



NOT one-to-one

③ $f(x) = 1 - \sin(x)$. $f(0) = 1$
$f(2\pi) = 1$ } NOT one-to-one

$f(x) = x^3$



$x_1 \neq x_2$

$$\boxed{f(x_1) = f(x_2)}$$

$$x_1^3 = x_2^3$$

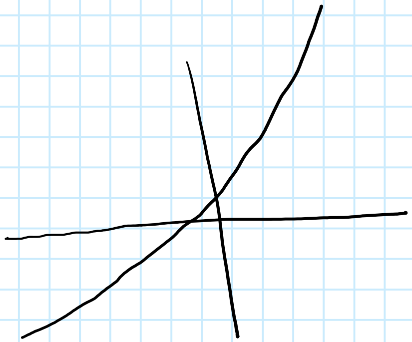
$$\boxed{x_1 = x_2}$$

For a one-to-one function $f(x)$, we can define an inverse $f^{-1}(x)$ (NOT THE SAME AS $\frac{1}{f(x)}$)

Domain of f = Range of f^{-1}

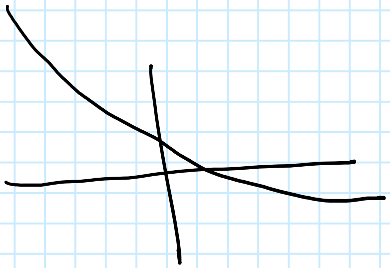
Range of f = Domain of f^{-1}.

Increasing function: $x_1 < x_2$ then $f(x_1) < f(x_2)$



DECREASING FN. : $x_1 < x_2$

then $f(x_1) > f(x_2)$.



one-to-one.

FINDING THE INVERSE

① $f(x) = x^5$

$y = x^5$ (solve for x)

new input

$$y^{1/5} = x$$

$$\boxed{f^{-1}(x) = x^{1/5}}$$

② $y = 1 + \sqrt{2+3x}$

(solve for x)

$$y - 1 = \sqrt{2+3x}$$

$$(y-1)^2 = 2 + 3x$$

$$(y-1)^2 - 2 = 3x$$

$$\frac{(y-1)^2 - 2}{3} = x$$

$$\boxed{y = \frac{(x-1)^2 - 2}{3}}$$

③ $y = x^2 - x$ $(x \geq 1/2)$

$[1/2, \infty)$

(solve for x)

$0 = x^2 - x - y$

$$x = \frac{1 \pm \sqrt{1+4y}}{2} = \frac{1}{2} \pm \frac{\sqrt{1+4y}}{2}$$

≥ 0

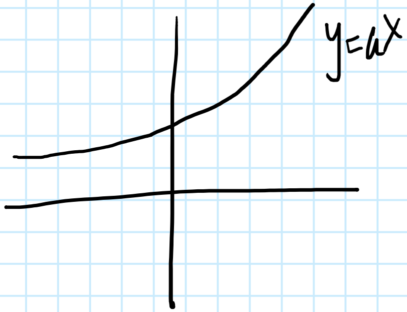
Domain of f^{-1} = range of f

Range has to be $[1/2, \infty)$

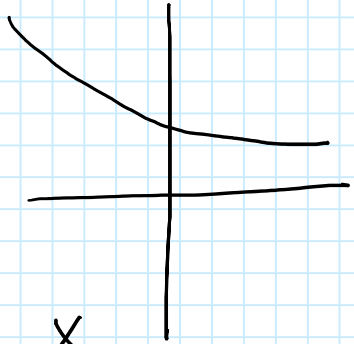
$$x = \frac{1}{2} + \frac{\sqrt{1+4y}}{2}$$

$$\boxed{y = \frac{1}{2} + \frac{\sqrt{1+4x}}{2}}$$

Logarithms



$a > 1$



$a < 1$

$f(x) = a^x$

$f^{-1}(x) = \log_a(x)$

[domain : $(0, \infty)$
range : $(-\infty, \infty)$ $\mathbb{R}$]

x	0	1	-1	2	-2
2^x	1	2	1/2	4	1/4

x	1/4	1/2	1	2	4
$\log_2(x)$	-2	-1	0	1	2

① $\log_4(1/2) = x$ $[4^x = 1/2]$

$$4^{-1/2} = \frac{1}{2}$$

$x = -\frac{1}{2}$ $\boxed{\log_4(1/2) = -\frac{1}{2}}$

② $\log_3(81) = x$ $[3^x = 81]$

$x = 4$

$\boxed{\log_3(81) = 4}$

③ $\log_8(4) = x$ $[8^x = 4]$

$$(2^3)^x = 2^{3x}$$

(But, $2^2 = 4$)

$3x = 2$

$x = 2/3$ (check: $8^{2/3} = 4$)

$\boxed{\log_8(4) = 2/3}$

$\log_a a = 1$ $(a^1 = a)$

$\log_a 1 = 0$ $(a^0 = 1)$

$\log_e(x)$ we use the symbol $\ln(x)$

NATURAL LOG. FUNCTION

Find the inverse of

$$y = \frac{3 - e^{-x}}{e^{-x}}$$

$$ye^{-x} = 3 - e^{-x}$$

$$ye^{-x} + e^{-x} = 3$$

$$e^{-x}(y+1) = 3$$

$$e^{-x} = \frac{3}{y+1}$$

$$\ln(e^{-x}) = \ln\left(\frac{3}{y+1}\right)$$

$$-x = \ln\left(\frac{3}{y+1}\right)$$

$$\log_a(a^x) = x \quad f^{-1}(f(x)) = x$$

$$a^{\log_a x} = x \quad f(f^{-1}(x)) = x$$

$$x = -\ln\left(\frac{3}{y+1}\right)$$

$$\boxed{y = -\ln\left(\frac{3}{x+1}\right)}$$

Rules for log.

① $\log_a(xy) = \log_a(x) + \log_a(y)$

② $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

③ $\log_a(x^p) = p\log_a(x)$.

④ $\log_a(x) = \frac{\ln(x)}{\ln(a)}$.

Write the following expression as a single log term

① $$\frac{1}{3}\ln\left((x-2)^3\right)+2\ln(x)$$

$\left(\log_a(x^p)=p\log_a(x)\right)$

$$=\ln\left(\left((x-2)^3\right)^{1/3}\right)+\ln(x^2)$$

$$=\ln(x-2)+\ln(x^2)$$

$$=\boxed{\ln\left((x-2)x^2\right)}$$

② **Prove:**

$$\log_{a^q}(x)=\frac{1}{q}\log_a(x).$$

L.H.S. $$\log_{a^q}(x)=\frac{\ln(x)}{\ln(a^q)}\qquad\left(\log_a(x)=\frac{\ln(x)}{\ln(a)}\right)$$

$$=\frac{\ln(x)}{q\cdot\ln(a)}.$$

$$=\frac{1}{q}\,\frac{\ln(x)}{\ln(a)}$$

$$=\frac{1}{q}\log_a(x)=\text{R.H.S.}$$

③ $\ln(p) - 5\ln(q) + \ln(p^5)$

$= \ln(p) - \ln(q^5) + \ln(p^5)$

$= \ln\left(\frac{p}{q^5}\right) + \ln(p^5)$

$= \ln\left(\frac{p}{q^5} \cdot p^5\right) = \ln\left(\frac{p^6}{q^5}\right)$

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④ Find the inverse of.

$$y = 2\ln(x) + \ln\left(\frac{x}{2}\right)$$

(solve for x)

$$y = 2\ln(x) + \ln(x) - \ln(2)$$

$$y + \ln(2) = 3\ln(x)$$

$$e^{\frac{y+\ln(2)}{3}} = e^{\ln(x)}$$

$[e^{\ln(x)} = x]$

$$e^{\frac{y+\ln(2)}{3}} = x$$

$$\boxed{y = e^{\frac{x+\ln(2)}{3}}}$$

To get the graph of f^{-1}: reflect the graph of f abt the line $x=y$.