Injective Hulls of a Ring with Compatible Ring Structures

Barbara L. Osofsky

THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS

June 18, 2008

Joint work with G. Birkenmeier, J. Park, S. T. Rizvi

Introductory comments

We begin with a ring R with identity. It has an injective hull $E = E(R_R)$ with a fixed embedding of R into E. Let $\Lambda = \mathrm{End}_R(E)$. We will be exploring possible ring structures on E which preserve the R-module structure. Specifically we will look at

- What general theorems are known about when E has such a compatible ring structure.
- Several examples illustrating the problems that can arise.
- A formal definition of a compatible ring structure
- Some elementary new results that illustrate why this problem is both nice in theory and so difficult to get satisfying example oriented results.

Known theorems concerning such a ring structure on E

Theorem

Assume the singular submodule of R is 0. Then there is a natural ring structure on E extending R-module multiplication making E a right self injective von Neumann regular ring.

Theorem

Assume $\hom_R(E/R, E) = 0$. *Then there is a natural ring structure on E extending R-module multiplication.*

Proof.

Let $\mu \in \Lambda$. Consider the abelian group map ϕ, $\mu \mapsto \mu(1)$, from Λ to E. Let $\mu(1) = \nu(1)$. Then $(\mu - \nu)(1) = 0$ so $\mu - \nu$ maps E/R to E. This implies $\mu = \nu$ so the map is 1 to 1. Let $m \in E$. Then the map $1 \mapsto m$ extends to a (unique) map m_λ in Λ so ϕ is onto. For m, $n \in E$, define $\cdot : E \times E \to E$, $m \cdot n = m_\lambda \circ n_\lambda(1)$ will give the ring required structure on E. □

Known theorems concerning such a ring structure on E

Theorem

Let A be a commutative artinian ring. Then $E(A)$ has a compatible ring structure $\iff$ A is quasiFrobenius.

Proof.

A is a product of local rings, so without loss of generality A is local. The socle of A is simple if and only if A is quasiFrobenius $\iff$ A is self injective.

So assume the socle of R has length $n > 1$. Then $E(R)_R$ is a direct sum of copies of $E(S)$ where S is the unique simple A-module. If E has a compatible ring structure, its 'top' must be $n \times n$ matrices over $\mathrm{End}_A(S)$ so by standard artinian ring theory $E_E = \oplus \sum_{i=1}^{n} e_i E$ for $\{e_i\}$ a complete set of orthogonal primitive idempotents, and its socle must have dimension n^2 over $\mathrm{End}_A(S)$ so it cannot be an essential extension of A.

$\rightarrow\leftarrow$ □

Examples

Example

Let $\mathbb{Z}_{(p)}$ denote the integers localized at a prime p, and $\mathbb{Z}_{p^\infty}$ the minimal injective cogenerator of $\mathbb{Z}_{(p)}$. Let R be the ring with additive group $\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{p^\infty}$ and $(\mathbb{Z}_{p^\infty})^2 = 0$. Then $E(R) = \hat{\mathbb{Z}}_{(p)} \oplus \mathbb{Z}_{p^\infty}$ has a compatible ring structure but is not a rational extension of R.

Crucial points

- $\mathbb{Z}_{p^\infty}$ is an injective cogenerator over $\mathbb{Z}_{(p)}$.
- $\hat{\mathbb{Z}}_{(p)}$ is its endomorphism ring.
- These facts tell you $E(R)$ is an injective cogenerator for R on the right.
- They do not tell you it is a ring.

Example

Dischinger and Müller have constructed a one-sided injective cogenerator as follows. Let K be a field such that there exists an endomorphism σ of K whose image is of finite codimension in K. Let T be the ring $T = K\,[[X\ ;\ \sigma]]$. the twisted power series ring with coefficients on the left and $X \cdot \kappa = \sigma(\kappa)\,X$ for all $\kappa \in K$. T is left Öre, so it has a field of left quotients $Q = \left\{X^{-i}\sum_{n=1}^{\infty}\kappa_n X^n : \kappa_n \in K,\ i \in \mathbb{N}\right\}$ which contains T as a subring (with $i = 0$). Since every left ideal of T is of the form TX^i, T is left hereditary so $Q\,/\,T\ X$ is injective and contains a left T-cogenerator U which can also be made into a right T-module. Set $R = T \oplus {}_T U_T$ where $U^2 = 0$.Then R is left self injective, and indeed a cogenerator on the left, but it is not right self injective. If one takes the classical left localization of $K\,[X\ ;\ \sigma]$ at the Öre monoid $\left\{X^i\right\}$, R is a ring structure on the injective hull of the resulting ring.

– Query: does this example have a compatible ring structure on the injective hull of R_R?

Example

Let $\mathbb{H}$ be the real quarternions, and $\mathbb{C}$ the complex numbers. Then the finite dimension algebra over $\mathbb{R}$,

$$R = \begin{bmatrix} \mathbb{C} & {}_{\mathbb{C}}\mathbb{H}_{\mathbb{H}} \\ 0 & \mathbb{H} \end{bmatrix}$$

is nonsingular on both sides so its injective hull on either side will be semisimple Artinian incorporating all endomorphisms of its socle. Its right socle $\begin{bmatrix} 0 & \mathbb{H} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{H} \end{bmatrix}$ is homogeneous of composition length 2 so $E(R_R) \cong \begin{bmatrix} \mathbb{H} & \mathbb{H} \\ \mathbb{H} & \mathbb{H} \end{bmatrix}$ and its left socle $\begin{bmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathbb{C}j \\ 0 & 0 \end{bmatrix}$ is homogeneous of composition length 3 so $E({}_R R) \cong \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} & \mathbb{C} \end{bmatrix}$.

Example

Let K be a field with an endomorphism σ of K whose image is of finite codimension in K. Set $F = \sigma(K)$ and let

$$\langle R\ ,\ +\rangle = F \oplus {}_{\sigma(F)}F_F$$

where for α, β, γ, $\delta \in F$, $\begin{bmatrix}\alpha & \beta\end{bmatrix}\begin{bmatrix}\gamma & \delta\end{bmatrix} = \begin{bmatrix}\alpha\gamma & \sigma(\alpha)\delta + \beta\gamma\end{bmatrix}$. Then R does not have a compatible ring structure on its injective hull on either side.

$$\langle R\ ,\ +\rangle = F \oplus {}_{\sigma(F)}F_F$$

Proof.

The right injective hull of R_R looks like $K \oplus F$, where R embeds in K as $\sigma(K) + F$. We cannot multiply $\begin{bmatrix} 0 & F \end{bmatrix} \cdot \begin{bmatrix} K = \sigma^{-1}(F) & 0 \end{bmatrix}$ preserving the R-module structure on $\begin{bmatrix} 0 & F \end{bmatrix}$.
The left injective hull $E({}_RR)$ of R looks like

$$E\left(\begin{bmatrix} 0 & {}_{\sigma(F)}\sigma(F) \end{bmatrix}\right) \oplus E\left(\begin{bmatrix} 0 & {}_{\sigma(F)}U \end{bmatrix}\right)$$

where U is a $\sigma(F)$-subspace of F complementary to $\sigma(F) \cdot 1$. Let $n > 1$ be the $\sigma(F)$-dimension of F over $\sigma(F)$. If $E({}_RR)$ has a ring structure, it is as an artinian ring $\mathcal{E}$ with $\mathcal{E}/J(\mathcal{E})$ $n \times n$ matrices over F, and ${}_{\mathcal{E}}J(\mathcal{E}) = \text{socle}({}_RR)$ which is has left dimension n over $\sigma(F)$. It must be the unique simple. Standard artinian ring theory shows this cannot happen. □

A very significant example

Example (BOPR)

Let A be a commutative local quasi-Frobenius ring with nonzero Jacobson radical J. Set

$$R = \begin{bmatrix} A & A/J \\ 0 & A/J \end{bmatrix}$$

Then $E(R) \cong A \oplus \begin{bmatrix} A/J & A/J \\ A/J & A/J \end{bmatrix}$ with the embedding $\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} \mapsto \alpha + \begin{bmatrix} \alpha + J & \beta \\ 0 & \delta \end{bmatrix}$ has a ring structure as a quasiFrobenius ring compatible with the R-module structure, and

$$\hom_R(E/R, E) \cong \hom_R\left(\begin{bmatrix} A/J & 0 \\ A/J & 0 \end{bmatrix}, A\right) \neq 0.$$

$$R = \begin{bmatrix} A & A/J \\ 0 & A/J \end{bmatrix} \text{ and } E(R) \cong A \oplus \begin{bmatrix} A/J & A/J \\ A/J & A/J \end{bmatrix}$$

- R is an Artin algebra. A is a finitely generated injective cogenerator over itself, so just as for finite dimensional algebras over a field, $\mathrm{Hom}_A(\_\,,\ A)$ induces a Morita duality showing that the injective hulls of simple R-modules are $A \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and the rows of the 2×2 matrix ring.
- $E(R)$ with the obvious multiplication is quasiFrobenius.
- E has more than one (set theoretic) ring structure as one can take, for example, the element $j \oplus \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ as the idempotent to be used in the $1,1$ position of the matrices for any $j \in \mathrm{socle}(A)$.
- All ring multiplications on $E(R)$ are isomorphic as rings.

Multiplicative structures

Definition (A)

A distributive multiplicative structure on an abelian group G is a group homomorphism λ from $G \otimes_{\mathbb{Z}} G$ to G.

Definition (B)

A distributive multiplicative structure on an abelian group G is a group homomorphism λ from G to the additive group of $\text{End}_{\mathbb{Z}}(G)$.

Definition (C)

A *ring structure* on an abelian group G with distinguished element $1 \neq 0$ is a group homomorphism λ from G to the additive group of $\mathrm{End}_{\mathbb{Z}}(G)$ such that the image of λ is closed under composition of functions, 1 maps to the identity of Λ, and for all $g \in G$, $\lambda(g)(1) = g$.

Notation

If $\lambda : G \to \mathrm{End}_Z(G)$ is a ring structure on the abelian group G, then for $g \in G$ we denote $\lambda(g)$ by g_λ and call the endomorphism g_λ left multiplication by g. For g and h in G, let $g \cdot h = g_\lambda(h)$.
Since the image of λ is closed under composition, there is a $k \in G$ such that $\lambda(k) = \lambda(g) \circ \lambda(h)$, namely $k = \lambda(g) \circ \lambda(h)(1)$. This structure is associative because composition of functions is associative, and 1 serves as a two-sided identity.

Example

Example

Let us look at the ring R of 2×2 matrices over a field. When one uses ordinary matrix multiplication, one thinks of this as a ring in the standard definition. The endomorphism ring of $\langle R, + \rangle$ is 4×4 matrices, and, using the ordered basis $\left\{ \begin{bmatrix} \varepsilon_{1,1} & \varepsilon_{1,2} & \varepsilon_{2,1} & \varepsilon_{2,2} \end{bmatrix} \right\}$, the left multiplications are given by

$$\lambda\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$$

The key concept

There are two ways to look at the ring structure, namely

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} au + bw & av + bx \\ cu + dw & cv + dx \end{bmatrix}$$

or

$$\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ x \end{bmatrix} = \begin{bmatrix} au + bw \\ av + bx \\ cu + dw \\ cv + dx \end{bmatrix}$$

Compatible ring structures on E

Definition

Let R be a ring with 1, $E = E(R_R)$ an injective hull of R_R for some fixed embedding $R \hookrightarrow E$. A compatible ring structure on E is (given by) an abelian group embedding $\lambda : E \to \Lambda = \mathrm{End}_R(E)$ closed under composition such that $(1_R)_\lambda =$ the identity of Λ.

- A priori Λ need not be an R-module. Indeed it 'usually' is not.
- The image of λ, E_λ, has an induced R-module structure.
- Since $(1_R)_\lambda = 1_\Lambda$ and for all $m, n \in E$ $(m_\lambda(n))_\lambda = m_\lambda \circ n_\lambda$, $(m_\lambda(1_R))_\lambda = m_\lambda$ so since λ is monic, $m = m_\lambda(1_R)$.
- Since Λ consists of R-maps, for all $m \in E$ and $r \in R$,

$$m_\lambda(r) = m_\lambda(1_R)\, r = mr$$

  That is, restricting the right factor to R in this multiplication on E gives the R-module multiplication on E.

The BOPR example

E has ordered basis

$$\left\{A \quad A/J_{1,1} \quad A/J_{1,2} \quad A/J_{2,1} \quad A/J_{2,2}\right\}$$

so

$$E_\lambda = \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 \\ 0 & 0 & a & 0 & b \\ 0 & c & 0 & d & 0 \\ 0 & 0 & c & 0 & d \end{bmatrix}$$

We now look at a new A-basis for E obtained by adding elements in the socle of A to the first column of the matrix ring while fixing a basis for R. That is, we take as a new basis the columns of

$$\mathbf{P}_{\mu,\nu} = \begin{bmatrix} 1 & \mu & 0 & \nu & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\mu & 0 & -\nu & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 \\ 0 & 0 & a & 0 & b \\ 0 & c & 0 & d & 0 \\ 0 & 0 & c & 0 & d \end{bmatrix} \begin{bmatrix} 1 & \mu & 0 & \nu & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s & -a\mu - c\nu + s\mu & 0 & -b\mu - d\nu + s\nu & 0 \\ 0 & a & 0 & b & 0 \\ 0 & 0 & a & 0 & b \\ 0 & c & 0 & d & 0 \\ 0 & 0 & c & 0 & d \end{bmatrix}$$

Our new theorem

Now assume we have a ring structure E_λ. Λ operates on the left of E and contains E_λ as a subring.

Theorem

Let E, R, and Λ be as before. Set

$$\mathfrak{I} = \{\mu \in \Lambda : \mu(1_R) = 0\}.$$

Then:

1. $\mathfrak{I}$ *is a left ideal of* Λ. *It is a right ideal if and only if* $\mathfrak{I} = 0$. *In particular,* $\Lambda/\mathfrak{I}$ *is a ring if and only if* $\mathfrak{I} = 0$.
2. *As left* E_λ*-modules,* $\langle \Lambda ; + \rangle = \langle \mathfrak{I} ; + \rangle \oplus \langle E_\lambda + \rangle$.
3. $E_\lambda / J(E_\lambda) \cong \Lambda / J(\Lambda)$.

Proof.

Re 1. Statement: $\mathfrak{I}$ is a left ideal of Λ. It is a right ideal if and only if $\mathfrak{I} = 0$. In particular, $\Lambda/\mathfrak{I}$ is a ring if and only if $\mathfrak{I} = 0$.

Proof: Let $\mu \in \Lambda$ and $\eta \in \mathfrak{I}$. Then $\mu\eta\iota(1) = 0$. If $\eta \neq 0$. Since E is essential over R_R, there exists a $m \in E$ such that $0 \neq \eta(m) \in R$. The map $1 \mapsto m$ from R to Λ extends to a map $\nu \in \Lambda$ such that $\eta\nu \notin \mathfrak{I}$.

Re 2. Statement: As left E_λ-modules, $\langle \Lambda\,; +\rangle = \langle \mathfrak{I}+\rangle \oplus \langle E_\lambda\,; +\rangle$.

Proof: For $\mu \in \Lambda$, set $\mu\iota(1) = m \in E$. Then $\mu - m_\lambda \in \mathfrak{I}$. Thus $\Lambda = E_\lambda + \mathfrak{I}$. If $m \neq 0$, $m_\lambda \notin \mathfrak{I}$, so the sum is direct.

Re 3. Statement: $E_\lambda / J(E_\lambda) \cong \Lambda / J(\Lambda)$.

$\mathfrak{I} \subseteq J(\Lambda)$ so $J(\Lambda) = \mathfrak{I} \oplus J(\Lambda) \cap E_\lambda$, and $J(\Lambda) \cap E_\lambda = J(E_\lambda)$.

□

Corollaries

Corollary (1)

$E_\lambda / J(E_\lambda)$ is a right self injective von Neumann regular ring, and idempotents lift modulo $J(E_\lambda)$.

Proof.

The ring properties are inherited from $\Lambda / J(\Lambda)$ since $\mathfrak{I} \subseteq J(\Lambda)$. The lifting of idempotents for both rings has exactly the same proof associating idempotents with closed submodules of R. □

Corollary (2)

Let $I_R \subseteq R_R$ have two set theoretically distinct injective hulls M_1 and M_2 in E_R. Then there is a compatible ring structure $E_{\lambda'}$ on E such that some element e of E idempotent in E_λ is not idempotent in $E_{\lambda'}$. Hence the ring structure is not set theoretically unique.

Proof.

Let $E = M_1 \oplus K$, Then $K \neq 0$. Let $e \in E_\lambda$ be the projection of E to M_1 with kernel K,so $K = (1 - e) E_\lambda$. Then $E = M_2 \oplus K$ also, and e maps M_2 isomorphically onto M_1, so M_2 is generated by an idempotent of the form $e + (1 - e) ke$ with $(1 - e) ke \neq 0$. Let ϕ be the R-module automorphism of E_R taking e to $e + (1 - e) ke$ and $1 - e$ to itself. Then the conjugation $m_\lambda \mapsto \phi^{-1} m_\lambda \phi$ which is a composition of elements of Λ and fixes $\iota(1)$ gives another compatible ring structure on E in which e is no longer idempotent. □

Corollary (3)

Let R be any ring such that $E = E(R_R)$ has a compatible ring structure. Then any two compatible ring structures on E are isomorphic.

Proof.

Let E_λ and $E_{\lambda'}$ be ring structures on the injective hull $E(R_R)$. By our theorem, E_λ and $E_{\lambda'}$ are both direct sum complements of $\mathfrak{I}$. Let π be the projection of Λ onto $E_{\lambda'}$ with kernel $\mathfrak{I}$. Then $\pi|_{E_\lambda}$ is an abelian group isomorphism from E_λ to $E_{\lambda'}$. Note that Λ is in general not an R-module, so $\pi(1_\Lambda)$ need not be 1_Λ, but it must be of the form $1_\Lambda + \alpha$ where α is an element of $\mathfrak{I}$ and therefore it must be a unit in Λ. For example, in BOPR, if E_λ corresponds to the original multiplication on the injective hull, all other multiplications will have $\pi(1_\lambda) = (1+\kappa)_{\lambda'}$ for some κ in the socle of $\mathcal{A}$, and $\kappa = 0 \iff \mu = -\nu$.

Set $\mathcal{S} = \left\{(\pi\lambda)^{-1} u_{\lambda'} (\pi\lambda) : u_{\lambda'} \in E_{\lambda'}\right\}$. Then $\mathcal{S} \subseteq \Lambda$ is closed under composition and addition, and $1_\Lambda = (\pi\lambda)^{-1} (1_R)_{\lambda'} (\pi\lambda)$; therefore $\mathcal{S}$ is a subring of Λ isomorphic to $E_{\lambda'}$.

Let $s = (\pi\lambda)^{-1} u_{\lambda'} (\pi\lambda) \in \mathcal{S}$. Note that λ maps E isomorphically to E_λ, π takes E_λ to $E_{\lambda'}$, $u_{\lambda'}$ takes $E_{\lambda'}$ to $E_{\lambda'}$ without sending any units of Λ to zero, π^{-1} then takes $E_{\lambda'}$ to E_λ. and then λ^{-1} maps E_λ back to E by evaluating at 1. Now

$$\begin{aligned} s(1) &= (\pi\lambda)^{-1} u_{\lambda'} (\pi(1_\Lambda)) = \lambda^{-1}\left(\pi^{-1}\left(u_{\lambda'}(\pi(1_\Lambda))\right)\right) \\ &= \lambda^{-1}(k_\lambda) = k \qquad \text{for some } k \in E \end{aligned}$$

and $s(1) = 0 \iff s = 0$. The map $\mathcal{S} \longrightarrow E_\lambda$, $s \mapsto (s(1))_\lambda$ is then a ring isomorphism between $\mathcal{S}$ and E_λ so both E_λ and $E_{\lambda'}$ are ring isomorphic to $\mathcal{S}$.

Corollary (4)

If E_λ is two-sided artinian, then E_λ is quasiFrobenius.

Proof.

Since E_R is injective and a finite sum of nonisomorphic indecomposable injective modules with simple socles, the socle of every principal indecomposable projective right E-module is simple. Since for any two primitive idempotents e and f, $eE \cong fE \iff \text{socle}\,(eE) \cong \text{socle}\,(fE)$, every simple right E-module appears in the socle of E. Hence the right annihilator of the socle of E, $r\,(\text{socle}\,(E_R)) = J$.
For any primitive idempotent e, let $sE \cong eE\,/eJ\,(E)$ for some $0 \neq s \in R$. Any E-map from sE to E is determined by its image on sR, so is given by multiplication by an element in E_λ, so $\Lambda s = Es$ is its own double annihilator, and $\sum_{i=1}^{j} \Lambda s_i = \sum_{i=1}^{j} Es_i$ also is its own double annihilator. Moreover, the right annihilator of s_i is a maximal right ideal, so the left annihilator of that right annihilator, namely Es_i, must be simple. In particular, the right socle is contained in the left socle.

Now for each isomorphism class of primitive idempotents $[e]$, there is a unique isomorphism class of primitive idempotents $[f]$ such that there is a nonzero element $fre \in \text{socle}(E_E)$. Then fre is in the left socle of E, so every simple left E-module appears in the left socle of E and its dual is simple because the socle of eE is simple.

What's left?

– Can a ring have a compatible multiplication on its injective hull on one side but not the other?

??? When does a ring have a compatible ring structure on its injective hull?

Very Early Papers

Bae40 R. Baer, *Abelian groups which are direct summands of every containing group*, Proc. Amer. Math. Soc. **46**, 800–806.

Joh51 R. E. Johnson, *The extended centralizer of a ring over a module*, Proc. Amer. Math. Soc. **2** (1951), 891–895.

Utu56 Yuzo Utumi, *On quotient rings*, Osaka Math. J. **8** (1956), 1–18.

FL58 G. D. Findlay and J. Lambek, *A generalized ring of quotients. I, II*, Canad. Math. Bull. **1** (1958), 77–85, 155–167.

WJ59 E. T. Wong and R. E. Johnson, *Self-injective rings*, Canad. Math. Bull. **2** (1959), 167–173.

Joh61 R. E. Johnson, *Quotient rings of rings with zero singular ideal*, Pacific J. Math. **11** (1961), 1385–1392.

JW61 R. E. Johnson and E. T. Wong, *Quasi-injective modules and irreducible rings*, J. London Math. Soc. **36** (1961), 260–268.

Utu63a Y. Utumi, *On rings of which any one-sided quotient rings are two-sided*, Proc. Amer. Math. Soc. **14** (1963), 141–147.

Utu63b Yuzo Utumi, *A note on rings of which any one-sided quotient rings are two-sided*, Proc. Japan Acad. **39** (1963), 287–288.

FU65 Carl Faith and Yuzo Utumi, *Maximal quotient rings*, Proc. Amer. Math. Soc. **16** (1965), 1084–1089.

Later Papers

Oso67 B. L. Osofsky, *A non-trivial ring with non-rational injective hull*, Canad. Math. Bull. **10** (1967), 275–282.

Oso68 B. L. Osofsky, *Endomorphism rings of quasi-injective modules*, Canad. J. Math. **20** (1968), 895–903.

Lan75 N. C. Lang, *On ring properties of injective hulls*, Canad. Math. Bull. **18** (1975), no. 2, 233–239.

DM86 Friedrich Dischinger and Wolfgang Müller, *Left PF is not right PF*, Comm. Algebra **14** (1986), no. 7, 1223–1227.

BOPR Gary F. Birkenmeier, Barbara L. Osofsky, Jae Keol Park, and S. Tariq Rizvi *Injective hulls with distinct ring structures*, To Appear.

Reference Books

Fai67 Carl Faith, *Lectures on injective modules and quotient rings*, Lecture Notes in Mathematics, No. 49, Springer-Verlag, Berlin, 1967.

AF92 Frank W. Anderson and Kent R. Fuller, *Rings and categories of modules*, second ed., Graduate Texts in Mathematics, vol. 13, Springer-Verlag, New York, 1992.

Lam99 T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999.

Thank You

Figure: Happy Birthday S K and many more