

# Remarks About the Relationship Between Relational Physics and a Large Kantian Component of the Laws of Nature

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## Abstract

Relational mechanics is a reformulation of mechanics (classical or quantum) for which space is relational. This means that the configuration of an  $N$ -particle system is a shape, which is what remains when the effects of rotations, translations and dilations are quotiented out. This reformulation of mechanics naturally leads to a relational notion of time as well, in which a history of the universe is just a curve in shape space without any reference to a special parametrization of the curve given by an absolute Newtonian time. When relational mechanics (classical or quantum) is regarded as fundamental, the usual descriptions in terms of absolute space and absolute time emerge merely as corresponding to the choice of a gauge. This gauge freedom forces us to recognize that what we have traditionally regarded as fundamental in physics might in fact be imposed by us through our choice of gauge. It thus imparts a somewhat Kantian aspect to physical theory.

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# 1 Relational Physics

We shall address here three basic questions concerning relational physics: What is it? What is its point? What can we learn from it? These issues have been addressed in a long paper [1] published in 2020 with our friend and colleague Detlef Dürr, who passed away in January 2021.

Relational physics has been revived in relatively recent times by Julian Barbour and Bruno Bertotti, who, in a very inspiring and influential paper published at the beginning of the eighties [2], transformed a long standing philosophical controversy about the nature of space and time into a well-defined physical problem. Our main focus here will be on gauge freedom and objectivity versus subjectivity, probability, and the non-normalizability of the quantum state.

## 1.1 Relational Space

The philosophical issue about the nature of space dates back to the dispute between Isaac Newton, who favored and argued for the need of an absolute theory of space and time, and Gottfried Wilhelm Leibniz, who insisted upon a relational approach,

also defended by Ernst Mach in the 19th century. To put it briefly, the gist of the relational approach is that overall position, orientation, and size are not relevant. This can be clarified by means of a very elementary and simplified model of the universe.

Suppose we are given the configuration of a universe of  $N$  particles. And suppose we translate every particle of the configuration in the same direction by the same amount. From a physical point of view it seems rather natural to take the relational perspective that the two configurations of the universe so obtained are physically equivalent or identical. Similarly for any rotation. Going one step further, one regards two configurations of the universe differing only by a dilation, i.e. by a uniform expansion or contraction, as representing in fact the same physical state of the universe. The space of all genuinely physically different possible configurations so obtained—taking into account translations, rotations, and dilations—is usually called *shape space*. The name shape space is indeed natural: only the shape of a configuration of particles is relevant, not its position or orientation or overall size.

Shape space should be contrasted with *absolute configuration space*, the totality of configurations of  $N$  points in Euclidean three-dimensional space.

## 1.2 Relational Time

Once shape space is taken as fundamental, one should address how shapes evolve. While positing an absolute Newtonian time as in classical mechanics is compatible with relational space, it is natural to consider a relational notion of time as well. According to relational time, global changes of speed of the history of the universe give physically equivalent representations, so that a history of the universe is just a curve in shape space without any reference to a special parametrization of the curve given by any sort of absolute Newtonian time.

Relational time should be contrasted with absolute or *metrical time*. By metrical time we refer to any objective physical coordination of the configurations along a geometrical path in a configuration space with the points of a one-dimensional continuum: a (continuous) mapping from the continuum onto the path. The continuum is usually represented by the real numbers, but it need not be. However, it should be physically distinct from the particular continuum that is the path itself.

Understood in this way, metrical time does not exist, from the relational point of view, for the universe as a whole. However, for subsystems of the universe metrical time naturally emerges: the continuum with which the geometrical path corresponding to the evolution of the subsystem is coordinated can be taken to be the path of its environment, with the obvious mapping between the paths.

### 1.3 Relational versus Relativistic

Relational physics should be contrasted with relativistic physics. A simple point: In relational physics the traditional separation of space and time is retained. While configuration space is replaced by shape space, and time becomes non-metrical, shape space retains an identity separate and distinct from that of (non-metrical) time. This is in obvious contrast with relativistic physics, in which space and time lose their separate identities and are merged into a space-time.

Perhaps the most characteristic feature of relativity is the absence of absolute simultaneity. Not so for relational physics. Since it retains the separation of space and time, an absolute simultaneity is built into the very structure of relational physics as described here. Nonetheless, there is a sense in which simultaneity is lost. With relational time the notion of the shape of the universe at “time  $t$ ” is not physically meaningful. And with what is meaningful—geometrical paths in the space of possible shapes—one can no longer meaningfully compare or ask about the configurations for two different possible histories at the same time. Given the actual configuration of the universe, it is not meaningful to ask about the configuration of an alternative history at that time without further specification of exactly what that should mean.

One may inquire whether the relational point of view can be merged with or extended to relativity. Can we achieve a relational understanding of space-time? General relativity is certainly a step in that direction, but it does not get us there. Space-time in general relativity is metrical—in a way that neither space nor time are in relational physics. A complete extension, if at all possible, is a real challenge.

Another possibility is that relativity is not fundamental, but is, instead, a consequence of a suitable choice of gauge. This possibility, which is suggested by the work of Bryce DeWitt [10] and Barbour and coworkers [5] would be worth carefully exploring. For a bit of elaboration on this, see Section 7.5.3.

### 1.4 Relational Space, Relational Time, and Gauge Freedom

When physical theories are formulated in shape space, one should consider first the simplest ones, namely the “free” theories based only on the geometrical structures provided by the metric, without invoking any potential. This is in contrast with theories formulated in absolute space, for which free theories can’t begin to account for the experimental data. It is then natural to ask: when we represent the theories in absolute space, what form do the laws of motion take? Is the representation unique or are there various representations yielding different-looking laws of motion, some unfamiliar and some more or less familiar? Moreover, do interacting theories emerge with nontrivial interactions, although in shape space the motion is free?

To answer these questions it is helpful to represent absolute configuration space in geometrical terms as a fiber bundle, with shape space as base manifold and the fibers generated by the similarity group, i.e, by translations, rotations and dilations, which acting on configurations yield, from a relational point of view, physically equivalent states. A representation in absolute configuration space of the motion in shape space is then given by a “lift” of the motion from the base into the fibers (more details below).

Such lifts can rightly be called gauges. In the classical case it turns out that in some gauges the law looks unfamiliar but there is (at least) one gauge in which, after performing a time change (representing indeed another gauge freedom when also time is seen as relational), the law of motion is Newtonian with a potential appearing. So, it turns out that relational space and relational time form a harmonious combination. More or less the same is true for the quantum case. And similarly, as hinted above, the same might be true for general relativity.

## 1.5 Why Go Relational?

One reason for turning to relational physics is this. Only relational distinctions concerning space and time are observable. Thus, so the argument might continue, since the positions of bodies are individuated only by relative distances and angles, from an observational point of view, only shapes or relational times are meaningful, while absolute space, absolute time, and even absolute spacetime, do not have any clear observational meaning. While having some force, this argument is too positivistic to be taken as the main reason to go relational.

Here is another reason. Since the long-standing philosophical controversy about the nature of space and time has now become a well-defined problem in theoretical physics, we have a professional reason to go relational. It seems worth exploring how far one can go in formulating physics in minimal relational terms.

A stronger reason, which is indeed the one we favor, has to do with the status and the nature of the wave function. Since at this stage, this sounds totally incomprehensible, we shift gears and turn to quantum mechanics.

## 2 Quantum Mechanics

Standard quantum mechanics is problematic. It is often said that the fundamental problem with quantum mechanics is the *measurement problem*, or, more or less equivalently, Schrödinger’s cat paradox. However, the measurement problem, as important as it is, is nonetheless but a manifestation of a more basic difficulty with

standard quantum mechanics: it is not at all clear what quantum theory is about. Indeed, it is not at all clear what quantum theory actually says. Is quantum mechanics fundamentally about measurement and observation? Or is it about the behavior of suitable fundamental microscopic entities, elementary particles and/or fields? To use the words of Bell, standard formulation of quantum mechanics are “unprofessionally vague and ambiguous.” A formulation free of these shortcomings is Bohmian mechanics [7, 6, 11, 12, 8].

## 2.1 Bohmian Mechanics

Bohmian mechanics is a theory providing a description of reality, compatible with all of the quantum formalism, but free of any reference to observables or observers in its formulation. In Bohmian mechanics a system of particles is described in part by its wave function, evolving according to Schrödinger’s equation, the central equation of quantum theory. However, the wave function provides only a partial description of the system. This description is completed by the specification of the actual positions of the particles. The latter evolve according to the “guiding equation,” which expresses the velocities of the particles in terms of the wave function. Thus in Bohmian mechanics the configuration of a system of particles evolves via a deterministic motion choreographed by the wave function.

Bohmian mechanics might be regarded as the minimal completion of Schrödinger’s equation, for a non-relativistic system of particles, to a theory describing a genuine motion of particles. For Bohmian mechanics the state of a system of  $N$  particles is described by its wave function  $\Psi = \Psi(\vec{q}_1, \dots, \vec{q}_N) = \Psi(\mathbf{q})$ , a complex- (or spinor-) valued function on the space of possible configurations  $\mathbf{q}$  of the system, together with its actual configuration  $\mathbf{Q}$  defined by the actual positions  $\vec{Q}_1, \dots, \vec{Q}_N$  of its particles. The theory is then defined by two evolution laws. One is *Schrödinger’s equation*

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi, \quad (1)$$

for  $\Psi = \Psi_t$ , the wave function at time  $t$ , where  $H$  is the non-relativistic (Schrödinger) Hamiltonian, containing the masses  $m_\alpha$ ,  $\alpha = 1, \dots, N$ , of the particles and a potential energy term  $V$ . For spinless particles, it is of the form

$$H = - \sum_{\alpha=1}^N \frac{\hbar^2}{2m_\alpha} \vec{\nabla}_\alpha^2 + V, \quad (2)$$

where  $\vec{\nabla}_\alpha = \frac{\partial}{\partial \vec{q}_\alpha}$  is the gradient with respect to the position of the  $\alpha$ -th particle. The

other law is the *the guiding law*, given by the equation

$$\frac{d\vec{Q}_\alpha}{dt} = \frac{\hbar}{m_\alpha} \text{Im} \frac{\Psi^* \vec{\nabla}_\alpha \Psi}{\Psi^* \Psi}(\vec{Q}_1, \dots, \vec{Q}_N). \quad (3)$$

for  $\mathbf{Q} = \mathbf{Q}(t) = (\vec{Q}_1(t), \dots, \vec{Q}_N(t))$ , the configuration at time  $t$  (with  $\text{Im}$  denoting the imaginary part of the complex quantity that follows it). If  $\Psi$  is spinor-valued, the products in numerator and denominator in (3) should be understood as scalar products. If external magnetic fields are present, the gradient should be understood as the covariant derivative, involving the vector potential. For spinless particles, setting  $\Psi = R e^{iS/\hbar}$  (the polar representation of  $\Psi$ ), the guiding equation (3) becomes

$$\frac{d\vec{Q}_\alpha}{dt} = \frac{\hbar}{m_\alpha} \text{Im} \frac{\vec{\nabla}_\alpha \Psi}{\Psi}(\vec{Q}_1, \dots, \vec{Q}_N) = \frac{1}{m_\alpha} \vec{\nabla}_\alpha S(\vec{Q}_1, \dots, \vec{Q}_N) \quad (4)$$

For an  $N$ -particle system Schrödinger's equation and the guiding equation, together with the detailed specification of the Hamiltonian  $H$ , completely define the Bohmian motion of the system.

## 2.2 OOEOW

Bohmian mechanics is for us the simplest version of quantum mechanics: Given Schrödinger's equation, it has the Obvious Ontology Evolving in the Obvious Way.

For a theory of *particles*, in addition to the wave function you've got the actual positions of the particles. This is a rather obvious ontology. We usually emphasize this point by saying that the positions of the particles provide the *primitive ontology* of the theory. In so saying we wish to convey that the whole point of the theory—and the whole point of the wave function—is to define a motion for the particles, and it's in terms of this motion that pointers end up pointing and experiments end up having results, the kinds of results that it was the whole point of quantum mechanics to explain. So the connection to physical reality in the theory is via what we're calling the primitive ontology of the theory, in Bohmian mechanics the positions of the particles.

The only thing Bohmian mechanics adds is a first-order equation of motion for how the positions of particles evolve. This equation, the new equation in Bohmian mechanics, is kind of an obvious equation. It is more or less the first thing you would guess if you asked yourself, What is the simplest motion of the particles that could reasonably be defined in terms of the wave function?

There are indeed many (more or less) obvious ways of guessing the velocity field function  $\mathbf{v} = \mathbf{v}^\Psi$  in the right hand side of (3). The simplest way, for particles without



spin, is the following: Begin with the de Broglie relation  $\mathbf{p} = \hbar\mathbf{k}$ , a remarkable and mysterious distillation of the experimental facts associated with the beginnings of quantum theory—and itself a relativistic reflection of the first quantum equation, namely the Planck relation  $E = h\nu$ . The de Broglie relation connects a particle property, the momentum  $\mathbf{p} = m\mathbf{v}$ , with a wave property, the wave vector  $\mathbf{k}$ . Understood most simply, it says that the velocity of a particle should be the ratio of  $\hbar\mathbf{k}$  to the mass of the particle. But the wave vector  $\mathbf{k}$  is defined only for a plane wave. For a general wave  $\psi$ , the obvious generalization of  $\mathbf{k}$  is the local wave vector  $\nabla S(\mathbf{q})/\hbar$ , where  $S$  is the phase of the wave function (defined by its polar representation, see above). With this choice the de Broglie relation becomes  $\mathbf{v} = \nabla S/m$ , the right hand side of which is our first guess for  $\mathbf{v}^\psi$ .

We note that the de Broglie relation also immediately yields Schrödinger's equation, giving the time evolution for  $\psi$ , as the simplest wave equation that reflects this relationship. This is completely standard. In this simple way, the defining equations of Bohmian mechanics can be regarded as flowing in a natural manner from the first quantum equation  $E = h\nu$ .

The quantum continuity equation is the key for a route which is meaningful also for particles with spin. This equation, an immediate consequence of Schrödinger's equation, involves a quantum probability density  $\rho$  and a quantum probability current  $J$ . Since densities and currents are classically related by  $J = \rho\mathbf{v}$ , it requires little imagination to set  $\mathbf{v}^\psi = J/\rho$ .

Another way to arrive at a formula for  $\mathbf{v}^\psi$  is to invoke symmetry. Since the space-time symmetry of the non-relativistic Schrödinger equation is that of rotations, translations, time-reversal, and invariance under Galilean boosts, it is natural to demand that this Galilean symmetry be retained when Schrödinger's equation is combined with the guiding equation. As described in [11], this leads to a specific formula for  $\mathbf{v}^\psi$  as the simplest possibility.

There are several other natural ways to arrive at  $\mathbf{v}^\psi$ , but we shall give no more. All these routes, those we've explicitly mentioned and those to which we've alluded, yield in fact exactly the same formula for  $\mathbf{v}^\psi$ , though the explicit form may appear different in some cases.

## 2.3 The Implications of Bohmian Mechanics

While the *formulation* of Bohmian mechanics does not involve the notion of quantum observables, as given by self-adjoint operators—so that its relationship to the quantum formalism (the familiar axioms of quantum theory) may at first appear somewhat obscure—it can in fact be shown that Bohmian mechanics embodies quantum

randomness, as expressed by Born’s rule, and the entire quantum formalism based on self-adjoint operators as observables as the very expression of its empirical import [12, Ch.2 and 3]. The most notable implication of Bohmian mechanics is the fact that in a universe governed by Bohmian mechanics there are sharp, precise, and irreducible limitations on the possibility of obtaining knowledge, limitations that can in no way be diminished through technological progress leading to better means of measurement. This *absolute uncertainty* is in precise agreement with Heisenberg’s uncertainty principle.

We shall review below the key points that lead to these implications.

### 2.3.1 The Conditional Wave Function

In physics we are usually concerned not with the entire universe but with subsystems of the universe, for example with a hydrogen atom or a pair of entangled photons. The quantum mechanical treatment of such systems involves the quantum state of that system, often given by its wave function—not the wave function of the universe. Yet, the notion of the wave function of a subsystem is rather elusive. In standard quantum mechanics the state of a subsystem is usually described in terms of its reduced density matrix.

However, Bohmian mechanics provides a precise formulation and understanding of this notion in terms of the conditional wave function [11]

$$\psi(x) = \Psi(x, Y), \tag{5}$$

where  $\Psi = \Psi(q) = \Psi(x, y)$  is the wave function of the universe, with  $x$  and  $y$  the generic variables for the configurations of the system and its environment (everything else), respectively, and where  $Y$  is the actual configuration of the environment. The conditional wave function of a Bohmian system behaves exactly as one would expect the wave function of a system to behave, with respect to both dynamics and statistics.

### 2.3.2 The Conditional Probability Formula

In Bohmian mechanics, for a non-relativistic system of particles, the configuration of a system is regarded as random, with randomness corresponding to the quantum equilibrium distribution  $\mu^\psi$  given by  $|\psi|^2 dq$ . What this actually means, in a deterministic theory such as Bohmian mechanics, is a delicate matter, involving a long story [11] with details and distinctions that we shall ignore here. However, a crucial ingredient for that analysis—for an understanding of the origin of quantum randomness in a universe governed by Bohmian mechanics—is the *fundamental conditional probability formula* for the conditional distribution of the configuration  $X_t$

of a system at time  $t$  given that of its environment  $Y_t$  at that time:

$$P^{\Psi_0}(X_t \in dx | Y_t) = |\psi_t(x)|^2 dx, \quad (6)$$

where  $\Psi_0$  is the initial wave function of the universe and  $P^{\Psi_0}$  is the probability distribution on trajectories arising from the Bohmian dynamics with an initial quantum equilibrium distribution, and  $\psi_t$  is the (normalized) conditional wave function (5) of the system at time  $t$ .

A crucial ingredient in proving (6) is *equivariance* [11]: if at any time the configuration of a Bohmian system is randomly distributed according to  $|\Psi|^2$  at that time then at any other time  $t$  it will be distributed according to  $|\Psi_t|^2$ . Equivariance is an immediate consequence of the continuity equation arising from the Schrödinger equation:

$$\frac{\partial \rho^\Psi}{\partial t} + \operatorname{div} J^\Psi = 0, \quad (7)$$

with  $\rho^\Psi = |\Psi|^2$ , the quantum equilibrium distribution, and

$$J^\Psi = \rho^\Psi \mathbf{v}^\Psi, \quad (8)$$

the quantum probability current, where  $\mathbf{v}^\Psi$  is the Bohmian velocity in the right hand side of (4).

## 2.4 Bohmian Motion on a Riemannian Manifold

We conclude this section with a mathematical observation, whose importance will become clear later on when we discuss the quantum mechanics of shapes. Note that, given  $V$ , the Bohmian mechanics defined by equations (1), (2), and (4) depends only upon the Riemannian structure  $g = g_e$  given by the standard Euclidean metric on configuration space  $[g_e]_{ij} = m_{\alpha_i} \delta_{ij}$ , where the  $i$ -th component refers to the  $\alpha_i$ -th particle. In terms of this Riemannian structure, the evolution equations (4) and (2) become

$$\frac{dQ}{dt} = \hbar \operatorname{Im} \frac{\nabla_g \Psi}{\Psi} \quad (9)$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2} \Delta_g \Psi + V \Psi, \quad (10)$$

where  $\Delta_g$  and  $\nabla_g$  are, respectively, the Laplace-Beltrami operator and the gradient on the configuration space equipped with this Riemannian structure. But there is nothing special about this particular Riemannian structure. Indeed, equations (9) and (10) as such hold very generally on *any* Riemannian manifold. Thus, the formulation of a Bohmian dynamics on a Riemannian manifold requires only as basic ingredients the differentiable and metric structure of the manifold.

### 3 Quantum Puzzles

The Bohmian perspective allows us to illuminate some quantum puzzles, somewhat hidden in other formulations of quantum mechanics. These puzzles find their natural resolution in a proper relational reformulation of quantum mechanics. We shall elaborate in later sections.

#### 3.1 The Wave Function of the Universe and Entropy

The fundamental equation for the wave function of the universe in canonical quantum cosmology is the Wheeler-DeWitt equation [9]

$$\mathcal{H}\Psi = 0, \tag{11}$$

for a wave function  $\Psi(q)$  of the universe, where  $q$  refers to 3-geometries and to whatever other stuff is involved, all of which correspond to structures on a 3-dimensional space. In this equation  $\mathcal{H}$  is a sort of generalized Laplacian, a cosmological version of a Schrödinger Hamiltonian  $H$ . And like a typical  $H$ , it involves nothing like an explicit time-dependence. But unlike Schrödinger's equation, the Wheeler-DeWitt equation has on one side, instead of a time derivative of  $\Psi$ , simply 0. Its natural solutions are thus time-independent.

That this is so is in fact the *problem of time* in quantum cosmology. We live in a world where things change. But if the basic object in the world is a timeless wave function, how does change come about? Much has been written about this problem of time. A great many answers have been proposed. But what we want to emphasize here is that from a Bohmian perspective the timelessness of  $\Psi$  is not a problem. Rather it is just what the doctor ordered.

The fundamental role of the wave function in Bohmian mechanics is to govern the motion of something else. Change fundamentally occurs in Bohmian mechanics not so much because the wave function changes but because the thing  $Q$  that it governs changes, according to a law

$$dQ/dt = v^\Psi(Q) \tag{12}$$

determined by the wave function. The problem of time vanishes entirely from a Bohmian point of view.

But there is a difficulty that arises in connection with the stationarity of the universal wave function. According to standard formulations of quantum statistical mechanics, thermodynamics seems to depend only upon the wave function. But if the initial wave function is stationary, it can't be responsible for the irreversible

behavior of our universe, and in particular of entropy increase. So what is the origin of the arrow of time? And why does entropy increase?

### **3.2 Nonnormalizability of the Wave Function of the Universe**

The solutions of Wheeler-DeWitt equation typically fail to be normalizable. So what could the associated non-normalizable “probabilities” given by  $|\Psi|^2$  physically mean? How can the quantum equilibrium analysis we hinted at above, that allows us to recover the entire quantum formalism, get off the ground for such a non-normalizable measure?

### **3.3 Why vs. What**

From a Bohmian perspective there is another puzzle. One may ask, Why should the motion of configurations be governed by a wave function?

This problem is rather subtle and elusive, hardly understandable at all in the standard formulation of quantum mechanics, as well as in other more precise versions of quantum mechanics, such as Many Worlds or GRW, in which the wave function is all there is. We believe that relational physics sheds light on this problem and shall elaborate on this in Section 5. We also wish to note, concerning quantum mechanics, the contrast between asking why and asking what. One often finds derivations of quantum mechanics from some principle or other, which thus seem to answer the why question. But in most, if not all, such cases, the answer leaves us with little or no understanding of what quantum mechanics actually says about physical reality. This is not good. The appropriate time to ask why is after the what question has been satisfactorily addressed. With Bohmian mechanics we are at such a time.

## **4 Wave Function as Law**

The most puzzling issue in the foundations of quantum mechanics is perhaps that of the status of the wave function of a system in a quantum universe. We have suggested elsewhere that the wave function should be regarded as nomological, nomic—that it’s really more in the nature of a law than a concrete physical reality.

Thoughts in this direction might arise when you consider the unusual way Bohmian mechanics is formulated, and the unusual way the wave function behaves in Bohmian mechanics. The wave function of course affects the behavior of the configuration,

i.e., of the particles. This is expressed by the guiding equation (12). But in Bohmian mechanics there's no back action, no effect in the other direction, of the configuration upon the wave function, which evolves autonomously via Schrödinger's equation, in which the actual configuration  $Q$  does not appear.

A second point is that for a multi-particle system the wave function  $\Psi(\mathbf{q}) = \Psi(\vec{\mathbf{q}}_1, \dots, \vec{\mathbf{q}}_N)$  is not a field on physical space, but on configuration space, the set of all hypothetical configurations of the system. For a system of more than one particle that space is not physical space. What this suggests to us is that you should think of the wave function as describing a law and not as some sort of concrete physical reality. After all (12) is an equation of motion, a law of motion, and the whole point of the wave function here is to provide us with the law, i.e., with the right hand side of this equation.

#### 4.1 $\psi$ vs. $\Psi$

There are, however, problems with regarding the wave function as nomological. Laws aren't supposed to be dynamical objects, they aren't supposed to change with time, but the wave function of a system typically does. And laws are not supposed to be things that we can control. But the wave function is often an initial condition for a quantum system. We often, in fact, prepare a system in a certain quantum state, that is, with a certain wave function. We can in this sense control the wave function of a system. But we don't control a law of nature. This makes it a bit difficult to regard the wave function as nomological.

But with regard to this difficulty it's important to recognize, as already suggested, that there's only one wave function we should be worrying about, the fundamental one, the wave function  $\Psi$  of the universe. In Bohmian mechanics, the wave function  $\psi$  of a subsystem of the universe is defined in terms of the universal wave function  $\Psi$  according to (5). Thus, to the extent that we can grasp the nature of the universal wave function, we should understand as well, by direct analysis, also the nature of the objects that are defined in terms of it, and in particular we should have no further fundamental question about the nature of the wave function of a subsystem of the universe.

Note that from this perspective, the most important equation in quantum mechanics, the time-dependent Schrödinger's equation should be regarded as a phenomenological equation describing how the conditional wave function  $\psi$  of a subsystem evolves—under suitable conditions of decoupling from its environment—in a universe having a suitable stationary universal wave function  $\Psi$ .

## 4.2 A Kantian Component of Laws of Nature

Laws are supposed to be simple. But here a distinction should be made. Consider the velocity field in the right hand side of (12),

$$v^\Psi(Q) = v(\Psi, Q).$$

If we regard  $\Psi$  as a law,  $v^\Psi$  should be a simple function of  $Q$ . On the other hand, if we regard  $\Psi$  as *part* of a law,  $v(\Psi, Q)$  should be a simple function of  $\Psi$  and  $Q$ .

So we would like to find a compelling law of motion, or a compelling principle yielding such a law, that can be expressed in terms of a  $v^\Psi$  for a suitable  $\Psi$ . We don't have that. But, as we shall explain, the flexibility and gauge freedom afforded by relational physics might help us reach our goal.

Ideally, the suitable  $\Psi$  referred to above should be simple. Failing that, it should at least obey a simple law, such as the Wheeler-DeWitt equation. From a fundamental point of view, it might be a complete accident that  $\Psi$  obeys such an equation. It might just happen to do so. The fact that the equation is satisfied might have nothing to do with why the fundamental dynamics is of the form (12). But as long as  $\Psi$  does satisfy the equation, by accident or not, all the consequences of satisfying it would follow.

The point here is to distinguish what is really fundamental from what we have traditionally regarded as fundamental. The latter might, in fact, be imposed by us. We express this by saying that the laws of nature might have a Kantian component. And that Kantian component might be much larger than suggested so far, as will become clearer in the next section devoted to relational quantum mechanics.

## 5 Relational Quantum Mechanics

At the end of Section 1 we said that the stronger reason for us to go relational has to do with understanding the status and the nature of the wave function. At the beginning of Section 3, we claimed that the quantum puzzles presented there find their natural resolution in a proper relational reformulation of quantum mechanics.

It's time to intertwine all these threads. And the key question to start with is the one we asked in the last paragraph of the previous section, What is really fundamental?

## 5.1 Shape Space as the Fundamental Configuration Space

Let us go back to the toy model of an  $N$ -particle universe introduced in Section 1.1. The space of all its genuinely physically different possible configurations is *shape space*. Mathematically, this is described as follows.

### 5.1.1 Shape Space

The totality of configurations  $\mathbf{q} = (\vec{\mathbf{q}}_1, \dots, \vec{\mathbf{q}}_N)$  of  $N$  points in Euclidean three-dimensional space forms the configuration space  $\mathcal{Q} = \{\mathbf{q}\} = \mathbb{R}^{3N}$  of an  $N$ -particle system. We shall call  $\mathcal{Q}$  the *absolute configuration space*. On  $\mathcal{Q}$  there is a natural action of the similarity transformations of Euclidean space, namely rotations, translations and dilations, since each of them acts naturally on each component of the configuration vector in physical Euclidean space. The totality of such transformations form the group  $G$  of *similarity transformations* of Euclidean space. Since the shape of a configuration is what remains when the effects associated with rotations, translations and dilations are filtered away, the totality of shapes, i.e., the *shape space*, is the quotient space  $\mathcal{Q} \equiv \mathcal{Q}/G$ , the set of equivalence classes with respect to the equivalence relations provided by the similarity transformations of Euclidean space.

As such, shape space is not in general a manifold. To transform it into a manifold some massaging is needed (e.g., by excluding from  $\mathcal{Q}$  coincidence points and collinear configurations), but we shall not enter into this.<sup>1</sup> Here we shall assume that the appropriate massaging of  $\mathcal{Q}$  has been performed and that  $\mathcal{Q}$  is a manifold. Since the group of similarity transformations has dimension 7 (3 for rotations + 3 for translations + 1 for dilations), the dimension of  $\mathcal{Q} \equiv \mathcal{Q}/G$  is  $3N - 7$  for  $N \geq 3$ . For  $N = 1$  and  $N = 2$  shape space is trivial (it contains just a single point).  $N = 3$  corresponds to the simplest non-trivial shape space; it has dimension 2.

Accordingly, it is helpful to represent absolute configuration space  $\mathcal{Q}$  in geometrical terms as a fiber bundle, with shape space  $\mathcal{Q}$  as the base manifold and with the fibers generated by the the similarity group  $G$  group. The points on each fiber represent physically equivalent states. See Fig. 1.

### 5.1.2 Metrics on Shape Space and Best Matching

The topology of shape space is well defined by its construction as a quotient space, but topology, of course, does not fix a metric. A metric should provide more, namely a natural notion of distance on  $\mathcal{Q}$ . And since each point in  $\mathcal{Q}$  represents a class

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<sup>1</sup>For more details on this issue, see, e.g., [14] and the references therein.



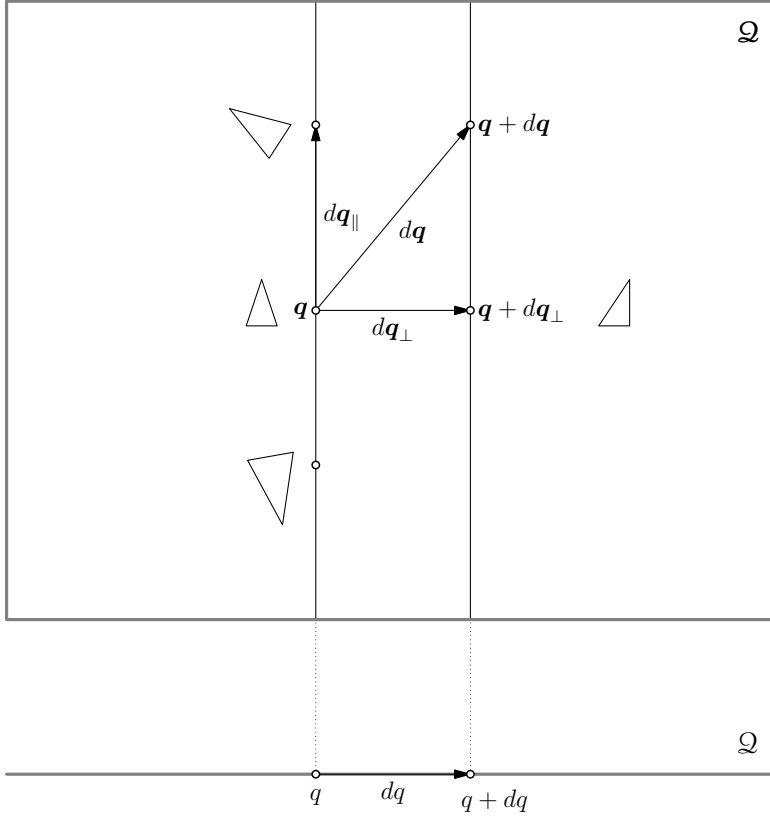


Figure 1: Absolute configuration space  $\mathcal{Q}$  and shape space  $\mathcal{Q}$  (for a system of three particles). The fiber above shape  $q$  consists of absolute configurations  $\mathbf{q}$  differing by a similarity transformation of Euclidean space and thus representing the same shape  $q$ . Real change of shape occurs only by a displacement to a neighboring fiber  $q + dq$ . Only the orthogonal component  $d\mathbf{q}_\perp$  of  $d\mathbf{q}$  represents real change, while the vertical displacement  $d\mathbf{q}_\parallel$  does not contribute;  $\mathbf{q} + d\mathbf{q}_\perp$  is the absolute configuration in the fiber above  $q + dq$  closest to  $q$  in the sense of the  $g_B$ -distance (best matching).

of configurations of  $N$  particles related by a similarity transformation, the distance between two elements of  $\mathcal{Q}$  induced by the metric should not recognize any absolute configurational difference due to an overall translation, or rotation, or dilation. In other words, it should provide a measure of the *intrinsic* difference between two absolute configurations (that is, not involving any consideration regarding how such configurations are embedded in Euclidean space).

It turns out [1] that a metric on absolute configuration space  $\mathcal{Q}$  that is invariant under the group  $G$  of similarity transformations of Euclidean space, given by a

suitable “conformal factor” (to be explained below), defines canonically a metric on shape space  $\mathcal{Q}$ .

To understand why this is so, consider the representation of absolute configuration space  $\mathcal{Q}$  as a fiber bundle with each fiber being homeomorphic to  $G$  and  $\mathcal{Q}$  being its base space (see Fig. 1). So, if  $g$  is a metric invariant under any element of  $G$ , the tangent vectors at each point  $\mathbf{q} \in \mathcal{Q}$  are *naturally* split into “vertical” and “horizontal,” where by “naturally” we mean that the splitting itself is invariant under the action of  $G$ . The vertical ones correspond to (infinitesimal) displacements along the fiber through  $\mathbf{q}$  and the horizontal ones are those that are orthogonal to the fiber, i.e., to the vertical ones, according to the relation of orthogonality defined by  $g$ . More precisely, if  $d\mathbf{q}$  is an infinitesimal displacement at  $\mathbf{q}$ , we have

$$d\mathbf{q} = d\mathbf{q}_{\parallel} + d\mathbf{q}_{\perp} \quad \text{with} \quad g(d\mathbf{q}_{\parallel}, d\mathbf{q}_{\perp}) = 0$$

(see Fig. 1), with  $d\mathbf{q}_{\parallel}$  vertical and  $d\mathbf{q}_{\perp}$  horizontal.

The corresponding Riemannian metric on  $\mathcal{Q}$  is defined as follows. Let  $q$  be a shape,  $\mathbf{q}$  any absolute configuration in the fiber above  $q$ , and  $d\mathbf{q}$  any displacement at  $\mathbf{q}$ . Since  $g$  is invariant under the group  $G$ , the length of  $d\mathbf{q}_{\perp}$  has the same value for all absolute configurations  $\mathbf{q}$  above  $q$ . Thus we may set the length of  $d\mathbf{q}$  equal to that of  $d\mathbf{q}_{\perp}$  and hence obtain the Riemannian metric  $g_B$  on  $\mathcal{Q}$

$$g_B(d\mathbf{q}, d\mathbf{q}) = g(d\mathbf{q}_{\perp}, d\mathbf{q}_{\perp}). \quad (13)$$

The subscript  $B$  stands for Barbour and Bertotti, as well as base and best matching. In fact, The distance on  $\mathcal{Q}$  induced by  $g_B$  is exactly the one resulting from applying Barbour’s best matching procedure. Consider two infinitesimally close shapes,  $q$  and  $q + dq$ , and let  $\mathbf{q}$  be any absolute representative of  $q$ , i.e., any point in the fiber above  $q$ . The  $g_B$ -distance between these shapes is then given by the  $g$ -length of the vector  $d\mathbf{q}$  such that (i)  $d\mathbf{q}$  is orthogonal to the fiber above  $q$  and (ii)  $\mathbf{q} + d\mathbf{q}$  is an absolute representative of  $q + dq$ . It follows that  $\mathbf{q} + d\mathbf{q}$  is the absolute configuration closest to  $\mathbf{q}$  in the fiber above  $q + dq$ . Thus the  $g_B$ -distance is the “best matching” distance.

### 5.1.3 Invariant Metrics on Absolute Configuration Space

So we need to show how to construct an invariant metric  $g$  on  $\mathcal{Q}$ . Let  $g_e$  be the mass-weighted Euclidean metric on  $\mathcal{Q}$  with positive weights  $m_{\alpha}$ ,  $\alpha = 1, \dots, N$ , (the masses of the particles), in particle coordinates  $\mathbf{q} = (\vec{\mathbf{q}}_1, \dots, \vec{\mathbf{q}}_{\alpha}, \dots, \vec{\mathbf{q}}_N)$  given by

$$ds^2 = \sum_{\alpha=1}^N m_{\alpha} d\vec{\mathbf{q}}_{\alpha} \cdot d\vec{\mathbf{q}}_{\alpha}, \quad (14)$$

i.e., with  $[g_e]_{ij} = m_{\alpha_i} \delta_{ij}$ , where the  $i$ -th component refers to the  $\alpha_i$ -th particle. The corresponding line element is

$$|d\mathbf{q}| = \sqrt{\sum_{\alpha=1}^N m_{\alpha} d\vec{\mathbf{q}}_{\alpha} \cdot d\vec{\mathbf{q}}_{\alpha}}. \quad (15)$$

The metric defined by (14) is invariant under rotations and translations, but not under a dilation  $\mathbf{q} \rightarrow \lambda \mathbf{q}$ , where  $\lambda$  is a positive constant. Invariance under dilations is achieved by multiplying  $|d\mathbf{q}|^2$  by a scalar function  $f(\mathbf{q})$  that is invariant under rotations and translations and is homogeneous of degree  $-2$ . We call  $f$  a *conformal factor*. So, for any choice of  $f$ ,

$$g = f g_e, \quad \text{i.e.,} \quad g(d\mathbf{q}, d\mathbf{q}) = f(\mathbf{q}) |d\mathbf{q}|^2, \quad (16)$$

is an invariant metric on  $\mathcal{Q}$ , yielding the metric on shape space

$$g_B(dq, dq) = f(\mathbf{q}) |d\mathbf{q}_{\perp}|^2. \quad (17)$$

For the associated line element we shall write

$$ds = |dq| = \sqrt{g_B(dq, dq)} = \sqrt{f(\mathbf{q})} |d\mathbf{q}_{\perp}|. \quad (18)$$

#### 5.1.4 Conformal Factors

Many choices of conformal factors are possible. One that was originally suggested by Barbour and Bertotti is<sup>2</sup>

$$f(\mathbf{q}) = f_a(\mathbf{q}) \equiv \left( \sum_{\alpha < \beta} \frac{m_{\alpha} m_{\beta}}{|\vec{\mathbf{q}}_{\alpha} - \vec{\mathbf{q}}_{\beta}|} \right)^2. \quad (19)$$

Another example is

$$f(\mathbf{q}) = f_b(\mathbf{q}) \equiv L^{-2}, \quad (20)$$

where

$$L^2 = \sum_{\alpha} m_{\alpha} \vec{\Pi}_{\alpha}^2 = \frac{1}{\sum_{\alpha} m_{\alpha}} \sum_{\alpha < \beta} m_{\alpha} m_{\beta} |\vec{\mathbf{q}}_{\alpha} - \vec{\mathbf{q}}_{\beta}|^2 \quad (21)$$

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<sup>2</sup>Here and in the following examples the conformal factors are modulo dimensional factors.

with  $\vec{\Pi}_\alpha = \vec{q}_\alpha - \vec{q}_{\text{cm}}$ , the coordinates relative to the center of mass

$$\vec{q}_{\text{cm}} = \frac{\sum_\alpha m_\alpha \vec{q}_\alpha}{\sum_\alpha m_\alpha}. \quad (22)$$

$I \equiv L^2$  is sometimes called (but the terminology is not universal) the moment of inertia of the configuration  $\mathbf{q}$  about its center of mass. This quantity is half the trace of the moment of inertia tensor  $\mathbf{M}$ ,

$$L^2 = \frac{1}{2} \text{Tr } \mathbf{M}. \quad (23)$$

We recall that  $\mathbf{M} = \mathbf{M}(\mathbf{q})$ , the tensor of inertia of the configuration  $\mathbf{q}$  about any orthogonal cartesian system  $x, y, z$  with origin at the center of mass of the configuration  $\mathbf{q}$ , has matrix elements given by the standard formula

$$M_{ij} = \sum_{\alpha=1}^N m_\alpha (\rho_\alpha^2 \delta_{ij} - \rho_{\alpha i} \rho_{\alpha j}), \quad (24)$$

where  $i, j = x, y, z$ ,  $\rho_{\alpha x} \equiv x_\alpha$ ,  $\rho_{\alpha y} \equiv y_\alpha$ ,  $\rho_{\alpha z} \equiv z_\alpha$ , and  $\rho_\alpha^2 = x_\alpha^2 + y_\alpha^2 + z_\alpha^2$ .

A choice of conformal factor that has not been considered in the literature is

$$f(\mathbf{q}) = f_c(\mathbf{q}) \equiv L^{-\frac{8}{7}} (\det \mathbf{M})^{-\frac{1}{7}}. \quad (25)$$

Since  $\det \mathbf{M}$  scales as  $L^6$ ,  $f(\mathbf{q})$  given by (25) scales as it should, namely, as  $L^{-2}$ . Though at first glance this choice seems unnatural, it is in fact very natural once the motion of shapes is analyzed from a quantum perspective [1].

Finally, we give other two examples:

$$f(\mathbf{q}) = f_d(\mathbf{q}) \equiv \sum_{\alpha < \beta} \frac{m_\alpha m_\beta}{|\vec{q}_\alpha - \vec{q}_\beta|^2} \quad (26)$$

$$f(\mathbf{q}) = f_g(\mathbf{q}) \equiv L^{-1} \sum_{\alpha < \beta} \frac{m_\alpha m_\beta}{|\vec{q}_\alpha - \vec{q}_\beta|}. \quad (27)$$

The first one corresponds to a natural modification of the Newtonian gravitational potential and the second, discussed in [3], corresponds to a dynamics very close to that of Newtonian gravity.

## 5.2 Free Motion on Shape Space

All one needs to define the simplest motion on shape space, be it classical or quantum (Bohmian), is the metric  $g_B$  on shape space. In the classical case, such a metric  $g_B$  directly yields a law of *free motion* on shape space, that is, geodesic motion with constant speed.

In the quantum (Bohmian) case, the situation is similar. Equations (9) and (10) define immediately free quantum motion on shape space with Riemannian metric  $g = g_B$  as the motion on shape space given by the evolution equations

$$\frac{dQ}{dt} = \hbar \operatorname{Im} \frac{\nabla_B \Psi}{\Psi} \quad (28)$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2} \Delta_B \Psi \quad (29)$$

where  $\Delta_B$  and  $\nabla_B$  are, respectively, the Laplace-Beltrami operator and the gradient on the configuration space equipped with the Riemannian metric (17).

We have insisted on free motion, because it is the simplest motion on shape space, which has, surprisingly, a very rich structure. Of course, one may introduce a potential term both in the classical and the quantum case in a straightforward way: (minus) the gradient of the potential in the right hand side of the geodesic equation for the classical case and the usual potential term in the right hand side of Schrödinger's equation.

This is all there is to say about the *formulations* of classical and quantum theories on shape space, modulo a caveat: so defined, classical and quantum motion rely on an external absolute time, which, from a relational perspective, is problematical, see below.

## 5.3 Fundamental Level vs. “Human” Level

Let us draw some morals from the foregoing. First of all, it is rather clear that to define motion on shape space is very easy. The hard work, basically in the quantum case [1], involves the transition from the fundamental level of shape space to what could be called the “human” level of absolute configuration space. We say “human” because it is the description we usually use. Even if shape space is fundamental, we humans don't typically formulate things in terms of shape space. That doesn't come quite so naturally to us, even if shape space is in a deeper sense more fundamental and natural. Traditionally we do things in this way for good human reasons and probably for many others.

The level of absolute configuration space is a sort of emergent level because it is not fundamental. How things are described at that level is imposed by us. It is “human” in the sense that it is based on our choice of description. This is why we consider it a Kantian component of physical theory. As we shall elaborate below, much of what we’ve taken as fundamental is not fundamental, but imposed by us by a proper choice of gauge. Interactions provide an important example.

## 5.4 Emergence of Interactions

Given a motion in shape space, there is a huge host of motions in absolute space that are compatible with it, the only constraint being that they should project down to the motion in shape space. In other words, recalling the fiber bundle structure of absolute configuration space described above, we have that a representation in absolute configuration space of the motion in shape space is given by a “lift” of the motion from the base into the fibers.

Such lifts can rightly be called gauges. In the classical case it turns out that in some gauges the law looks unfamiliar but there is (at least) one gauge in which, after performing a time change—representing indeed another gauge freedom when also time is seen as relational—the law of motion is Newtonian with a potential appearing. We call it the *Newton gauge*. The potential depends on the choice of the invariant metric (16) in absolute configuration space, with the various possibilities for the conformal factor, some of them listed at the end of Section 5.1.

More or less the same is true for the quantum case, where however the gauge yielding ordinary Bohmian mechanics in absolute configuration space—which we call the *Schrödinger gauge*—emerges only for a stationary, i.e. time-independent, wave function (such as with the Wheeler-DeWitt equation) on shape space. This again involves regarding time as being relational, with an external absolute time playing no physical role.

Also here, while the fundamental physics is given by a free Bohmian dynamics in shape space, in the Schrödinger gauge potential terms appear. One potential term is determined by the scalar curvature induced by the invariant metric on absolute configuration space. Another potential term arises from the gauge freedom we have to lift the Laplace-Beltrami operator from shape space to absolute configuration space, where more gauge freedom arises from allowing transformations of the lifted wave function.

## 6 Sources of Gauge Freedom

The main sources of gauge freedom are a *lift* from the base into the fibers and a *time change*, both for the classical and quantum case. For the quantum case there is in addition a gauge transformation

$$\Psi \mapsto \Psi' = F\Psi, \quad \text{with } F \text{ real.} \quad (30)$$

The implications of this will be discussed below in detail.

### 6.1 Classical Gauges

In the classical case, consider the free motion  $Q = Q(t)$  on shape space, that is, the geodesic motion with constant speed. A very natural lift of this motion in absolute configuration space is its *horizontal lift*, that is, a motion  $\mathbf{Q} = \mathbf{Q}(t)$  in absolute configuration space that starts at some point  $\mathbf{q}_1$  on the fiber above  $q_1$  and is *horizontal*, i.e., the infinitesimal displacements  $d\mathbf{Q}$  are all horizontal. (Note that the final point  $\mathbf{q}_2$  in the fiber above  $q_2$  is then uniquely determined.) We call this the *invariant (or horizontal) gauge*.

It can be shown [1] that in this gauge, the familiar total momentum  $\vec{\mathbf{P}}$ , total angular momentum  $\vec{\mathbf{J}}$  and (maybe less familiar) dilational momentum  $D$  are all zero, i.e.,

$$\vec{\mathbf{P}} = \sum_{\alpha=1}^N m_{\alpha} \frac{d\vec{\mathbf{Q}}_{\alpha}}{dt} = 0 \quad (31)$$

$$\vec{\mathbf{J}} = \sum_{\alpha=1}^N m_{\alpha} \vec{\mathbf{Q}}_{\alpha} \times \frac{d\vec{\mathbf{Q}}_{\alpha}}{dt} = 0 \quad (32)$$

$$D = \sum_{\alpha=1}^N m_{\alpha} \vec{\mathbf{Q}}_{\alpha} \cdot \frac{d\vec{\mathbf{Q}}_{\alpha}}{dt} = 0. \quad (33)$$

#### 6.1.1 The Newton Gauge

By a suitable change of speed, that is by a suitable time change  $t' = t'(t)$ , namely,

$$\frac{v}{\sqrt{f}} \frac{dt}{dt'} = \sqrt{2f}, \quad \text{i.e.,} \quad \frac{dt'}{dt} = \frac{v}{\sqrt{2f}}, \quad (34)$$

we obtain another gauge that we call the *Newton gauge*, a gauge in which the motion is Newtonian, i.e., it satisfies Newton's equation  $F = ma$  for suitable  $F$ . More

precisely, the particle positions  $\vec{Q}_\alpha$ ,  $\alpha = 1, \dots, N$ , forming the configuration  $\mathbf{Q}$  then satisfy Newton's equations

$$m_\alpha \frac{d^2 \vec{Q}_\alpha}{dt'^2} = -\vec{\nabla}_\alpha V(\vec{Q}_1, \dots, \vec{Q}_N). \quad (35)$$

with  $V(\mathbf{q}) = -f(\mathbf{q})$ , the conformal factor.

In this regard, one should observe that if time is relational, changes of speed, such as that given by (34), provide equivalent representations of the same motion. Accordingly, the use of one time variable instead of another is a matter of convenience, analogous to the choice of a gauge. The choice of time variable that leads from the invariant gauge to the Newton gauge is the gauge fixing condition

$$E = \frac{1}{2} \left| \frac{d\mathbf{Q}}{dt'} \right|^2 + V = 0, \quad (36)$$

that is, that the total Newtonian energy be zero. Of course, other gauges could be useful, as discussed next.

### 6.1.2 The Expansion Gauge

Another gauge that could be useful is what could be called the *expansion gauge*, a gauge in which you do not use a horizontal lift, but instead have a uniform expansion of configurations taking place—in the horizontal gauge the total overall size does not change: the universe would not expand since the total dilational momentum is zero in that gauge[3]. In contrast, in the expansion gauge configurations and thus the universe would expand. Why one would consider such a gauge? One reason is that in this way one might simplify the form of the gravitational forces (see below), so that they would become more familiar in this gauge. In this regard, note that since the motion is not horizontal, after suitable time change the Newtonian form of the equation of motion would involve other terms, in addition to the forces generated by the conformal factor. Another reason is that one would obtain in this way a striking explanation of why we think that the universe is expanding.

### 6.1.3 A Note on Newtonian Gravitation

We have seen that in the Newton gauge, when the physical law on shape space is free motion (or even non-free motion), the potential  $V = -f$  appears, where  $f$  is the conformal factor. We mentioned some choices for  $f$  in Section 5.1.4. No such choices, which are necessarily functions homogenous of degree  $-2$ , seem to yield exactly the



Newtonian gravitational potential  $U_g$ . While we believe the detailed exploration of the implications of the models discussed here is worthwhile, we nonetheless regard the models explored in this paper, both classical and quantum, as toy models, so that such an analysis of them, with the expectation of recovering well established physics, might be somewhat inappropriate or premature.

However, it should be observed that some of the conformal factors given in Sect. 5.1.4, e.g.,  $f_a$  and  $f_g$ , indeed give rise to a force law in the Newton gauge that is very close to that of the Newtonian gravitational force. Note for example that for the conformal factor  $f_g$  the corresponding potential is of the form  $V_g = L^{-1}U_g$ , where  $U_g$  is the Newton gravitational potential and  $L$  in the Newton gauge is a constant of the motion. The force arising from this potential adds to the Newtonian force a very small centripetal correction that allows  $I = L^2$ , the moment of inertia about the center of mass, to remain constant [3]. It is worth exploring these possibilities in the expansion gauge.

## 6.2 Quantum Gauges

Let  $Q = Q(t)$  be a Bohmian motion in shape space, that is, a solution of (28) with the wave function  $\Psi$  being a solution of Schrödinger's equation (29) on shape space. As in the classical case, we wish to characterize motions in absolute space that are compatible with motions in shape space, that is, motions  $\mathcal{Q} = \mathcal{Q}(t)$  in  $\mathcal{Q}$  that project down to  $Q = Q(t)$  in  $\mathcal{Q}$ , i.e., such that

$$\pi(\mathcal{Q}(t)) = Q(t), \quad (37)$$

where  $\pi$  is the canonical projection from  $\mathcal{Q}$  to  $\mathcal{Q}$ . Clearly, there are a great many possibilities for compatible motions in absolute configuration space.

As in the classical case, one may restrict the possibilities by considering natural gauges. And as in the classical case, where one looks for gauges such that the absolute motions satisfy Newton's equations, in the quantum case we now look for gauges such that the compatible motions on  $\mathcal{Q}$  are themselves Bohmian motions, i.e. motions generated by a wave function in the usual sort of way.

### 6.2.1 Three Quantum Gauges

Suppose that we proceed as in the classical case and take a horizontal lift of a motion  $Q = Q(t)$  in shape space, that is, an absolute motion for which the infinitesimal displacements  $d\mathcal{Q}$  are all horizontal. Let us now consider the lift to  $\mathcal{Q}$  of a wave

function  $\Psi$  on  $\mathcal{Q}$ , namely, the wave function  $\widehat{\Psi}_1$  on absolute configuration space such that

$$\widehat{\Psi}_1(\mathbf{q}) = \Psi(q) \quad (38)$$

for any point  $\mathbf{q}$  on the fiber above  $q$ . Let  $\nabla_g$  be the gradient with respect to the invariant measure (16). Then the vector  $\nabla_g \widehat{\Psi}_1(\mathbf{q})$  in  $\mathcal{Q}$  is horizontal and the motions on  $\mathcal{Q}$  defined by

$$\frac{d\mathbf{Q}}{dt} = \hbar \operatorname{Im} \frac{\nabla_g \widehat{\Psi}_1}{\widehat{\Psi}_1} \quad (39)$$

are horizontal lifts of motions on  $\mathcal{Q}$ . So, in the quantum case, horizontality is immediate.

Let us now consider the time evolution of the lifted wave function  $\widehat{\Psi}_1$  on  $\mathcal{Q}$ . Let  $\widehat{\Delta}_B$  be a lift to absolute configuration space of the Laplace-Beltrami operator  $\Delta_B$  on shape space, namely an operator on  $\mathcal{Q}$  such that

$$\widehat{\Delta}_B \widehat{\Psi}_1 = \Delta_B \Psi. \quad (40)$$

Then

$$i\hbar \frac{\partial \widehat{\Psi}_1}{\partial t} = \widehat{H}_1 \widehat{\Psi}_1, \quad \text{with} \quad \widehat{H}_1 = -\frac{\hbar^2}{2} \widehat{\Delta}_B. \quad (41)$$

It might seem natural to guess that  $\widehat{\Delta}_B$  coincides with  $\Delta_g$ , the Laplace-Beltrami operator with respect to  $g$ , but this is wrong; nor is  $\widehat{H}_1$  a familiar sort of Schrödinger Hamiltonian, with or without a potential term.

While  $\widehat{\Psi}_1$  need not obey any familiar Schrödinger-type equation, one may ask whether there exists a gauge equivalent wave function that does. By gauge equivalent wave function we mean this: If one writes  $\widehat{\Psi}_1$  as  $Re^{(i/\hbar)S}$  one sees that the velocity field given by (39) is just  $\nabla_g S$ , so transformations of the wave function

$$\widehat{\Psi}_1 \rightarrow \widehat{\Psi}'_1 = F \widehat{\Psi}_1, \quad (42)$$

where  $F$  is a positive function, do not change its phase and thus the velocity.

It turns out that there exists a positive function  $F$  such that  $\widehat{\Psi}_3 = F \widehat{\Psi}_1$  satisfies a Schrödinger-type equation on absolute configuration space for a suitable potential  $V$ , namely,

$$i\hbar \frac{\partial \widehat{\Psi}_3}{\partial t} = \widehat{H}_3 \widehat{\Psi}_3 \quad (43)$$

with

$$\widehat{H}_3 = -\frac{\hbar^2}{2} \sum_{\alpha=1}^N \vec{\nabla}_\alpha \cdot \frac{1}{f m_\alpha} \vec{\nabla}_\alpha + V \quad (44)$$

$$= -\frac{\hbar^2}{2} \nabla \cdot \frac{1}{f} \nabla + V, \quad (45)$$

where  $\nabla$  and  $\nabla \cdot$  are the gradient and divergence with respect to the mass-weighted Euclidean metric (14), i.e.,

$$\nabla = \left( \frac{1}{m_1} \vec{\nabla}_1, \dots, \frac{1}{m_N} \vec{\nabla}_N \right) \quad (46)$$

and

$$\nabla \cdot = \left( \vec{\nabla}_1, \dots, \vec{\nabla}_N \right) \cdot \cdot \quad (47)$$

Here  $f$  is the conformal factor, and

$$V = V_1 + V_2 \quad (48)$$

$$V_1 = -\frac{\hbar^2}{2} \frac{\widehat{\Delta}_B J^{1/2}}{J^{1/2}} \quad (49)$$

$$V_2 = -\frac{\hbar^2}{2} f^{\frac{n}{4}} \Delta_g (f^{-\frac{n}{4}}), \quad (50)$$

with

$$J = L f^{7/2} \sqrt{\det \mathbf{M}}, \quad (51)$$

where  $L = L(\mathbf{q})$  is given by equation (21) and  $\mathbf{M} = \mathbf{M}(\mathbf{q})$  is the tensor of inertia of the configuration  $\mathbf{q}$  about any orthogonal cartesian system  $x, y, z$  with origin in its center of mass and with matrix elements given by (24), and where  $\widehat{\Delta}_B$  is the canonical lift of  $\Delta_B$  (see [1] for details on this and a proof of the aforementioned statements).

There is a 2-gauge, intermediate between the 1-gauge and the 3-gauge, that we shall not describe further here.

### 6.2.2 The Schrödinger gauge

We shall now describe what we think is appropriate to be called the *Schrödinger gauge*, the true quantum analogue of the Newton gauge. If we take into account that

time is relational, as we should, the fundamental equation for the wave function on shape space is presumably the stationary equation

$$-\frac{\hbar^2}{2}\Delta_B\Psi = \mathcal{E}\Psi, \quad (52)$$

where  $\mathcal{E}$  is any given fixed constant (for example  $\mathcal{E} = 0$ ).

As before, let  $\widehat{\Psi}_1$  be the lift of  $\Psi$  to  $\mathcal{Q}$ , so that  $\widehat{\Psi}_1$  satisfies the equation

$$(\widehat{H}_1 - \mathcal{E})\widehat{\Psi}_1 = 0, \quad (53)$$

with  $\widehat{H}_1$  a lift of  $H$  as in (41), and the evolution on the absolute configuration space  $\mathcal{Q}$  is still given by (39). But now, for relational time, motions following the same path with different speeds are the same motion. So, in the formula for the gradient on the right hand side of (39),  $\nabla_g = f^{-1}\nabla$ , we may regard  $f$  as a change of speed defining a new time variable that for the sake of simplicity we shall still call  $t$  (a “time change”). Then in absolute space the guiding equation (39) becomes

$$\frac{d\vec{Q}_\alpha}{dt} = \frac{\hbar}{m_\alpha} \text{Im} \frac{\vec{\nabla}_\alpha \widehat{\Psi}_1}{\widehat{\Psi}_1}. \quad (54)$$

Again,  $\widehat{\Psi}_1$  need not obey any familiar stationary Schrödinger-type equation. However, as before, we may exploit gauge freedom to transform (53) into a stationary Schrödinger-type equation. Indeed, we now have an even greater gauge freedom in changing the wave function and the Hamiltonian, see [1]. In particular, there is a gauge, the Schrödinger gauge, in which (52) becomes

$$\widehat{H}_S\Phi = 0 \quad (55)$$

with

$$\widehat{H}_S = -\frac{\hbar^2}{2}\nabla^2 + U, \quad (56)$$

where  $\nabla^2 = \nabla \cdot \nabla$  is the mass-weighted Euclidean Laplacian, and

$$U = f(V_1 - \mathcal{E}) - \frac{\hbar^2}{8} \frac{n-2}{n-1} f R_g, \quad (57)$$

where  $R_g$  is the scalar curvature of the invariant metric  $g$  and  $n = 3N$ .

## 7 Lessons from Relational Physics

In this closing section we shall further consider what we can learn from relational physics.

### 7.1 Local Beables and Primitive Ontology

We have stressed elsewhere the importance of certain local beables, what we call the primitive ontology of a theory—stuff in space evolving in time—which is what the theory is fundamentally about. Relational physics requires a more flexible understanding of this notion.

To appreciate this point, note that for relational space the state of the universe at a particular location is not, in and of itself, meaningful. In that sense, for relational physics, there are no local beables in the usual sense, and locality itself can't easily be meaningfully formulated.

Similarly one can't meaningfully consider the behavior of individual particles without reference to other particles, since there is no absolute space in which an individual particle could be regarded as moving. And even for a pair of particles, to speak meaningfully of the distance between them, a third particle would be required, to establish a scale of distance. And similarly for galaxies.

Relational physics is, in fact, genuinely holistic, and suggests the holistic character of quantum physics associated with entanglement and quantum nonlocality.

### 7.2 Projectivity

Compare the guiding equation (3) with the same equation but with the denominator  $\Psi^*\Psi$  omitted. The latter simpler equation might be regarded as in some sense more fundamental, with (3) itself regarded as arising from a convenient choice of gauge corresponding to the freedom of time-change associated with relational time.

The time-parameter corresponding to the use of the denominator in (3) has the nice feature that the dynamics using that time-parameter depends on fewer details of the wave function than would be the case if the denominator were deleted: with the denominator the dynamics depends only on the ray of  $\Psi$ , with  $\Psi$  and  $c\Psi$  yielding the same dynamics for any constant  $c \neq 0$ . This has a particularly nice implication for the behavior of subsystems.

With this choice of time-parameter the dynamics for a subsystem will often not depend upon the configuration of its environment, with the subsystem evolving according to an autonomous evolution involving only the configuration and the (condi-

tional) wave function of the subsystem itself [11]. This would happen when the subsystem is suitably decoupled from its environment, for example for a product wave function when there is no interaction between system and environment. Without the denominator this would not be true, and there would appear to be an additional nonlocal dependence of the behavior of a subsystem on that of its environment that would not be present with a time-parameter associated with the use of the usual denominator.

### 7.3 Non-Normalizability of the Wave function

Note that since it is translation and scaling invariant, the wave function in the Schrödinger gauge or in any of the gauges discussed in Sect.6.2) are not normalizable. In other words, the corresponding  $|\widehat{\Psi}|^2$  measures are non-normalizable. However, since the non-normalizability arises from unobservable (and, from a shape space point of view, unphysical) differences and dimensions it should somehow not be a problem.

Nonetheless, the real question is how the empirical distributions arising from the fundamental shape space level are related to those coming from the physics in a gauge. While the different gauges, such as the Schrödinger gauge, correspond to theories that, we argued, are empirically equivalent to the fundamental shape space theory, that was only in purely dynamical terms. We have not yet addressed the possible differences in empirical distributions that may arise. We would like to see that they don't, i.e., that probabilities naturally associated with relational physics on shape space yield familiar Born-rule probabilities when the physics is lifted to absolute configuration space.

There are several considerations that suggest that the non-normalizability should not be a genuine problem. First of all, as just mentioned, the non-normalizability arises only from non-observable dimensions, suggesting that it should be physically irrelevant. Moreover, it is the universal wave function  $\widehat{\Psi}$  (in any of the gauges) that is not normalizable. But the universal wave function is rarely used in practice. In quantum mechanics we usually deal, not with the entire universe, but with small subsystems of the universe. The wave functions with which we usually deal are thus conditional wave functions, and there seems to be no reason why these should fail to be normalizable.

As already said, non-normalizable wave functions tend to occur in quantum cosmology. Such wave functions would normally be regarded as problematical and unphysical (since the formal structures of orthodox quantum mechanics, with their associated probabilities, are crucially based on the notion of a Hilbert space of square-

integrable, i.e. normalizable, wave functions). However, for our analysis in [1] the nonnormalizability of the measure on configuration space turned out to be crucial for its success (see below). Hence what from an orthodox perspective is a vice is transformed into a virtue in relational Bohmian mechanics.

## 7.4 Conditional Wave Function and Path Space

To substantiate the last statement we mention a few ingredients of our analysis in [1], without any pretension of completeness.

### 7.4.1 The Conditional Wave Function

The very notion of conditional wave function (5) presupposes the possibility of expressing the configuration  $Q$  of the universe as a pair  $(X, Y)$  consisting of the configuration  $X$  of a system, the  $x$ -system, and the configuration  $Y$  of its environment (the  $y$ -system). The holistic character of shape space physics emphasized above does not allow for this, that is, there is no natural product structure

$$\mathcal{Q} = \mathcal{Q}_{sys} \times \mathcal{Q}_{env}$$

for shape space: Here the system is a collection of (labelled) particles with its own shape space  $\mathcal{Q}_{sys} = \mathcal{X}$ , the set of possible shapes  $X$  of the system, and the environment consists of the rest of the particles of the universe, with shape space  $\mathcal{Q}_{env} = \mathcal{Y} = \{Q_{env} = Y\}$ , with  $Y$  the shape associated with the particles (labels) of the environment. The crucial fact is that it is not true that

$$\mathcal{Q} = \mathcal{X} \times \mathcal{Y}.$$

$X$  and  $Y$  don't involve sufficient information to determine the complete shape  $Q$ . What is missing is the spatial relationship between these shapes.

Nonetheless we have that  $\mathcal{Q}$  can be identified with  $\mathcal{X}_Y \times_y \mathcal{Y} = \{(X, Y) | Y \in \mathcal{Q}_{env}, X \in \mathcal{Q}_Y\}$ , where  $\mathcal{Q}_Y = \{Q \in \mathcal{Q} | Q_{env} = Y\}$ . We may then define the conditional wave function for the subsystem, for  $Y \in \mathcal{Q}_{env}$  and universal wave function  $\Psi$ , by

$$\psi(x) = \Psi(x, Y), \quad x \in \mathcal{Q}_Y. \quad (58)$$

This looks like the usual conditional wave function, but it is important to bear in mind that, unlike with the usual conditional wave function, here  $x$  represents the shape of the universe for a fixed  $Y$  and there are no obvious natural coordinates to efficiently describe it.

The conditional wave function (58) can be lifted to absolute configuration space and one has the formula

$$\widehat{\psi}(\mathbf{x}) = \widehat{\Psi}(\mathbf{x}, \mathbf{Y}), \quad (59)$$

Here the lifts  $\mathbf{x}$  of  $x$  and  $\mathbf{Y}$  of  $Y$  involve something akin to the choice of a frame of reference in absolute space, and the “hat” on  $\Psi$  refers to any of the gauge equivalent representations of the universal wave function that we have described in Section 6.2.

#### 7.4.2 The Conditional Probability Formula

One might wonder whether something analogous to (6) holds true. The answer is yes, but with several provisos.

It turns out that in the gauges 1 and 3 described in Section 6.2, indeed something similar holds. More precisely, in the 3-gauge the conditional distribution is exactly given by the modulus square of the conditional wave function (59) as in (6). The proof of this is rather intricate and lengthy[1].

In the Schrödinger gauge, however, a conceptual issue arises as to exactly what of physical significance this conditional distribution represents. After all, the transition to the Schrödingergauge required relational time, but if we take relational time seriously, what is physical is not the configuration  $Q_t$  of the universe at some time  $t$ , but the geometrical path of the full history of the configuration, with no special association of the configurations along a path with times. In this (more physical) framework, the conditional distribution of the configuration  $X_t$  of a subsystem given the configuration  $Y_t$  of its environment is not meaningful.

What is meaningful is (i) a probability distribution  $\mathbb{P}^\Psi$  on the space  $\mathcal{P}$  of (geometrical, i.e. unparametrized) paths and (ii) the conditional distribution relative to  $\mathbb{P}$  of the configuration  $X_Y$  of the subsystem when the path  $\gamma \in \mathcal{P}$  has environmental configuration  $Y$ , *given* that the path passes through a configuration with environment  $Y$ ,  $Y \in \gamma$ . In this more general framework, one arrives at a conditional probability formula analogous to (6), namely,

$$\mathbb{P}^\Psi(X_Y \in dx | Y \in \gamma) = |\widehat{\psi}_S(\mathbf{x})|^2 d\mathbf{x}, \quad (60)$$

where the subscript  $S$  refers to the Schrödinger gauge. Also the proof of this theorem is rather intricate and lengthy. An assumption of this theorem is what we have called in [1] the “existence of clock variables” for the Bohmian dynamics, which a mild assumption for a sufficiently structured universe.



### 7.4.3 Non-Normalizable Measures

A crucial ingredient of our analysis is the connection between a measure on path space and a (stationary) non-normalizable measure on configuration space. By the very nature of this connection the measure on configuration space and its associated wave function in fact has to be non-normalizable—just to rehearse the point made at the end of Section 7.3 that the nonnormalizability of the universal wave function is good!

### 7.4.4 The Physics of Sub-Systems

On the absolute configuration space level the dynamics and the probabilities for subsystems should be of the usual form. While it is true that on the universal level the connection between  $|\Psi|^2$  and probability, or, more precisely, typicality, would be broken, this would not be visible in any of the familiar every-day applications of quantum mechanics, which are concerned only with subsystems and not with the entire universe. This is the key to approach the problem of entropy increase raised in Section 3.1.

### 7.4.5 $|\psi|^2$ vs. $|\Psi|^2$

The patterns described by the quantum equilibrium hypothesis will be typical with respect to a measure, not on absolute configuration space, but on shape space, on the fundamental level, which is fine. There is a widespread misconception with respect to Bohmian mechanics that  $|\Psi|^2$  for the universe and  $|\psi|^2$  for subsystems play, physically and conceptually, similar roles. They do not, since the role of  $|\Psi|^2$  is typicality while that of  $|\psi|^2$  is probability. If this distinction is too subtle, the fact that, from a relational perspective, these objects live on entirely different levels of description,  $|\Psi|^2$  on the fundamental level, i.e., on shape space, and  $|\psi|^2$  on absolute configuration space, might make it easier to appreciate how very different they are.

## 7.5 More Relational Points

### 7.5.1 Identical Particles

There is one rather conspicuous relational aspect that we've ignored. For indistinguishable particles we should have taken one further quotient and enlarged the similarity group  $G$  to include the relevant permutations of particle labels. Presumably, this would not have any relevant implications in the classical case, but in the

quantum case would lead to the correct quantum description of the dynamics of bosonic and fermionic shapes [16].

### 7.5.2 Different Ontologies

We do not regard particles as the inevitable ultimate building blocks of the universe. Far from it. Our insistence on particles is mostly methodological, not metaphysical. The point is that the particle ontology already suffices to highlight the relevant conceptual issues of relational physics. Our analysis—in particular that concerning the wave function of the universe, subsystems, and probability—really depends only on rather general qualitative features of the structure of quantum relational physics, not on the details of any specific ontology.

### 7.5.3 Geometry

We have assumed that the geometrical relations among the particles are those of Euclidean geometry. But we know that the geometry of space need not be Euclidean and may not be fixed at all, but be dynamically determined by the matter content of the universe.

One is thus naturally lead to geometrodynamics, the dynamics of 3-geometries. According to John Archibald Wheeler, superspace, the totality  $\mathcal{Q}$  of 3-geometries, should be regarded as the true arena of general relativity (see, e.g., [17]).

In many respects, superspace is analogous to shape space. The totality  $\mathcal{Q}$  of Riemannian metrics on 3-space is analogous to absolute configuration space, and the group  $G$  of diffeomorphisms of 3-space is analogous to the similarity group of Euclidean space. The fiber bundle structure  $\mathcal{Q} = \mathcal{Q}/G$  (modulo singularities) has been highlighted by Bryce DeWitt [10].

Now DeWitt in [10] and many others after him have regarded four-dimensional spacetime as fundamental and the dynamics of 3-geometries as gauge, arising from the gauge freedom given by the so-called lapse and shift functions. However, one could take the opposite view and consider the 3-geometry as fundamental (analogously to a shape of particles) and four-dimensional spacetime as arising in a suitable gauge from a timeless history of the universe, i.e., a path, in superspace. This view, that has been explored by Barbour and collaborators [5], leads to the conclusion that relativity (general and special) need not be regarded as fundamental, but can be regarded as arising from a suitable choice of gauge. Again we encounter here a striking manifestation of a Kantian component of physical laws.

#### 7.5.4 Larger Groups

For the case of an  $N$ -particle universe, we have already indicated that a natural extension is to include in  $G$  the relevant permutations of particle labels. Are there other natural extensions? How far can the gauge group  $G$  be extended?

In the case of 3-geometries, Barbour and collaborators have explored the possibility of extending the group of diffeomorphism to include the conformal transformations of the 3-geometries. One arrives in this way at a conformal superspace [4, 13, 15]. They perform some massaging to arrive at general relativity (in a suitable gauge). But maybe there is no need of any massaging, nor any need of arriving exactly at general relativity—to recover it as an approximation of a more fundamental theory is sufficient.

### 7.6 The Quantum Gauge?

Is there a gauge that could be called the *quantum gauge*? We do not know the answer, but we find that the question itself is very interesting. It is related to the “why”-question we asked at the end of Section 3.3.

Let us explain what we mean. We started with the Bohmian motion on shape space governed by the universal wave function  $\Psi$ . Everything we discussed in the quantum case had as a basic ingredient the universal wave function  $\Psi$ . But one should contemplate the possibility of starting with something that is not really quantum, not a motion guided by a suitable  $\Psi$ .

Suppose we have a motion in shape space that is not of the usual Bohmian form, with no wave function  $\Psi$  involved in defining the law of motion of the configuration. In other words, suppose we discover a natural motion on shape space that is not quantum at all. But suppose also that when one takes into account the freedom of time change and the freedom of how the motion is lifted to absolute space, that is, that when one exploits all the gauge freedom we have discussed above, it turns out that the law of motion can indeed be cast into the usual Bohmian form, with a wave function guiding it in the usual way. If that were the case, quantum mechanics, while entirely absent from the fundamental (shape space) level, would arise as a choice of gauge in absolute configuration space. If this were the case, we would have another instance, maybe the most surprising, of what we have called a Kantian component of physical laws—that much of what we take as fundamental is not fundamental, but imposed by us by a proper choice of gauge.

In view of this—which at the moment is very speculative—and in view of what has been said before about relativity—much less speculative—it seems to us not unreasonable to entertain the following possibility: that quantum mechanics and relativity,

the two great revolutions of twentieth century physics, may not be fundamental at all but imposed on nature by us.

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