

Approach to Hyperuniformity in the One-Dimensional Facilitated Exclusion Process

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Abstract

For the one-dimensional Facilitated Exclusion Process with initial state a product measure of density $\rho = 1/2 - \delta$, $\delta \geq 0$, there exists an infinite-time limiting state ν_ρ in which all particles are isolated and hence cannot move. We study the variance $V(L)$, under ν_ρ , of the number of particles in an interval of L sites. Under $\nu_{1/2}$ either all odd or all even sites are occupied, so that $V(L) = 0$ for L even and $V(L) = 1/4$ for L odd: the state is *hyperuniform* [21], since $V(L)$ grows more slowly than L . We prove that for densities approaching $1/2$ from below there exist three regimes in L , in which the variance grows at different rates: for $L \gg \delta^{-2}$, $V(L) \simeq \rho(1 - \rho)L$, just as in the initial state; for $A(\delta) \ll L \ll \delta^{-2}$, with $A(\delta) = \delta^{-2/3}$ for L odd and $A(\delta) = 1$ for L even, $V(L) \simeq CL^{3/2}$ with $C = 2\sqrt{2/\pi}/3$; and for $L \ll \delta^{-2/3}$ with L odd, $V(L) \simeq 1/4$. The analysis is based on a careful study of a renewal process with a long tail. Our study is motivated by simulation results showing similar behavior in higher dimensions; we discuss this background briefly.

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1 Introduction

In the Facilitated Exclusion Process (FEP) on \mathbb{Z}^d , also known as the Conserved Lattice Gas process, each site of the lattice can be occupied by at most one particle, so that a configuration η is an element of the configuration space $X = \{0, 1\}^{\mathbb{Z}^d}$. At Poisson-distributed times a particle chooses one of its nearest-neighbor sites at random and attempts to jump to it, succeeding only if the target site is unoccupied and the original site has at least one occupied (*facilitating*) neighbor. (Variations of this dynamics, with simultaneous updating or with some other rule for choosing the target site, have also been considered [13].) Note that when $d = 1$ the need for a facilitating particle implies that no choice step is necessary: a particle can jump in at most one direction. We will always assume that the system is started in a Bernoulli initial state $\mu_0^{(\rho)}$ of density $\rho < 1$, that is, a product measure in which the $\eta(x)$ are independent and take value 1 with probability ρ . The evolved state at time t will then be denoted $\mu_t^{(\rho)}$; it is clearly translation invariant (TI).

The evolution of this system, or of minor variations of it, has been investigated for $d = 1$ [1–4, 7, 8, 10–13, 17, 22], primarily theoretically, and for $d \geq 2$ [13, 14, 16, 19], primarily via simulation in a cubical box with periodic boundary conditions. These investigations strongly suggest, for $d \geq 2$, and prove, for $d = 1$, the existence of a TI limiting state (first described for $d = 1$ in [22])

$$\nu_\rho := \lim_{t \rightarrow \infty} \mu_t^{(\rho)}. \quad (1)$$

Moreover, there appears to be a *critical density* ρ_c such that if $\rho \leq \rho_c$ then ν_ρ is a *frozen* state in which all particles are isolated and hence unable to move, while if $\rho > \rho_c$ then ν_ρ is an *active* stationary state in which there is a finite density of particles with an occupied neighboring site. Necessarily $\rho_c \leq 1/2$, since for $\rho > 1/2$ it is geometrically impossible for all the particles to be isolated, and indeed equality holds for $d = 1$. But for $d \geq 2$, simulations suggest values of ρ_c which are much smaller, for example, $\rho_c \approx 0.3308$ for $d = 2$.

Our main interest here will be in *fluctuations* in the measure ν_ρ , that is, in the variance $V_\rho(L) := \text{Var}_{\nu_\rho}(N(L))$ of the number $N(L)$ of particles in a cubical box of side L . In general, if such a variance computed from a TI measure μ grows as L^d when $L \nearrow \infty$, we say that μ has *normal fluctuations*. Hexner and Levine [14] observe that, in 2d and 3d, ν_{ρ_c} is not of this

character but rather is *hyperuniform* [21]: $V_{\rho_c}(L)$ grows more slowly than L^d , specifically, $V_{\rho_c}(L) \sim L^{\lambda_1}$, with $\lambda_1 \approx 1.57$ in 2d and $\lambda_1 \approx 2.76$ in 3d [14].

Hexner and Levine also discuss the behavior of $V_\rho(L)$ as $\rho \nearrow \rho_c$. Further consideration of this behavior has led us [13] to the following conjecture, which we state for $d \geq 2$; the $d = 1$ version is Theorem 1 below. We introduce the notation $\delta = \rho_c - \rho > 0$.

Fluctuation Conjecture. For the FEP with $d \geq 2$ a critical density ρ_c as described above exists, and ν_{ρ_c} is hyperuniform. For ρ less than but close to ρ_c , three regimes in L may be identified. For *small* L (but still with $L \gg 1$) the variances grow approximately as in the hyperuniform state at ρ_c : $V_\rho(L) \simeq C_1 L^{\lambda_1}$. At some (approximately defined) scale $L_1(\delta)$ the variances enter the regime of *intermediate* L , in which they grow as $V_\rho(L) \simeq C_2(\delta) L^{\lambda_2}$ with $\lambda_2 > d > \lambda_1$ and $C_2(\delta) > 0$. Then above an (approximate) scale $L_2 = L_2(\delta)$ the growth is as $V_\rho(L) \simeq \rho(1 - \rho)L^d$, that is, exactly as in the initial Bernoulli measure $\mu_0^{(\rho)}$. Finally, as $\rho \nearrow \rho_c$, $L_1(\delta)$ and $L_2(\delta)$ increase as $L_i \sim \delta^{-\gamma_i}$ for some exponents γ_1, γ_2 satisfying $\gamma_2 > \gamma_1 > 0$.

In the remainder of the paper we restrict our consideration to the $d = 1$ model. In Section 2 we state in Theorem 1 our main result, the one-dimensional version of the conjecture, and in Section 3 we describe the 1d limiting measure ν_ρ . The proof of Theorem 1 is given in Section 4.

2 Statement of the result

The key to the rigorous establishment of a version of the Fluctuation Conjecture in dimension $d = 1$ is that there the existence and exact value of the critical density are known— $\rho_c = 1/2$ —and that we also have a complete description of the limiting measure ν_ρ for $\rho \leq 1/2$ [1, 10, 11, 22]. (This measure was first identified and discussed in [22], a reference which has just come to our attention. We regret that we did not properly credit this work in earlier papers.) We will discuss ν_ρ for $\rho < 1/2$ in Section 3; for the moment let us note that the measure at the critical density is particularly simple:

$$\nu_{1/2} = \frac{1}{2}(\delta_{\eta^*} + \delta_{\eta^\dagger}), \quad (2)$$

where η^* and η^\dagger are the two configurations in $X = \{0, 1\}^{\mathbb{Z}}$ in which holes and particles strictly alternate.

The Fluctuation Conjecture concerns the asymptotic behavior of the quantity $V_\rho(L)$ with L “small,” “intermediate,” or “large,” yet in each case also sufficiently large. To give a precise result in 1d we introduce some notation for the description of this behavior.

Definition 1. Assume that L is a positive integer and δ a positive real number (in applications we will have $\delta = \rho_c - \rho$), that $A(\delta, L)$ and $B(\delta, L)$ are real valued functions whose asymptotic behavior in L we wish to compare, and that $L_1(\delta)$ and $L_2(\delta)$ are positive functions (these play the role of setting the scales of the various regions). Then we write respectively

$$A(\delta, L) \simeq B(\delta, L) \quad \text{for} \quad \begin{cases} L \ll L_1(\delta), \\ L_1(\delta) \ll L \ll L_2(\delta), \\ L \gg L_2(\delta), \end{cases} \quad (3)$$

if for any $\epsilon > 0$ there exists a $\delta_0 > 0$, together with a (small) number $s > 0$ and/or a (large) number $l > 0$, such that for $\delta < \delta_0$ we have, respectively,

$$1 - \epsilon < \frac{A(\delta, L)}{B(\delta, L)} < 1 + \epsilon \quad \begin{cases} \text{for } L < sL_1(\delta), \\ \text{for } lL_1(\delta) < L < sL_2(\delta), \\ \text{for } L > lL_2(\delta). \end{cases} \quad (4)$$

If A and B depend also on some additional parameter(s) α we say that (3) holds *uniformly* for α in some (possibly L - or δ -dependent) set if s and/or l may be chosen so that (4) holds for all such α .

With this notation established we may state our main result; we assume that $0 < \rho < 1/2$, that $\delta = 1/2 - \rho$, and that ν_ρ is the measure (1) of the 1d FEP. In contrast to the $d \geq 2$ behavior described in the the Fluctuation Conjecture, for $d = 1$ the variances $V_\rho(L)$ behave differently for L odd and L even.

Theorem 1. *Let $N^{(\delta)}(L)$ be the number of particles on the sites $1, 2, \dots, L$, with distribution determined by ν_ρ . Then:*

(a) *For L odd,*

$$V_\rho(L) = \text{Var}(N^{(\delta)}(L)) \simeq \begin{cases} \frac{1}{4}, & \text{for } L \ll \delta^{-2/3}, \\ \frac{2}{3} \sqrt{\frac{2}{\pi}} \delta L^{3/2}, & \text{for } \delta^{-2/3} \ll L \ll \delta^{-2}, \\ \frac{1}{4} L, & \text{for } L \gg \delta^{-2}. \end{cases} \quad (5)$$

(b) For L even,

$$V_\rho(L) = \text{Var}(N^{(\delta)}(L)) \simeq \begin{cases} \frac{2}{3}\sqrt{\frac{2}{\pi}}\delta L^{3/2}, & \text{for } 1 \ll L \ll \delta^{-2}, \\ \frac{1}{4}L, & \text{for } L \gg \delta^{-2}. \end{cases} \quad (6)$$

We actually have a stronger result for the asymptotics in the large- L region. The “right” estimate for $\text{Var}(N^{(\delta)}(L))$ there is $\rho(1-\rho)L$, as stated in the Fluctuation Conjecture and as we discuss further in Remark 1 below, and this is true for all, not just small, δ :

Corollary 2 (to the proof of Theorem 1). *For any $\epsilon > 0$ there exists an $l > 0$ such that*

$$1 - \epsilon < \frac{\text{Var}(N^{(\delta)}(L))}{\rho(1-\rho)L} < 1 + \epsilon \quad \text{for } L > l\delta^{-2}. \quad (7)$$

A comparison of Theorem 1(a) with the Fluctuation Conjecture shows that for L odd the behavior of $V_\rho(L)$ in one dimension corresponds directly to the conjectured behavior in higher dimension (but without the condition $L \gg 1$). In particular it follows from (2) that

$$V_{1/2}(L) = \begin{cases} \frac{1}{4}, & \text{if } L \text{ is odd,} \\ 0, & \text{if } L \text{ is even,} \end{cases} \quad (8)$$

which implies that $\nu_{1/2}$ is hyperuniform and also explains the $V_\rho(L) \simeq 1/4$ behavior in Theorem 1 for small odd L . The variables introduced in the conjecture become $\lambda_1 = 0$, $C_1 = 1/4$, $\lambda_2 = 3/2$, $C_2(\delta) = \sqrt{8/\pi}\delta/3$, $\gamma_1 = 2/3$, and $\gamma_2 = 2$. On the other hand, for even L the “small” growth region is absent in one dimension: for small and moderate values of L the variances grow as $C_2(\delta)L^{3/2}$. This odd/even distinction may be regarded as a legacy of (8) when δ is perturbed away from 0.

Remark 1. Corollary 2 certainly implies that $\lim_{L \rightarrow \infty} V_\rho(L)/L = \rho(1-\rho)$ for all $\rho < 1/2$, and this part of the result, although not the scale δ^{-2} at which the limit is achieved, may be obtained by an elementary argument [11]. For with probability 1 each particle will move only a finite distance during the evolution, so that for L sufficiently large $N(L)$ will, to high relative accuracy, be the same at the end of the evolution as it was at the beginning, and $\text{Var}(N(L))$ will be the same as for the original Bernoulli measure.

Remark 2. There are several one-dimensional models with exclusion and facilitation, closely related to the FEP, for which also $\rho_c = 1/2$ and for which $\lim_{t \rightarrow \infty} \mu_t^{(\rho)}$ is for $\rho \leq 1/2$ the measure ν_ρ that we are considering here, and hence for which the fluctuations $V_\rho(L)$ satisfy Theorem 1. In particular, this is true of the totally asymmetric, discrete-time (parallel) updating in which all particles attempt to jump at the same time, and only to the right [11]. It is also true of an asymmetric version of the continuous-time model of Section 1 in which particles attempt to jump to the left or right at different rates [1].

3 The limiting measure ν_ρ for $\rho < 1/2$

A key ingredient for understanding the behavior described in Theorem 1, especially the behavior in the intermediate regime, is the renewal structure of the stationary state ν_ρ , $\rho < 1/2$. In this state adjacent 1's have probability zero; thus the state is supported on configurations of the form

$$\dots 1 0 1 0 1 0 1 0 \hat{0} 1 0 1 0 \dots 1 0 \hat{0} \hat{0} \hat{0} 1 0 1 0 1 0 \dots 1 0 \hat{0} \dots \quad (9)$$

$$= \dots 0 (1 0)^{X_{-1}} 0 (1 0)^{X_0} 0 (1 0)^{X_1} 0 (1 0)^{X_2} 0 \dots \quad (10)$$

For ν_ρ the 00's in (9)—or more specifically the second 0 of each such pair, marked as $\hat{0}$ in (9) and corresponding to a 0 outside the parentheses in (10)—are *renewal events*, as is shown in [11, 22]. (Note that adjacent renewal events correspond to a zero value for the corresponding X_i .) This means that, if we let $\hat{\nu}_\rho$ be the measure ν_ρ conditioned on the occurrence of a renewal event at the origin, then under $\hat{\nu}_\rho$ the X_i 's in (10) are independent random variables that are identically distributed.

Remark 3. The density of the renewal events, i.e., the probability of finding adjacent zeros at, say, sites 0 and 1, is $1 - 2\rho = 2\delta$, since under ν_ρ the probability of adjacent ones is zero.

It is shown in [11, 22] that the distribution of the X_i 's under $\hat{\nu}_\rho$ is that of a random variable $\hat{X}^{(\delta)}$ for which

$$\begin{aligned} P(\hat{X}^{(\delta)} = n) &= C_n \rho^n (1 - \rho)^{n+1} \\ &= \frac{1 + 2\delta}{2 \cdot 4^n} C_n (1 - 4\delta^2)^n, \quad n = 0, 1, 2, \dots, \end{aligned} \quad (11)$$

with C_n the n^{th} Catalan number [20]:

$$C_n := \frac{1}{n+1} \binom{2n}{n} = \frac{4^n}{n^{3/2}\sqrt{\pi}} \left(1 + O\left(\frac{1}{n}\right)\right). \quad (12)$$

Here we have used Stirling's formula with error bounds. Thus for $n \gg 1$,

$$P(\widehat{X}^{(\delta)} = n) \simeq \frac{1+2\delta}{2n^{3/2}\sqrt{\pi}} (1-4\delta^2)^n \simeq \frac{1}{2n^{3/2}\sqrt{\pi}} e^{-4\delta^2 n}. \quad (13)$$

(The Catalan number C_n arises here as the number of random walks of length $2n$, with steps ± 1 , which begin and end at the origin and take only nonnegative values.)

While $\nu_{1/2} = \lim_{\delta \rightarrow 0} \nu_\rho$ is rather trivial (see (2)), and in particular contains no renewal events, the limit $\hat{\nu}_{1/2} = \lim_{\delta \rightarrow 0} \hat{\nu}_\rho$ is not: $\hat{\nu}_{1/2}$ is the probability distribution on configurations of the form (10) for which there is a renewal event at the origin and the i.i.d. random variables X_i have the distribution of $\widehat{X}^{(0)}$:

$$P(\widehat{X}^{(0)} = n) = \lim_{\delta \rightarrow 0} P(\widehat{X}^{(\delta)} = n) = \frac{C_n}{2 \cdot 4^n} \simeq \frac{1}{2\sqrt{\pi}n^{3/2}}. \quad (14)$$

Note that $\widehat{X}^{(0)} = \lim_{\delta \rightarrow 0} \widehat{X}^{(\delta)}$ (limit in distribution); note also that although $\hat{\nu}_\rho$ for $\rho < 1/2$ was obtained from ν_ρ by conditioning on a renewal event at the origin, $\hat{\nu}_{1/2}$ cannot be so obtained from $\nu_{1/2}$.

$\widehat{X}^{(0)}$ has a $3/2$ power-law tail, and this $3/2$ is, as we shall show, the origin of the $3/2$ in the $L^{3/2}$ behavior of the variance in the intermediate regime. (If $3/2$ were replaced by γ , with $1 < \gamma \leq 2$, we would have had L^γ behavior there [13]). Further, the fact that the exponential decay in (13) becomes significant when n is of order δ^{-2} is the origin of the fact that the transition to the large L regime occurs for L of order δ^{-2} .

Notation: Here, for the reader's convenience, we summarize our notation, reviewing some that was introduced earlier and also defining some new notation that will be used in the sequel. We write $\rho = 1/2 - \delta$, with $0 \leq \delta < 1/2$, define $J_L = \{1, 2, \dots, L\}$, and call the second of a pair of consecutive empty sites a renewal event. ν_ρ denotes the infinite-time limit state (1) for the one-dimensional FEP at density ρ , and $\hat{\nu}_\rho$ the state defined for $\delta > 0$ by conditioning ν_ρ on the occurrence of a renewal event at the origin, and for $\delta = 0$ as $\lim_{\delta \searrow 0} \hat{\nu}_\rho$. We will use the following random variables:

- $N^{(\delta)}(L)$, the number of particles in J_L under ν_ρ ;
 - $N_{\text{ren}}^{(\delta)}(L)$ and $\widehat{N}_{\text{ren}}^{(\delta)}(L)$, the number of renewal events in J_L under ν_ρ and $\hat{\nu}_\rho$, respectively;
 - $\widehat{X}^{(\delta)}$, a random variable with the distribution under $\hat{\nu}_\rho$ of the X_i in (10);
 - $\widehat{Y}^{(\delta)} = 2\widehat{X}^{(\delta)} + 1$, the distance between renewal events under $\hat{\nu}_\rho$.
- Note that from Remark 3,

$$E(N_{\text{ren}}^{(\delta)}(L)) = 2\delta L. \quad (15)$$

Note also that while $N_{\text{ren}}^{(0)}(L) = \lim_{\delta \rightarrow 0} N_{\text{ren}}^{(\delta)}(L)$ is the zero random variable, since in $\nu_{1/2}$ there are no renewal events, $\widehat{N}_{\text{ren}}^{(0)}(L) = \lim_{\delta \rightarrow 0} \widehat{N}_{\text{ren}}^{(\delta)}(L)$ is non-trivial. We stress that the key to the $\delta \searrow 0$ asymptotics described in Theorem 1 lies not in $\nu_{1/2}$ but in $\hat{\nu}_{1/2}$.

4 Proof of Theorem 1

The proof of Theorem 1 is broken into nine steps, as follows:

- Step 1:** Express $N^{(\delta)}(L)$ in terms of $N_{\text{ren}}^{(\delta)}(L)$.
- Step 2:** Express the distribution and second moment of $N_{\text{ren}}^{(\delta)}(L)$ in terms of those of $\widehat{N}_{\text{ren}}^{(\delta)}(L)$.
- Step 3:** Approximate the expressions found in Step 2 by replacing $\widehat{N}_{\text{ren}}^{(\delta)}(L)$ by $\widehat{N}_{\text{ren}}^{(0)}(L)$. The resulting expressions involve the renewal random variable $\widehat{Y}^{(\delta)}$.
- Step 4:** Replace the occurrences of $\widehat{Y}^{(\delta)}$ in the expression of Step 3 by $\widehat{Y}^{(0)}$.
- Step 5:** Obtain the large- L asymptotics of the distribution of $\widehat{N}_{\text{ren}}^{(0)}(L)$ and of its second moment, and insert these into the expressions found in Step 4.
- Step 6:** Use the asymptotics of the distribution of $\widehat{Y}^{(0)}$ to further approximate the expressions found in Step 5.
- Step 7:** Obtain from the expressions found in Step 6 the asymptotics of $\text{Var}(N_{\text{ren}}^{(\delta)}(L))$.

Step 8: Use Step 1 to obtain the results of Theorem 1 for $L \ll \delta^{-2}$ from the expression found in Step 7 for $\text{Var}(N_{\text{ren}}^{(\delta)}(L))$.

Step 9: Use some facts about the truncated two-point correlation function for ν_ρ to obtain the results of Theorem 1 for $L \gg \delta^{-2}$.

We now consider these steps in order.

Step 1: In this step we again use the notation introduced in (9), so that the values which may be taken by a configuration η_i are $\hat{0}$, 0, and 1, where $\hat{0}$ denotes a renewal event, 0 an empty site preceded by a 1, and 1 an occupied site. Now we observe that $L - N_{\text{ren}}^{(\delta)}(L)$ is odd if and only if the pair (η_1, η_L) has value $(0, 0)$, $(0, \hat{0})$, $(\hat{0}, 1)$, or $(1, 1)$; moreover,

$$N^{(\delta)}(L) = \frac{1}{2} [L - (N_{\text{ren}}^{(\delta)}(L) + \sigma^{(\delta)}(L))], \quad (16)$$

where

$$\sigma^{(\delta)}(L) = \begin{cases} 0, & \text{if } L - N_{\text{ren}}^{(\delta)}(L) \text{ is even,} \\ 1, & \text{if } L - N_{\text{ren}}^{(\delta)}(L) \text{ is odd and } (\eta_1, \eta_L) \text{ is } (0, 0) \text{ or } (0, \hat{0}), \\ -1, & \text{if } L - N_{\text{ren}}^{(\delta)}(L) \text{ is odd and } (\eta_1, \eta_L) \text{ is } (\hat{0}, 1) \text{ or } (1, 1). \end{cases} \quad (17)$$

One checks this by induction on $N_{\text{ren}}^{(\delta)}(L)$; the case $N_{\text{ren}}^{(\delta)}(L) = 0$ is easy. For the induction step one passes from a configuration η to another η' by removing a $\hat{0}$ from some site i with $1 \leq i \leq L$ and setting $\eta'_j = \eta_j$ if $j < i$ and $\eta'_j = \eta_{j+1}$ if $j \geq i$; one then applies the induction assumption to η' on J_{L-1} , noting that then L and $N_{\text{ren}}^{(\delta)}(L)$ both decrease by 1, and observing that $(\eta'_1, \eta'_{L-1}) \neq (\eta_1, \eta_L)$ only if $i = 1$ and $\eta'_1 = 1$ or $i = L$ and $\eta'_{L-1} = 0$.

From (16) we have that

$$\text{Var}(N^{(\delta)}(L)) = \frac{1}{4} \text{Var}(N_{\text{ren}}^{(\delta)}(L) + \sigma^{(\delta)}(L)). \quad (18)$$

To simplify this expression further we note that, writing E for expectation, we have

$$E(\sigma^{(\delta)}(L) \mid N_{\text{ren}}^{(\delta)}(L) = n) = 0, \quad \text{for any } n \geq 0, \quad (19)$$

as we will argue shortly. But then

$$E(\sigma^{(\delta)}(L)) = 0 \quad (20)$$

and

$$E(N_{\text{ren}}^{(\delta)}(L)\sigma^{(\delta)}(L)) = \text{Cov}(N_{\text{ren}}^{(\delta)}(L), \sigma^{(\delta)}(L)) = 0, \quad (21)$$

so that

$$\text{Var}(N_{\text{ren}}^{(\delta)}(L) + \sigma^{(\delta)}(L)) = \text{Var}(N_{\text{ren}}^{(\delta)}(L)) + \text{Var}(\sigma^{(\delta)}(L)). \quad (22)$$

Thus from (18) and (20),

$$\begin{aligned} \text{Var}(N^{(\delta)}(L)) &= \frac{1}{4}(\text{Var}(N_{\text{ren}}^{(\delta)}(L)) + \text{Var}(\sigma^{(\delta)}(L))) \\ &= \frac{1}{4}(\text{Var}(N_{\text{ren}}^{(\delta)}(L)) + P(\sigma^{(\delta)}(L) \neq 0)). \end{aligned} \quad (23)$$

To verify (19) we note that ν_ρ is invariant under reflection about any (integer or half-integer) point. But reflection about $L/2$, the midpoint of the interval $\{0, 1, \dots, L\}$, leaves $N_{\text{ren}}^{(\delta)}(L)$ unchanged and, when $L - N_{\text{ren}}^{(\delta)}(L)$ is odd, changes the sign of $\sigma^{(\delta)}(L)$, as one sees by checking separately for the four possible values of (η_1, η_L) which can then occur.

Step 2: Let $F^{(\delta)}$ be the location of the first renewal event to the right of the origin, and let

$$p_\delta(l) = P(F^{(\delta)} = l). \quad (24)$$

The critical observation for Step 2 is that the number of renewal events in J_L , if there are any, is one more than the number of such events to the right of $F^{(\delta)}$. Thus for $n \geq 1$,

$$\begin{aligned} P(N_{\text{ren}}^{(\delta)}(L) = n) &= \sum_{l=1}^L p_\delta(l) P(N_{\text{ren}}^{(\delta)}(L) = n \mid F^{(\delta)} = l) \\ &= \sum_{l=1}^L p_\delta(l) P(\widehat{N}_{\text{ren}}^{(\delta)}(L-l) = n-1), \end{aligned} \quad (25)$$

and so

$$E(N_{\text{ren}}^{(\delta)}(L)^2) = \sum_{l=1}^L p_\delta(l) E((\widehat{N}_{\text{ren}}^{(\delta)}(L-l) + 1)^2). \quad (26)$$

Step 3: As indicated earlier, the next step is to control the approximation arising from the replacement of $\widehat{N}_{\text{ren}}^{(\delta)}$ by $\widehat{N}_{\text{ren}}^{(0)}$ in (25) and (26). Specifically,

we will show that for $n \geq 1$,

$$P(N_{\text{ren}}^{(\delta)}(L) = n) \simeq \sum_{l=1}^L p_\delta(l) P(\widehat{N}_{\text{ren}}^{(0)}(L-l) = n-1) \quad \text{for } L \ll \delta^{-2}, \quad (27)$$

uniformly in $n \leq k\sqrt{L}$ for k any fixed positive integer (see Definition 1), and also that

$$E(N_{\text{ren}}^{(\delta)}(L)^2) \simeq \sum_{l=1}^L p_\delta(l) E((\widehat{N}_{\text{ren}}^{(0)}(L-l) + 1)^2) \quad \text{for } L \ll \delta^{-2}. \quad (28)$$

Note that the right hand sides of (27) and (28) both mix quantities defined for $\delta > 0$ with those defined for $\delta = 0$ (p_δ and $\widehat{N}_{\text{ren}}^{(0)}$ respectively). These equations are more delicate than they may appear because they demand that we control the errors in these approximations by requiring merely that, for small δ , $L\delta^2$ be sufficiently small regardless of the size of L itself.

To do so we first note that from (11) and (14) we have that

$$P(\widehat{X}^{(\delta)} = l) = (1 + 2\delta)(1 - 4\delta^2)^l P(\widehat{X}^{(0)} = l). \quad (29)$$

Let η_L and $\widehat{\eta}^{(\delta)}(L)$ denote respectively a fixed and a random configuration on J_L , with the latter distributed according to $\hat{\nu}_\rho$, and suppose that η_L contains n renewal events and that $2l_i + 1$, $i = 1, \dots, n$, are the distances between these, with $2l_1 + 1$ the distance from the origin to the first renewal event. (Note that since under $\hat{\nu}_\rho$ there is a renewal event at the origin, if $\eta_L(1) = 0$ then this 0 at site 1 must also be a renewal event). Further, define

$$q_\delta(L) = P(\widehat{Y}^{(\delta)} > L) = P\left(\widehat{X}^{(\delta)} > \left\lfloor \frac{L-1}{2} \right\rfloor\right) \quad (30)$$

with $\widehat{Y}^{(\delta)} = 2\widehat{X}^{(\delta)} + 1$ as defined at the end of Section 3, so that $q_\delta(L - l(\eta_L))$ is the probability that, given that the last renewal event in η_L is at $l(\eta_L)$, the next renewal event is indeed beyond L . Finally, define

$$r_\delta(L) = \frac{q_\delta(L)}{q_0(L)}. \quad (31)$$

Then, writing $\widetilde{L} = L - l(\eta_L)$, we have from (29) that

$$\begin{aligned} P(\widehat{\eta}^{(\delta)}(L) = \eta_L) &= \prod_{i=1}^n P(X^{(\delta)} = l_i) q_\delta(\widetilde{L}) \\ &= (1 + 2\delta)^n (1 - 4\delta^2)^{\sum_{i=1}^n l_i} r_\delta(\widetilde{L}) P(\widehat{\eta}^{(0)}(L) = \eta_L). \end{aligned} \quad (32)$$

We next show that each of the first three factors in (32) is asymptotic to 1 for $L \ll \delta^{-2}$, uniformly in $n \leq k\sqrt{L}$ for any fixed positive k . First, under this condition,

$$1 \leq (1 + 2\delta)^n \leq (1 + 2\delta)^{k\sqrt{L}} \leq e^{2k\delta\sqrt{L}}. \quad (33)$$

Next, from $\sum_{i=1}^n l_i \leq L/2$ and $1 - x \geq e^{-2x}$ for $0 \leq x \leq 1/2$ we have

$$1 \geq (1 - 4\delta^2)^{\sum_{i=1}^n l_i} \geq e^{-4L\delta^2} \quad \text{for } 0 \leq \delta \leq \frac{1}{2\sqrt{2}}. \quad (34)$$

Finally, consider $r_\delta(\tilde{L})$, and notice that this is a ratio of probabilities of an event involving a single jump, of size \hat{Y} , between successive renewal events, for $\delta = 0$ and for $0 < \delta \ll 1$. If the event were that $\hat{Y} = \hat{L}$ with \hat{L} fixed and not too large, it would easily follow from (29) that for small δ the ratio is sufficiently close to 1. However, the relevant event is of the form $\{\hat{Y} > \hat{L}\}$, and its probability is given by a sum of probabilities of events $\{\hat{Y} = \hat{L}'\}$, with the sums for $\delta = 0$ and $\delta > 0$ involving the same events. The ratio of corresponding terms in the numerator and the denominator of the ratio will, for small δ , be uniformly close to 1—and hence so will the complete ratio itself—provided that the terms corresponding to jumps that are very large can be ignored. That the latter is so follows from the power law fall-off of the tail of the jump distribution as expressed in (14). It is straightforward to turn these consideration into a formal proof that for $\tilde{L} \leq L$,

$$r_\delta(\tilde{L}) \simeq 1 \quad \text{for } L \ll \delta^{-2}. \quad (35)$$

(27) then follows from (32)–(35).

We now turn to (28). For any positive k we write $\hat{N}_{\text{ren}}^{(\delta)}(L) = \hat{N}_{\leq k\sqrt{L}}^{(\delta)}(L) + \hat{N}_{> k\sqrt{L}}^{(\delta)}(L)$, where

$$\hat{N}_{\leq x}^{(\delta)}(L) := \hat{N}_{\text{ren}}^{(\delta)}(L) I_{\{\hat{N}_{\text{ren}}^{(\delta)}(L) \leq x\}} \quad (36)$$

and

$$\hat{N}_{> x}^{(\delta)}(L) := \hat{N}_{\text{ren}}^{(\delta)}(L) I_{\{\hat{N}_{\text{ren}}^{(\delta)}(L) > x\}}, \quad (37)$$

with $I_{\{\cdot\}}$ denoting the indicator function of the set $\{\cdot\}$. Then from (27),

$$E(\hat{N}_{\leq k\sqrt{L}}^{(\delta)}(L)^2) \simeq E(\hat{N}_{\leq k\sqrt{L}}^{(0)}(L)^2) \quad \text{for } L \ll \delta^{-2}. \quad (38)$$

(28) will follow easily once we strengthen (38) to

$$E(\hat{N}_{\text{ren}}^{(\delta)}(L)^2) \simeq E(\hat{N}_{\text{ren}}^{(0)}(L)^2) \quad \text{for } L \ll \delta^{-2}. \quad (39)$$

There are two crucial facts for obtaining (39). The first, to be proved shortly, is that for any $\epsilon > 0$ there exists a $k > 0$ such that

$$E(\widehat{N}_{>k\sqrt{L}}^{(\delta)}(L)^2) \leq \epsilon L \quad \text{for } L \ll \delta^{-2} \quad (40)$$

(where $L \ll \delta^{-2}$ holds for all L if $\delta = 0$). The second, to be proved in Step 5, is that $\lim_{L \rightarrow \infty} E(\widehat{N}_{\text{ren}}^{(0)}(L)^2)/L = 1$, so that for some constant $C > 0$,

$$E(\widehat{N}_{\text{ren}}^{(0)}(L)^2) \geq CL \quad \text{for all } L \geq 1. \quad (41)$$

(39) follows from (38), (40) (for both $\delta = 0$ and $\delta > 0$) and (41).

To see this, fix $\epsilon > 0$ and take k so that (40) holds. Then from (40) for $\delta = 0$ and (41) we have (for all L)

$$\left| \frac{E(\widehat{N}_{\leq k\sqrt{L}}^{(0)}(L)^2)}{E(\widehat{N}_{\text{ren}}^{(0)}(L)^2)} - 1 \right| = \frac{E(\widehat{N}_{>k\sqrt{L}}^{(0)}(L)^2)}{E(\widehat{N}_{\text{ren}}^{(0)}(L)^2)} < \frac{\epsilon}{C}. \quad (42)$$

Now by (38) we may take $L\delta^2$ so small that

$$\left| \frac{E(\widehat{N}_{\leq k\sqrt{L}}^{(\delta)}(L)^2)}{E(\widehat{N}_{\leq k\sqrt{L}}^{(0)}(L)^2)} - 1 \right| < \epsilon. \quad (43)$$

From this,

$$\begin{aligned} E(\widehat{N}_{\leq k\sqrt{L}}^{(\delta)}(L)^2) &> E(\widehat{N}_{\leq k\sqrt{L}}^{(0)}(L)^2)(1 - \epsilon) \\ &= [E(\widehat{N}_{\text{ren}}^{(0)}(L)^2) - E(\widehat{N}_{>k\sqrt{L}}^{(0)}(L)^2)](1 - \epsilon) \\ &> (C - \epsilon)(1 - \epsilon)L, \end{aligned} \quad (44)$$

and then we have from (40) (possibly with a further restriction on $L\delta^2$) that

$$\left| \frac{E(\widehat{N}_{\leq k\sqrt{L}}^{(\delta)}(L)^2)}{E(\widehat{N}_{\text{ren}}^{(\delta)}(L)^2)} - 1 \right| \leq \frac{E(\widehat{N}_{>k\sqrt{L}}^{(\delta)}(L)^2)}{E(\widehat{N}_{\leq k\sqrt{L}}^{(\delta)}(L)^2)} < \frac{\epsilon}{(C - \epsilon)(1 - \epsilon)}. \quad (45)$$

Since ϵ here is arbitrary, (42), (43), and (45) imply (39).

To conclude Step 3 we must establish (40). To do so we first note that for any integer $n \geq 1$,

$$P(\widehat{N}_{\text{ren}}^{(\delta)}(L) \geq n) \leq P(\widehat{Y}^{(\delta)} \leq L)^n = (1 - q_\delta(L))^n, \quad (46)$$

since the distances between the successive renewal events in J_L (including the distance of the first such event from the origin), which are independent, must each be no greater than L .

In the remainder of this section we write $q = q_\delta(L)$. For any integer-valued random variable N , and any integer $n_c \geq 1$, we have (as a consequence of summation by parts) that

$$E(N^2 I_{\{N \geq n_c\}}) = n_c^2 P(N \geq n_c) + \sum_{n=n_c+1}^{\infty} (2n-1)P(N \geq n), \quad (47)$$

provided that $n^2 P(N \geq n) \rightarrow 0$ as $n \rightarrow \infty$. Thus from (46),

$$\begin{aligned} E(\widehat{N}_{>n_c}^{(\delta)}(L)^2) &\leq n_c^2(1-q)^{n_c} + \sum_{n=n_c+1}^{\infty} (2n-1)(1-q)^n \\ &\leq \left(n_c^2 + \frac{2n_c}{q} + \frac{2}{q^2} \right) (1-q)^{n_c} \\ &\leq 2 \left(n_c + \frac{1}{q} \right)^2 e^{-qn_c}. \end{aligned} \quad (48)$$

If $\delta > 0$ then it follows from (13) that for $L\delta^2$ sufficiently small there is an $A > 0$ such that

$$q \geq \frac{A}{\sqrt{L}}. \quad (49)$$

Moreover, (14) implies the same conclusion, for any L , when $\delta = 0$. If now for any $k > 0$ we set we set $n_c = \lfloor k\sqrt{L} \rfloor$ then (48) and (49) yield

$$E(\widehat{N}_{>k\sqrt{L}}^{(\delta)}(L)^2) = E(\widehat{N}_{>n_c}^{(\delta)}(L)^2) \leq \left(k + \frac{1}{A} \right)^2 e^A e^{-Ak} L, \quad (50)$$

and (40) will hold for sufficiently large k .

Step 4: Equation (28), the starting point for our future investigations, involves $p_\delta(l) = P(F^{(\delta)} = l)$, the probability under ν_ρ that the first renewal event to the right of the origin occurs at site $l > 0$. This happens precisely when there is a renewal event at some site $-l' \leq 0$, an event with probability 2δ (see Remark 3), and the next renewal event to its right is at l , so that

$$p_\delta(l) = 2\delta \sum_{l' \geq 0} P(\widehat{Y}^{(\delta)} = l' + l) = 2\delta P(\widehat{Y}^{(\delta)} \geq l) (= 2\delta q_\delta(l-1)). \quad (51)$$

Then (35) yields

$$p_\delta(l) \simeq 2\delta P(\widehat{Y}^{(0)} \geq l) \quad \text{for } l \ll \delta^{-2}, \quad (52)$$

and thus from (28) we have that

$$E(N_{\text{ren}}^{(\delta)}(L)^2) \simeq 2\delta \sum_{l=1}^L P(\widehat{Y}^{(0)} \geq l) E((\widehat{N}_{\text{ren}}^{(0)}(L-l) + 1)^2) \quad \text{for } L \ll \delta^{-2}. \quad (53)$$

Step 5: In this step we obtain the large- L asymptotics of $E(\widehat{N}_{\text{ren}}^{(0)}(L)^2)$:

$$E(\widehat{N}_{\text{ren}}^{(0)}(L)^2) \simeq L, \quad \text{for } L \gg 1. \quad (54)$$

This formula may be obtained from [9], but we give a self-contained proof, arguing from the detailed form of the distribution of the renewal random variable $\widehat{Y}^{(0)} = 2\widehat{X}^{(0)} + 1$. Recall (see (11)) that $P(\widehat{X}^{(0)} = n) = C_n 2^{-(2n+1)}$; the Catalan number C_n counts the number of paths between time 0 and time $2n$ of a random walk which starts and ends at the origin while never taking any positive value. Thus $\widehat{Y}^{(0)}$ has the same distribution as the time of first arrival at site 1 of a simple symmetric random walk W_l , $l = 0, 1, \dots$, which starts at the origin. As a consequence, $\widehat{N}_{\text{ren}}^{(0)}(L)$ has the same distribution as the maximum value $M(L)$ of W_l over the interval $[0, L]$. From [6], Section III.7, Theorem 1 we then have

$$P(\widehat{N}_{\text{ren}}^{(0)}(L) = n) = \begin{cases} P(W_L = n), & \text{if } L - n \text{ is even,} \\ P(W_L = n + 1), & \text{if } L - n \text{ is odd.} \end{cases} \quad (55)$$

An easy calculation from (55) gives, for L odd,

$$E(\widehat{N}_{\text{ren}}^{(0)}(L)^2) = E(W_L^2) - E(|W_L|) + \frac{1}{2}(1 - P(W_L = 0)), \quad (56)$$

and this yields (54), since $E(W_L^2) = L$ and $E(|W_L|) \leq \sqrt{L}$, by the Schwarz inequality.

Remark 4. (a) From (55) one can show easily that $\widehat{N}_{\text{ren}}^{(0)}(L)/\sqrt{L}$ converges in distribution, as $L \rightarrow \infty$, to $|Z|$, with Z a standard normal random variable. (b) In [11, 22] a random walk representation of particle configurations (there called a height function or height process) was used to obtain the distribution (11). The Catalan numbers play the same role in this derivation that they do above.

Step 6: Equation (53) provides the leading order small- δ approximation to $E(N_{\text{ren}}^{(\delta)}(L)^2)$, valid for $L \ll \delta^{-2}$. In this step we use (54) and (57) below to approximate this moment to leading order in L , as well.

First, note that from (13) we have at once that

$$P(\widehat{Y}^{(0)} \geq l) = P\left(\widehat{X}^{(0)} \geq \frac{l-1}{2}\right) \simeq \sqrt{\frac{2}{\pi l}} \quad \text{for } l \gg 1. \quad (57)$$

Substituting (54) and (57) into (53) yields, at least formally,

$$E(N_{\text{ren}}^{(\delta)}(L)^2) \simeq 2\sqrt{\frac{2}{\pi}}\delta \sum_{l=1}^L \frac{L-l}{\sqrt{l}}, \quad \text{for } 1 \ll L \ll \delta^{-2}. \quad (58)$$

We will justify (58) shortly, but for the moment only note that the restriction $L \gg 1$, not present in (53), arises from (54) and (57). From (58) we have that for $1 \ll L \ll \delta^{-2}$,

$$\begin{aligned} E(N_{\text{ren}}^{(\delta)}(L)^2) &\simeq 2\sqrt{\frac{2}{\pi}}\delta \int_1^L \frac{L-x}{\sqrt{x}} dx \\ &= 2\sqrt{\frac{2}{\pi}}\delta L^{3/2} \int_{1/L}^1 \frac{1-y}{\sqrt{y}} dy \simeq \frac{8}{3}\sqrt{\frac{2}{\pi}}\delta L^{3/2}. \end{aligned} \quad (59)$$

This is our final approximation for $E(N_{\text{ren}}^{(\delta)}(L)^2)$.

We now return to (58). We are justified (when l is not too large) in replacing $E((\widehat{N}_{\text{ren}}^{(0)}(L-l)+1)^2)$ by $E((\widehat{N}_{\text{ren}}^{(0)}(L-l))^2)$ in passing from (53) to (58) since, from (54) and the Schwarz inequality, $E(\widehat{N}_{\text{ren}}^{(0)}(L)) \ll E(\widehat{N}_{\text{ren}}^{(0)}(L)^2)$ for $L \gg 1$. Further, although this replacement and the substitutions obtained from (54) and (57) are jointly valid, say to relative error ϵ , only in some range $l_\epsilon \leq l \leq L - l_\epsilon$, the sums in (53) and (58) over $1 \leq l < l_\epsilon$ and $L - l_\epsilon < l \leq L$ are $O(L)$ and $O(1)$, respectively, as $L \rightarrow \infty$, while the sum over $l_\epsilon \leq l \leq L - l_\epsilon$ is of order $L^{3/2}$ (see (59)). This completes the justification.

Step 7: From (15) and (59),

$$\frac{E(N_{\text{ren}}^{(\delta)}(L))^2}{E(N_{\text{ren}}^{(\delta)}(L)^2)} \simeq \frac{3}{2}\sqrt{\frac{\pi}{2}}\delta L^{1/2} \ll 1 \quad \text{for } 1 \ll L \ll \delta^{-2}. \quad (60)$$

Thus, again from (59),

$$\text{Var}(N_{\text{ren}}^{(\delta)}(L)) \simeq E(N_{\text{ren}}^{(\delta)}(L)^2) \simeq \frac{8}{3}\sqrt{\frac{2}{\pi}}\delta L^{3/2}, \quad \text{for } 1 \ll L \ll \delta^{-2}. \quad (61)$$

Step 8: We can now combine the results of Step 1 and Step 7 to obtain the parts of Theorem 1 which concern $L \ll \delta^{-2}$. For from (23) and (61) we have that

$$\text{Var}(N^{(\delta)}(L)) \simeq \frac{2}{3} \sqrt{\frac{2}{\pi}} \delta L^{3/2} + \frac{1}{4} P(\sigma^{(\delta)}(L) \neq 0) \quad \text{for } 1 \ll L \ll \delta^{-2}, \quad (62)$$

with $\sigma^{(\delta)}(L)$ defined in (17).

If L is even then, from (15),

$$\begin{aligned} P(\sigma^{(\delta)}(L) \neq 0) &= P(N_{\text{ren}}^{(\delta)}(L) \text{ is odd}) \\ &\leq P(N_{\text{ren}}^{(\delta)}(L) > 0) \leq E(N_{\text{ren}}^{(\delta)}(L)) = 2\delta L. \end{aligned} \quad (63)$$

Since $L \ll L^{3/2}$ for $L \gg 1$, the $L \ll \delta^{-2}$ part of Theorem 1(b) follows.

On the other hand, if L is odd then, again from (15),

$$\begin{aligned} P(\sigma^{(\delta)}(L) \neq 0) &= P(N_{\text{ren}}^{(\delta)}(L) \text{ is even}) \\ &\geq P(N_{\text{ren}}^{(\delta)}(L) = 0) \geq 1 - 2\delta L \simeq 1 \quad \text{for } L \ll \delta^{-1}. \end{aligned} \quad (64)$$

Since $\delta L^{3/2} \gg 1$ for $L \gg \delta^{-2/3}$ and $\delta L^{3/2} \ll 1$ for $L \ll \delta^{-2/3}$, we obtain from (64) and (62) the conclusions of Theorem 1(a) for $1 \ll L \ll \delta^{-2}$. To remove the restriction that $L \gg 1$ we note that

$$\text{Var}(N_{\text{ren}}^{(\delta)}(L)) \leq E(N_{\text{ren}}^{(\delta)}(L)^2) \leq E(N_{\text{ren}}^{(\delta)}(L')^2) \quad (65)$$

for $L \leq L'$. Choosing L' such that also $1 \ll L' \ll \delta^{-2/3}$, we see using (23), (59), and (64) that $\text{Var}(N^{(\delta)}(L)) \simeq \frac{1}{4}$ for $1 \leq L \ll \delta^{-2/3}$.

Remark 5. Concerning the estimate $P(N_{\text{ren}}^{(\delta)}(L) > 0) \leq 2\delta L$ used in (63) and (64), note that it follows from (24), (52), and (57) that in fact for $L \gg 1$,

$$P(N_{\text{ren}}^{(\delta)}(L) > 0) = P(F^{(\delta)} \leq L) \simeq 2\delta \int_0^L \sqrt{\frac{2}{\pi l}} dl \simeq 4\sqrt{\frac{2}{\pi}} \delta \sqrt{L}. \quad (66)$$

Step 9: We now turn to the $L \gg \delta^{-2}$ part of Theorem 1 and to the related Corollary 2 (recall also Remark 1). The argument is based on the standard formula

$$\text{Var}(N^{(\delta)}(L)) = \rho(1 - \rho)L + 2 \sum_{k=1}^{L-1} \sum_{j=1}^k g_{\rho}^T(j), \quad (67)$$

with g_ρ^T the truncated two-point correlation function for the TI state ν_ρ :

$$g_\rho^T(k) = E(\eta(j)\eta(j+k)) - \rho^2. \quad (68)$$

It is shown in [11] that for all $n \geq 0$,

$$g_\rho^T(2n+1) + g_\rho^T(2n+2) = 0, \quad (69)$$

so that (67) becomes

$$\text{Var}(N^{(\delta)}(L)) = \rho(1-\rho)L + 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{L-1} g_\rho^T(k). \quad (70)$$

We will show shortly that for $j \geq 0$,

$$|g_\rho^T(2j+1)| \leq \rho^2(1-4\delta^2)^j. \quad (71)$$

Then from (70),

$$\left| \frac{\text{Var}(N^{(\delta)}(L))}{\rho(1-\rho)L} - 1 \right| \leq \frac{2\rho}{(1-\rho)L} \sum_{j=0}^{\infty} (1-4\delta^2)^j < \frac{1}{2\delta^2 L}, \quad (72)$$

and this verifies the result stated in Corollary 2. The $L \gg \delta^{-2}$ cases of Theorem 1 then follow by taking the δ_0 of Definition 1 sufficiently small.

Consider now (71). The generating function of the g_ρ^T is computed in [11]:

$$G_\rho(z) := \sum_{k=1}^{\infty} g_\rho^T(k)z^k = \frac{z(\sqrt{1-z^2(1-4\delta^2)} - 2\delta)^2}{4(z-1)(z+1)^2}. \quad (73)$$

The numerator in (73) has double zeros (on its first sheet) at $z = \pm 1$, so that G_ρ is analytic at these points with singularities at $z = \pm z_*$, where $z_* := (1-4\delta^2)^{-1/2}$. Expressing $g_\rho^T(k)$ via Cauchy's formula as an integral over a small circle around the origin, distorting this contour to obtain the sum of integrals of the discontinuity of G_ρ across cuts on the real axis from z_* to ∞ and from $-z_*$ to $-\infty$, and making the change of variable $z \rightarrow -z$ in the second of these integrals, we obtain a representation of $g_\rho^T(k)$ which for k odd is

$$g_\rho^T(k) = -\frac{2\delta}{\pi z_*} \int_{z_*}^{\infty} \frac{\sqrt{z^2 - z_*^2}}{z^k(z^2 - 1)^2} dz, \quad k \text{ odd}. \quad (74)$$

This implies that for k odd, $|g_\rho^T(k)| \leq z_*^{-(k-1)}|g_\rho^T(1)|$, and since $g_\rho^T(1) = -\rho^2$, (71) follows.

This completes the proof of Theorem 1.

5 Concluding remarks

We note that on \mathbb{Z} the approach to hyperuniformity as $\rho \searrow \rho_c$ is different from that for $\rho \nearrow \rho_c$; in the former case the unique stationary measure for the FEP on \mathbb{Z} is known and [13, 15]

$$\lim_{L \rightarrow \infty} \frac{1}{L} V_\rho(L) = \rho(1 - \rho)(2\rho - 1) \quad \text{for } \rho > 1/2. \quad (75)$$

Thus from Theorem 1 and (8), $\lim_{L \rightarrow \infty} L^{-1} V_\rho(L)$ is continuous in ρ from above, but not from below, at $\rho = 1/2$. We expect similar behavior on \mathbb{Z}^d .

The case of the FEP on a *ladder*, a system consisting of two (infinite) rows of sites, was studied numerically in [17, 18]. Here again, as on \mathbb{Z}^d with $d \geq 2$, $\rho_c < 1/2$ (for the continuous-time symmetric FEP on the ladder, $\rho_c \approx 0.4755$ or, for the slightly different dynamics of [16], $\rho_c \approx 0.4874$ [18]). The results of [18] also suggest that the Fluctuation Conjecture holds for this model (although the scaling behavior of the $L_i(\delta)$ is not discussed). Rigorously, one may observe that for $\rho < \rho_c$ the portions of the system to the left and right of an empty square are independent under the $t \rightarrow \infty$ limiting measure ν_ρ , implying in particular that the locations of the empty squares (when these have a nonzero density) form a renewal process and that the portions of the system between them are jointly independent. Further, we see that the critical density must satisfy $\rho_c \geq 1/4$, since at smaller densities there would always be a finite density of empty squares and the stationary state would be frozen [18]. We have no such lower bound for ρ_c on \mathbb{Z}^d , $d \geq 2$.

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References

- [1] A. Ayyer, S. Goldstein, J. L. Lebowitz, and E. R. Speer, Stationary States of the One-dimensional Facilitated Asymmetric Exclusion Process. *Ann. I. H. Poincaré-Pr.* **59**, 728–742 (2023).
- [2] J. Baik, G. Barraquand, I. Corwin, and T. Suidan, Facilitated Exclusion Process. *Computation and Combinatorics in Dynamics, Stochastics and Control*, 1–35, Abel Symp. **13**, Springer, Cham, 2018.

- [3] O. Blondel, C. Erignoux, M. Sasada, and M. Simon, Hydrodynamic Limit for a Facilitated Exclusion Process. *Ann. I. H. Poincaré-Pr.* **56**, 667714 (2020).
- [4] O. Blondel, C. Erignoux, and M. Simon, Stefan Problem for a Non-Ergodic Facilitated Exclusion Process. *Prob. Math. Phys.* **2**, 127–178 (2021).
- [5] U. Basu and P. K. Mohanty, Active-Absorbing-State Phase Transition Beyond Directed Percolation: A Class of Exactly Solvable Models. *Phys. Rev. E* **79**, 041143 (2009).
- [6] W. Feller, *An Introduction to Probability Theory and its Applications, Volume I*, Third edition. John Wiley & Sons, New York, 1968.
- [7] A. Gabel, P. L. Krapivsky, and S. Redner, Facilitated Asymmetric Exclusion. *Phys. Rev. Lett.* **105**, 210603 (2010).
- [8] A. Gabel and S. Redner, Cooperativity-Driven Singularities in Asymmetric Exclusion, *J. Stat. Mech.* **2011**, P06008 (2011).
- [9] C. Godrèche and J. M. Luck, Statistics of Occupation Time of Renewal Processes. *J. Stat. Phys.* **104**, 489–524 (2001).
- [10] S. Goldstein, J. L. Lebowitz and E. R. Speer, Exact Solution of the F-TASEP. *J. Stat. Mech.* **2019** 123202 (2019).
- [11] S. Goldstein, J. L. Lebowitz and E. R. Speer, The Discrete-Time Facilitated Totally Asymmetric Simple Exclusion Process. *Pure Appl. Funct. Anal.* **6**, 177–203 (2021).
- [12] S. Goldstein, J. L. Lebowitz and E. R. Speer, Stationary States of the One-Dimensional Discrete-Time Facilitated Symmetric Exclusion Process. *J. Math. Phys.* **63**, 083301 (2022).
- [13] S. Goldstein, J. L. Lebowitz and E. R. Speer, Approach to Hyperuniformity of Steady States of Facilitated Exchange Processes. *J. Phys.: Condens. Matter* **36**, 345402 (2024).
- [14] D. Hexner and D. Levine, Hyperuniformity of Critical Absorbing States. *Phys. Rev. Lett.* **114**, 110602 (2015).

- [15] T. D. Lee and C. N. Yang, Statistical Theory of Equations of State and Phase Relations II: Lattice Gas and Ising Model. *Phys. Rev.* **87**, 410–419 (1952).
- [16] S. Lübeck, Scaling Behavior of the Absorbing Phase Transition in a Conserved Lattice Gas Around the Upper Critical Dimension. *Phys. Rev. E* **64**, 016123 (2001).
- [17] M. J. Oliveira, Conserved Lattice Gas Model with Infinitely Many Absorbing States in One Dimension. *Phys. Rev. E* **71**, 016112 (2005).
- [18] C. Ramirez-Ibanez, *Facilitated Exclusion Processes on a Ladder*. Doctoral dissertation, Rutgers (2023).
- [19] M. Rossi, R. Pastor-Satorras, and A. Vespignani, Universality Class of Absorbing Phase Transitions with a Conserved Field. *Phys. Rev. Lett.* **85**, 1803 (2000).
- [20] R. P. Stanley, *Catalan Numbers*. Cambridge University Press, Cambridge, 2015.
- [21] S. Torquato, Hyperuniform States of Matter. *Phys. Rep.* **745**, 1–95 (2018).
- [22] L. Zhao and D. Chen, The Invariant Measures and the Limiting Behaviors of the Facilitated TASEP. *Stat. Probabil. Lett.* **154**, 108557 (2019).