Canonical Typicality and GAP Measures for Quantum States

97th STATISTICAL MECHANICS CONFERENCE

Rutgers May 7, 2007

Joel Lebowitz, Christian Mastrodonato, Roderich Tumulka, and Nino Zanghì
Canonical Typicality

- Schrödinger (1927)
- Bocchieri and Loinger (1959)
- Lloyd (1988)
- Tasaki (1998)
- Popescu, Short, and Winter (2006)
“It is time for this discovery to stay discovered.” (Lloyd, Nature Physics, Nov. 2006)
“It is time for this discovery to stay discovered.” (Lloyd, Nature Physics, Nov. 2006)

“When I discovered it, it stayed discovered.” (Larry Shepp)
\[ H^B = H^{(2)} \]

\[ H^S = H^{(1)} \]

\[
H^{(1+2)} = H^{(1)} + H^{(2)} = H^{(1)} \otimes I^{(2)} + H^{(2)} \otimes I^{(1)}
\]

on

\[
\mathcal{H}^{(1+2)} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}
\]
Microcanonical\((1+2[\text{large}]) \Rightarrow \text{Canonical}(1)\)

\[
\rho_\beta = \exp(-\beta H^{(1)})/Z \quad \text{on } \mathcal{H}^{(1)}
\]

\[
\rho_{E,\delta} = P_{\mathcal{H}_{E,\delta}}/D = P_{E \leq H^{(1+2)} \leq E+\delta}/D \quad \text{on } \mathcal{H}^{(1+2)} \quad (D = \dim \mathcal{H}_{E,\delta})
\]

\[
(\rho_{E,\delta})_1 = \text{tr}^{(2)}(\rho_{E,\delta}) \simeq \rho_\beta \quad (\beta \leftrightarrow E)
\]

\[
\rho_1 = \text{tr}^{(2)}\rho, \quad \text{on } \mathcal{H}^{(1)}, \quad \text{reduced density matrix, etc.}
\]
Canonical Typicality

Fix $\psi \in \mathcal{H}_{E,\delta}$.

$$\rho^\psi = |\psi\rangle\langle\psi|$$ on $\mathcal{H}^{(1+2)}$

$$\rho_{\psi} \simeq \rho_{\beta}$$

for typical $\psi$, i.e., for the overwhelming majority of $\psi$'s in $\mathcal{H}_{E,\delta}$, defined in terms of the uniform distribution $u_{E,\delta}$ (the micro-canonical measure) on the unit sphere $\mathcal{S}(\mathcal{H}_{E,\delta})$ of $\mathcal{H}_{E,\delta}$.

Key: entanglement
Classically, randomness in yields randomness out. In quantum mechanics, we can have “randomness out, without randomness in.” With quantum mechanics we have “spontaneous uncertainty.” But the uncertainty is even greater than it seems, and that is the main issue I want to address.
Density matrix and random wave function

\[ \mu \mapsto \rho_\mu = \int_\mathcal{S} \mu(d\psi) |\psi\rangle \langle \psi| \]

probability measure on the unit sphere \( \mathcal{S} \) of a Hilbert space \( \mathcal{H} \)
\( \mapsto \) density matrix on that Hilbert space

- onto
- many-to-one

Example: \( \rho_{E,\delta} = \rho_\mu \) for \( \mu = u_{E,\delta} \) and for \( \mu = u^{eig}_{E,\delta} \)

where \( u^{eig}_{E,\delta} \) is the uniform distribution over the energy eigenstates in \( \mathcal{H}_{E,\delta} \).
“In equilibrium the state of a system is random. In quantum mechanics the state of a system is given by its wave function. If the system is in equilibrium, with density matrix $\rho_\beta$, its wave function should be random, with some distribution $\mu$ on $\mathcal{S}$. What is $\mu$?”
GAP Measures, $GAP(\rho)$

Gaussian

Adjusted

Projected
Density Matrix and Covariance

\[ \rho = \rho_\mu = \int_\mathcal{H} \mu(d\psi) |\psi\rangle \langle \psi| \leftrightarrow \mu \text{ has covariance } \rho \]

(assuming, as we will more or less always do, that \( \mu \) has mean 0.)

\[ \langle \phi | \rho | \phi' \rangle = \int_\mathcal{H} \mu(d\psi) \langle \phi | \psi \rangle \langle \psi| \phi' \rangle \]

Remark: \( \int_\mathcal{H} \mu(d\psi) \|\psi\|^2 = \int_\mathcal{H} \mu(d\psi) \text{tr} |\psi\rangle \langle \psi| = \text{tr} \rho = 1. \)
$G(\rho)$

$G(\rho)$ is Gaussian, with covariance $\rho$. 

\( G(\rho) \)

\( G(\rho) \) is Gaussian, with covariance \( \rho \). More explicitly, for

\[ \rho = \sum_n \rho_n |n\rangle\langle n| \]

\( G(\rho) \) is the distribution of

\[ \Psi^G = \sum_n Z_n |n\rangle \]

where the \( Z_n \) are independent complex-Gaussian with (mean 0 and) variance \( E(|Z_n|^2) = \rho_n \).
$GA(\rho)$

$G(\rho)$ is not supported by $\mathcal{S}$. We would therefore like to project $G(\rho)$ onto $\mathcal{S}$. But doing so would alter the covariance—unless we first “adjust” $G(\rho)$:

$$GA(\rho)(d\psi) = \|\psi\|^2 G(\rho)(d\psi)$$

(Since $\int G(\rho)(d\psi) \|\psi\|^2 = 1$, $GA(\rho)$ is properly normalized; it is a probability measure on $\mathcal{H}$).
$GAP(\rho)$

$GAP(\rho)$ is the image of $GA(\rho)$ under the “projection”

$\psi \mapsto \psi/\|\psi\|$ to $\mathcal{S}$.
GAP(\rho)

GAP(\rho) is the image of GA(\rho) under the “projection” \( \psi \mapsto \psi/\|\psi\| \) to \( \mathcal{I} \). It is the distribution of

\[ \Psi^{GAP} = \Psi^{GA}/\|\Psi^{GA}\| \]

where \( \Psi^{GA} \) has distribution GA(\rho).
\[ E \left( |\Psi^{\text{GAP}}\rangle\langle\Psi^{\text{GAP}}| \right) = \int G(\rho)(d\psi) \|\psi\|^2 \frac{|\langle\psi|\psi\rangle|}{\|\psi\|^2} = \rho \]
Examples \( (\rho = \sum_n \rho_n |n\rangle\langle n|) \)

- \( GAP(\rho_\beta) \leftrightarrow \Psi^G = \sum Z_n |n\rangle \) with \( |n\rangle \) the energy eigenstates and \( E(|Z_n|^2) = e^{-\beta E_n}/Z \) (for both G and for GAP).

- \( GAP(\rho_{E,\delta}) = u_{E,\delta} \)

- Not GAP: \( EIG(\rho)(|m\rangle) = \rho_m \)
  \[ \longleftrightarrow \Psi = \sum_n Z_n |n\rangle, \quad Z_n = \delta_{nm} \text{ with probability } \rho_m \]

\( EIG(\rho) \) is supported by the set of eigenvectors of \( \rho \).
\( GAP(\rho)(\{\text{eigenvectors of } \rho\}) = 0. \)
Properties of $GAP(\rho)$

- $\rho_{GAP(\rho)} = \rho$

- $GAP(\rho)$ is covariant:

  $$GAP(U\rho U^*) = U[GAP(\rho)]$$

  In particular, $GAP(\rho_\beta)$, $\rho_\beta \propto \exp(-\beta H)$, is a stationary measure for the Schrödinger dynamics generated by $H$.

- $GAP$ is hereditary: $[GAP(\rho^{(1)} \otimes \rho^{(2)})]_1 = GAP(\rho^{(1)})$
The wave function of a subsystem: the conditional wave function $\Psi_1$

For $\Psi \in \mathcal{S}(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)})$, and $\{\phi_n\}$ a basis of $\mathcal{H}^{(2)}$, $\Psi_1$ is random vector in $\mathcal{H}^{(1)}$, taking on the value

$$\Psi_1 = \langle \phi_n | \Psi \rangle / \| \langle \phi_n | \Psi \rangle \| \in \mathcal{S}(\mathcal{H}^{(1)})$$

with probability $\| \langle \phi_n | \Psi \rangle \|^2$. ($\Psi_1$ depends on the choice of basis.)

In other words, for $\Psi = \sum_n \psi_n \otimes \phi_n$

$$\text{Prob}(\Psi_1 = \psi_n / \| \psi_n \|) = \| \psi_n \|^2.$$
Ψ can be random: For any probability measure $\mu$ on $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$,

$$\mu_1 = \text{the distribution of } \Psi_1$$

when $\Psi$ has distribution $\mu$. 
Corollary of Property 3 (using equivalence of ensembles and continuity of $GAP(\rho)$):

$$(u_{E,\delta})_1 \simeq GAP(\rho_\beta)$$

$$\rho_{E,\delta} \simeq \rho_{\beta}^{(1+2)} = \rho_{\beta}^{(1)} \otimes \rho_{\beta}^{(2)}$$

$$u_{E,\delta} = GAP(\rho_{E,\delta}) \simeq GAP(\rho_{\beta}^{(1)} \otimes \rho_{\beta}^{(2)})$$

$$\rho_\beta = \rho_{\beta}^{(1)}$$
GAP Typicality
The wave function of a subsystem of a large system is typically GAP-distributed ("large": $\dim \mathcal{H}^{(2)} \gg \dim \mathcal{H}^{(1)}$):

\[
\begin{align*}
\Psi & \rightarrow \rho \Psi \\
& \rightarrow (\rho \Psi)_1 \\
\Psi_1 & \xrightarrow{\text{dist}} GAP[(\rho \Psi)_1]
\end{align*}
\]

\[\text{dist}(\Psi_1) \simeq GAP[(\rho \Psi)_1]\]

for typical $\Psi \in \mathcal{S}(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)})$. 

23
Too weak. \((\rho^\Psi)_1\) is typically \(\propto I^{(1)}\). For given \(\rho^{(1)}\), we need to condition on the event that \(\Psi\) is such that \((\rho^\Psi)_1 = \rho^{(1)}\).

\[
\begin{align*}
\Psi & \rightarrow \rho^\Psi \overset{1}{\rightarrow} \rho^{(1)} \\
\downarrow & \quad \downarrow \\
\Psi_1 & \overset{dist}{\rightarrow} GAP(\rho^{(1)})
\end{align*}
\]

\[dist(\Psi_1) \sim GAP(\rho^{(1)})\]

for typical \(\Psi \in \mathcal{I}(\rho^{(1)}) = \{\Psi \in \mathcal{I}(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}) \mid (\rho^\Psi)_1 = \rho^{(1)}\}\).
Corollary:

- For any $\Psi \in S(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)})$
  
  $\text{dist}(\Psi_1) \simeq \text{GAP}[(\rho_\Psi)_1]$  

  for a typical choice of basis for $\mathcal{H}^{(2)}$.

- Thus, using canonical typicality: For a typical $\Psi \in \mathcal{H}_{E,\delta}$
  
  $\text{dist}(\Psi_1) \simeq \text{GAP}(\rho_\beta)$  

  for a typical choice of basis for $\mathcal{H}^{(2)}$.  

GAP typicality →
GAP is the “canonical” wave function distribution.
Sketch of Proofs

\[ H^{(1)} = \sum_n E_n \vert n \rangle \langle n \vert \]

\[ \rho_{E,\delta} = D^{-1} \sum_n P_{\mathcal{H}^{(2)}_{E-E_n,\delta}} \otimes \vert n \rangle \langle n \vert \]

\[ \rho_1 = D^{-1} \sum_n \left( \text{dim } \mathcal{H}^{(2)}_{E-E_n,\delta} \right) \vert n \rangle \langle n \vert \simeq \rho_\beta \]

since \( \text{dim } \mathcal{H}^{(2)}_{E-E_n,\delta} \sim e^{S(E-E_N)} \propto e^{-\frac{\partial S}{\partial E} E_n} = e^{-\beta E_n} \)
Proof of CT

\[ \psi = \sum_n |n\rangle \otimes \Phi_n \]

\[ \Phi_n = \sum_{E - E_n \leq E_m^{(2)} \leq E - E_n + \delta} X_{nm} |m\rangle^{(2)} \]

We may assume that the \( X_{nm} \) are i.i.d. complex-Gaussian, with \( E(|X_{nm}|^2) = 1/D \). Then

\[ \left( \rho^\psi \right)_1 = \sum_{n,m} \langle \Phi_m | \Phi_n \rangle |n\rangle \langle m| \simeq \sum_n \rho_n |n\rangle \langle n| \]

\[ \rho_n = \sum_{E - E_n \leq E_m^{(2)} \leq E - E_n + \delta} |X_{nm}|^2 \simeq \dim \mathcal{H}_{E-E_n, \delta/D}^{(2)} \]
Proof of GT

Schmidt decomposition (biorthonormal decomposition)

For $\Psi \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)},$  $\dim \mathcal{H}^{(1)} \leq \dim \mathcal{H}^{(2)},$  
there exists a basis $\psi_n$ of $\mathcal{H}^{(1)}$ and an o.n. system $\chi_n$ for $\mathcal{H}^{(2)}$ such that

$$\Psi = \sum_n c_n \psi_n \otimes \chi_n, \quad c_n \geq 0.$$
Schmidt decomposition and \((\rho \Psi)_1\)

\[
(\rho \Psi)_1 = \rho^{(1)} = \sum_n \rho_n |n\rangle\langle n| \quad \leftrightarrow \quad \Psi \text{ has SD of the form }
\]

\[
\Psi = \sum_n \sqrt{\rho_n} |n\rangle \otimes \chi_n
\]

\[
\Rightarrow \mathcal{S}(\rho^{(1)})
\]

\(u_{\mathcal{S}(\rho^{(1)})} \leftrightarrow \text{ the o.n. system } \chi_n \text{ is random, with distribution }
\]

invariant under the action of the unitary group of \(\mathcal{H}^{(2)}\).
Lemma: Fix a basis $\phi_m$ of $\mathcal{H}^{(2)}$. The empirical distribution of $\sqrt{\dim \mathcal{H}^{(2)}} \psi_1$, assigning equal probabilities $1/\dim \mathcal{H}^{(2)}$ to the vectors $\sqrt{\dim \mathcal{H}^{(2)}} \langle \phi_m | \psi \rangle \in \mathcal{H}^{(1)}$, is $G(\rho^{(1)})$.

Proof:

$$\chi_n = \sum_{m} X_{nm} \phi_m / \sqrt{\dim \mathcal{H}^{(2)}}$$

$u_{\mathcal{S}(\rho^{(1)})} \rightarrow$ the $X_{nm}$ are approximately i.i.d. complex-Gaussian, with $E(|X_{nm}|^2) = 1$. 
With this approximation, the sequence of vectors
\[
\sqrt{\dim \mathcal{H}^{(2)}} \langle \phi_m | \Psi \rangle = \sum_n \sqrt{\rho_n} X_{nm} |n\rangle = \sum_n Z_{nm} |n\rangle \quad m = 1, \ldots, \dim \mathcal{H}^{(2)}
\]
is i.i.d., with common distribution that of \( \sum_n Z_n |n\rangle \) where the \( Z_n \) are independent complex-Gaussian with \( E(|Z_n|^2) = \rho_n \).

**LLN \( \Rightarrow \)** the empirical distribution of the sequence is the theoretical distribution \( G(\rho^{(1)}) \).

Empirical distribution, improper normalization, \( G \leftrightarrow \) quantum distribution, proper normalization, GAP.
In the new, post-1925 quantum theory the ‘anarchist’ position became dominant and modern quantum physics, in its ‘Copenhagen interpretation’, became one of the main standard bearers of philosophical obscurantism. In the new theory Bohr’s notorious ‘complementarity principle’ enthroned [weak] inconsistency as a basic ultimate feature of nature, and merged subjectivist positivism and antilogical dialectic and even ordinary language philosophy into one unholy alliance. After 1925 Bohr and his associates introduced a new and unprecedented lowering of critical standards for scientific theories. This led to a defeat of reason within modern physics and to an anarchist cult of incomprehensible chaos. (Lakatos, Criticism and the Growth of Knowledge, p. 145, 1965)