Bohmian Mechanics and Quantum Information*

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Abstract

Many recent results suggest that quantum theory is about information, and that quantum theory is best understood as arising from principles concerning information and information processing. At the same time, by far the simplest version of quantum mechanics, Bohmian mechanics, is concerned, not with information but with the behavior of an objective microscopic reality given by particles and their positions. What I would like to do here is to examine whether, and to what extent, the importance of information, observation, and the like in quantum theory can be understood from a Bohmian perspective. I would like to explore the hypothesis that the idea that information plays a special role in physics naturally emerges in a Bohmian universe.

1 Introduction: The Status of the Wave Function

Few people have struggled as long and as hard with the foundations of quantum mechanics as Jeffrey Bub, and even fewer have done so with as much seriousness, honesty, and gentleness. Jeffrey has in fact explored more or less all approaches to the interpretation of quantum mechanics, and has made

*Dedicated to Jeffrey Bub on the occasion of his 65th birthday.

seminal contributions to most of them. I am indeed pleased—and honored—to have been invited to contribute to this volume in honor of Jeffrey, and thank the organizers for having done so.

The question animating the foundations quantum mechanics, for Jeffrey and everyone else in the field, is this: What is the nature of the reality, if any, that lies behind the quantum mathematics? Now the reality issue, in quantum mechanics and in general, is difficult and controversial. Here are two quotations that, while expressing in very different ways the subtleties involved, nonetheless get to the core of the problem—and its solution—and indeed express pretty much the same thing:

> What if everything is an illusion and nothing exists? In that case, I definitely overpaid for my carpet. (*Woody Allen*)

> I did not grow up in the Kantian tradition, but came to understand the truly valuable which is to be found in his doctrine, alongside of errors which today are quite obvious, only quite late. It is contained in the sentence: "The real is not given to us, but put to us (by way of a riddle)." This obviously means: There is such a thing as a conceptual construction for the grasping of the inter-personal, the authority of which lies purely in its validation. This conceptual construction refers precisely to the "real" (by definition), and every further question concerning "the nature of the real" appears empty. (*Albert Einstein*)

Many readers will perhaps find what Einstein [1, page 680] says here too realistic. To others it will no doubt seem too positivistic. For me, however, it is right on target.

Perhaps the most puzzling object in quantum mechanics is the wave function, concerning which many basic questions can be asked:

- Is it subjective or objective?
- Does it merely represent information or does it describe an observer independent reality?
- If it is objective, does it represent a concrete material sort of reality, or does it somehow have an entirely different and perhaps novel nature?
- What's the deal with collapse?

There seems to be little agreement about the answers to these questions. But we can at least all agree that one of the following crudely expressed possibilities for the wave function must be correct:

1. The wave function is everything.

2. The wave function is something (but not everything).

3. The wave function is nothing.

The second possibility, which amounts to the suggestion that there are, in addition to the wave function, what are often called hidden variables, is regarded by the physics community as the least acceptable and most implausible of these three possibilities—the very terminology "hidden variables" points to the unease. This is interesting since it would also seem to be the most modest of the three.

The third possibility is best associated with the view that the wave function of a system is merely a representation of our information about that system. However, supporters of this view very often also seem to subscribe to the first possibility as well, at least insofar as microscopic reality is concerned. (I shall argue later that also (2) and (3) are not as incompatible as they seem to be.) But here we should recall the words of Bell [2, page 201], concerning the theories that reject (1) in favor of (2):

> Absurdly, such theories are known as "hidden variable" theories. Absurdly, for there it is not in the wavefunction that one finds an image of the visible world, and the results of experiments, but in the complementary "hidden"(!) variables. Of course the extra variables are not confined to the visible "macroscopic" scale. For no sharp definition of such a scale could be made. The "microscopic" aspect of the complementary variables is indeed hidden from us. But to admit things not visible to the gross creatures that we are is, in my opinion, to show a decent humility, and not just a lamentable addiction to metaphysics. In any case, the most hidden of all variables, in the pilot wave picture, is the wavefunction, which manifests itself to us only by its influence on the complementary variables.

The idea that the wave function merely represents information, and does not describe an objective state of affairs, raises many questions and problems:

- Information about what?

- What about quantum interference? How can the terms of a quantum superposition interfere with each other, producing an observable interference pattern, if such a superposition is just an expression of our ignorance?

- The problem of vagueness: Quantum mechanics is supposed to be a fundamental physical theory. As such it should be precise. But if it is fundamentally about information, then it is presumably concerned directly either with mental events or, more likely, with the behavior of macroscopic variables. But the notion of the macroscopic is intrinsically vague.

- Simple physical laws are to be expected, if at all, at the most fundamental level—of the basic microscopic entities—and that messy complications should arise at the level of larger complex systems. It is only at this level that talk of information, as opposed to microscopic reality, can become appropriate.

- The very form of the Hamiltonian and wave function strongly points to a microscopic level of description.

- There is a widespread belief that large things are built out of small ones, and that to understand even the large we need to understand the small.

Nonetheless, many arguments suggest that quantum mechanics is about information, or that the wave function represents information. (This suggestion is usually accompanied by the claim that if you ask for more—if you try to regard quantum mechanics or the wave function as describing an objective microscopic reality—you get into trouble.) I don't want to directly criticize these here. Rather I want to observe that Bohmian mechanics, the simplest version of quantum mechanics—discovered by Louis de Broglie [3, page 119] in 1927 and rediscovered by Jeffrey's mentor David Bohm [4] in 1952—does do more, and thus I want to try to understand how, from the perspective of Bohmian mechanics, the informational aspect of the wave function or the quantum state can seem natural. I wish to discuss in particular the following three informational aspects of the wave function in Bohmian mechanics:

- The wave function as a property of the environment.

- The wave function as providing the best possible information about the system (given by $|\psi|^2$).

- The wave function as nomological.

I note as well that Bohm and Hiley [5] wrote of the wave function as "active information."

Before proceeding to the description of Bohmian mechanics, I would like to recall the conventional wisdom on the subject. So here are three typical recent statements about hidden variables and the like, the second from a very popular textbook on quantum mechanics. The reader should bear these in mind when reading about Bohmian mechanics. In particular, he or she should contrast the simplicity of Bohmian mechanics with the complexity, implausibility or artificiality suggested by the quotations.

> Thus, unless one allows the existence of contextual hidden variables with very strange mutual influences, one has to abandon them—and, by extension, 'realism' in quantum physics—altogether. (*Gregor Weihs* [6, The truth about reality])

> Over the years, a number of hidden variable theories have been proposed, to supplement q.m.; they tend to be cumbersome and implausible, but never mind–until 1964 the program seemed eminently worth pursuing. But in that year J.S. Bell proved that *any* local hidden variable is *incompatible* with quantum mechanics.[1] (*D.J. Griffiths* [7, page 423])

> Attempts have been made by Broglie, David Bohm, and others to construct theories based on hidden variables, but the theories are very complicated and contrived. For example, the electron would definitely have to go through only one slit in the two-slit experiment. To explain that interference occurs only when the other slit is open, it is necessary to postulate a special force on the electron which exists only when that slit is open. Such artificial additions make hidden variable theories unattractive, and there is little support for them among physicists. (*Encyclopedia Britannica* [8])

2 Bohmian Mechanics

In Bohmian mechanics the state of an N-particle system is given by its wave function $\psi = \psi(\mathbf{q}_1, \ldots, \mathbf{q}_N) = \psi(q)$ together with the positions $\mathbf{Q}_1, \ldots, \mathbf{Q}_N$, forming the configuration Q, of its particles. The latter define the *primitive ontology* (PO) [9] of Bohmian mechanics, what the theory is fundamentally about. The wave function, in contrast, is not part of the PO of the theory, though that should not be taken to suggest that it is not objective or real. It plays a crucial role in expressing the dynamics for the particles, via a first-order differential equation of motion for the configuration Q, of the form $dQ/dt = v^\psi(Q)$.

The defining equations of Bohmian mechanics are Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi\,, \tag{1}$$

[1]I shall not address in this paper the issue of nonlocality. But what is misleading about the last sentence is its suggestion that the source of the incompatibility is the assumption of hidden variables [2, pages 143 and 150]. What Bell in fact showed is that the source of the difficulty is the assumption of locality. He showed that quantum theory is intrinsically nonlocal, and that this nonlocality can't be eliminated by the incorporation of hidden variables.

where

$$H = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k}\nabla_k^2 + V, \tag{2}$$

for the wave function, and the *guiding equation*

$$\frac{d\mathbf{Q}_k}{dt} = \frac{\hbar}{m_k}\mathrm{Im}\frac{\psi^*\nabla_k\psi}{\psi^*\psi}(\mathbf{Q}_1 \ldots, \mathbf{Q}_N) \tag{3}$$

for the configuration. In the Hamiltonian (2) the m_k are of course the masses of the particles and $V = V(q)$ is the potential energy function. For particles with spin, the products involving ψ in the numerator and the denominator of (3) should be understood as spinor inner products, and when magnetic fields are presents, the ∇_k in (2) and (3) should be understood as a covariant derivative, involving the vector potential $\boldsymbol{A} = \boldsymbol{A}(\mathbf{q}_k)$.

For particles without spin, the ψ^* in the guiding equation (3) cancels, and the equation assumes the more familiar form

$$\frac{d\mathbf{Q}_k}{dt} = \frac{\nabla_k S}{m_k} \tag{4}$$

where S arises from the polar decomposition $\psi = Re^{iS/\hbar}$ with S real and $R \geq 0$. Equation (3) however has two advantages:

- It is explicitly of the form $\frac{J_k}{\rho}$ with J_k the quantum probability current and $\rho = \psi^*\psi = |\psi|^2$ the quantum probability density, a fact of great importance for the statistical implications of Bohmian mechanics.

- With the guiding equation in this form, Bohmian mechanics applies without further ado also to particles with spin; in particular there is no need to associate any additional discrete spin degrees of freedom with the particles—the fact that the wave function is spinor valued entirely takes care of the phenomenon of spin.

A surprising and striking fact about Bohmian mechanics is its simplicity and obviousness. Indeed, given Schrödinger's equation, from which one immediately extracts J and ρ, related classically by $J = \rho v$, it takes little imagination when looking for an equation of motion for the positions of the particles in quantum mechanics to consider the possibility that $v = J/\rho$, which is precisely (3).

But even without having arrived at Schrödinger's equation, or parallel with doing so, we could easily guess the guiding equation (4) for particles without spin: The de Broglie relation $\mathbf{p} = \hbar\mathbf{k}$ is a remarkable and mysterious distillation of the experimental facts associated with the beginnings of quantum theory. This relation, connecting a particle property, the momentum $\mathbf{p} = m\mathbf{v}$, with a wave property, the wave vector $\mathbf{k}$, immediately yields

Schrödinger's equation, giving the time evolution for ψ, as the simplest wave equation that reflects this relationship. This is completely standard and very simple. Even simpler, but not at all standard, is the connection between the de Broglie relation and the guiding equation, giving the time evolution for Q: The de Broglie relation says that the velocity of a particle should be the ratio of $\hbar\mathbf{k}$ to the mass of the particle. But the wave vector $\mathbf{k}$ is defined for only for a plane wave. For a general wave ψ, the obvious generalization of $\mathbf{k}$ is the local wave vector $\nabla S(\mathbf{q})/\hbar$, and with this choice the de Broglie relation becomes the guiding equation $dQ/dt = \nabla S/m$.

3 The Implications of Bohmian Mechanics

That a theory is simple and obvious doesn't make it right. And in the case of Bohmian mechanics this fact suggests in the strongest possible terms that it must be wrong. If something so simple could account for quantum phenomena, it seems extremely unlikely that it would have been ignored or dismissed by almost the entire physics community for so many decades—and in favor of alternatives which seem at best far more radical.

Of course, one can see at a glance, see Fig. 1, that Bohmian mechanics

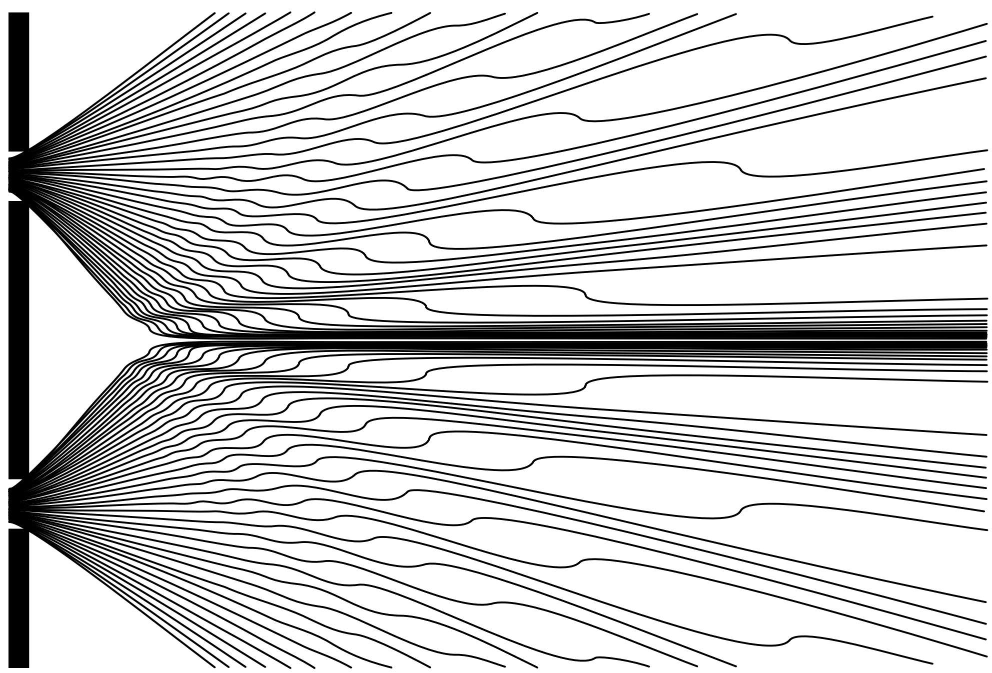

Figure 1: An ensemble of trajectories for the two-slit experiment, uniform in the slits. (Drawn by G. Bauer from [10].)

seems to handle one of the characteristic mysteries of quantum mechanics, the two-slit experiment, quite well. One sees in Fig. 1, in this ensemble of Bohmian trajectories with an approximately uniform distribution of initial positions in the slits, how an interference-like profile in the pattern of trajectories develops after the parts of the wave function emerging from the upper and lower slits begin to overlap.

This of course does not prove that Bohmian mechanics makes the same quantitative predictions for the two-slit experiment—let alone the same predictions for all quantum experiments—as orthodox quantum theory, but it in fact does. Bohmian mechanics is entirely empirically equivalent to orthodox quantum theory, as least insofar as the latter is unambiguous. This was basically shown by Bohm in his first papers [4, 11] on the subject, modulo the status in Bohmian mechanics of the Born probability formula $\rho = |\psi|^2$. That issue was addressed in [12] and is now completely understood. In particular, as a consequence of Bohmian mechanics one obtains the following:

1. familiar (macroscopic) reality
2. formal scattering theory [13]
3. operators as observables [4, 11, 14]
4. quantum randomness [12]
5. absolute uncertainty [12]
6. the wave function of a (sub)system [12]
7. collapse of the wave packet [14]

Concerning these, a few comments. Since macroscopic objects are normally regarded as built out of microscopic constituents, which of course could be point particles, there can be no problem of macroscopic reality per se in Bohmian mechanics. Less obvious, but reasonably clear [15], is the fact that in a Bohmian universe macroscopic objects behave classically, for example moving according to Newton's equations of motion as appropriate.

The picture of what occurs in a Bohmian scattering experiment, in which particles are directed at a target—or at each other—with which they collide and scatter in an apparently random direction, is exactly the picture that an experimentalist has in mind. Moreover, the additional structure (actual particles!) afforded by Bohmian mechanics allows one to considerably sharpen traditional scattering theory both conceptually and indeed mathematically.

It should be noted that operators as observables play no role whatsoever in the formulation of Bohmian mechanics. In fact the only quantum operator that appears in the defining equations of Bohmian mechanics is the Hamiltonian H, but merely as part of an evolution equation. Nonetheless, it turns out that operators on Hilbert space are exactly the right mathematical objects to provide a compact representation of the statistics for the results of experiments in a Bohmian universe.

I wish to focus here in more detail on items 4–7, which are quite relevant to my main concern here, the informational aspects of the wave function in Bohmian mechanics, and which, as it turns out, come together as a package. For example, the statistical properties of the collapse of the wave packet depend upon quantum randomness. It should be noted that the claim that the collapse of the wave packet is an implication of Bohmian mechanics should seem paradoxical, since Schrödinger's equation is an absolute equation of Bohmian mechanics, never to be violated—unlike the situation in orthodox quantum theory.

A crucial ingredient in the emergence of quantum randomness is the *equivariance* of the probability distribution on configuration space given by $\rho^\psi = |\psi|^2$. This means that

$$\left(\rho^\psi\right)_t = \rho^{\psi_t} \tag{5}$$

where on the left we have the evolution of the probability distribution under the Bohmian flow (3) and on the right the probability distribution associated with the evolved wave function ψ_t. That this is so for

$$\rho^\psi(q) = |\psi(q)|^2 \tag{6}$$

is, by (3), equivalent to the quantum continuity equation. The equivariance of $\rho^\psi = |\psi|^2$ means that *if* $\rho_{t_0}(q) = |\psi_{t_0}(q)|^2$ *at some time* t_0 *then* $\rho_t(q) = |\psi_t(q)|^2$ *for all* t. It says that Schrödinger's equation and the guiding equation are compatible modulo $\rho = |\psi|^2$.

The upshot of a long analysis [12] that begins with the equivariance of $\rho^\psi = |\psi|^2$ is that the *quantum equilibrium* given by $\rho_{qe}(q) = |\psi(q)|^2$ has a status very much the same as that of *thermodynamic equilibrium*, described in part by the Maxwellian velocity distribution $\rho_{eq}(\mathbf{v}) \propto e^{-\frac{1}{2}m\mathbf{v}^2/kT}$ for the molecules of a gas in a box in equilibrium at temperature T. It has recently been shown [16] that quantum equilibrium is unique. More precisely, it has been shown that $|\psi(q)|^2$ is the only equivariant distribution that is, in a natural sense, a local functional of the wave function.

In order to grasp the meaning of quantum equilibrium, to appreciate the physical significance $\rho_{qe}(q) = |\psi(q)|^2$, one must first address this question: in a Bohmian universe with wave function Ψ, what is to be meant by the wave function ψ of a subsystem of that universe?

4 The Wave Function of a Subsystem

Consider a Bohmian universe. This is completely described by its wave function Ψ, the wave function of the universe, and its configuration Q. Given an initial condition Ψ_0 and Q_0 for this universe, the equations of motion (1) and (3) determine the trajectories of all particles throughout all of time and hence everything that could be regarded as physical in this universe. However, we are rarely concerned with the entire universe. What we normally deal with in physics is the behavior of a system that is a subsystem of the universe, usually a small one such as a specific hydrogen atom.

It is important to appreciate that a subsystem of a Bohmian universe is not ipso facto itself a Bohmian system. After all, the behavior of a part is entirely determined by the behavior of the whole, so we are not free to stipulate the behavior of a subsystem of a Bohmian universe, in particular that it be Bohmian, having its own wave function that determines the motion of its configuration in a Bohmian way. Nonetheless, there is a rather obvious candidate for the wave function of a subsystem, at least for a universe of spinless particles, and this obvious candidate behaves in exactly the manner that one should expect for a quantum mechanical wave function. (For particles with spin the situation is a little more complicated, so I will confine the presentation here to the case of spinless particles.) This is the conditional wave function, to which I now turn.

Fig. 2 depicts a system corresponding to particles in a certain region (at a given time), a region surrounded by the rest of the universe, in which are contained (at that time) the particles of what we'll call the *environment* of the system. Corresponding to this system we we have a splitting

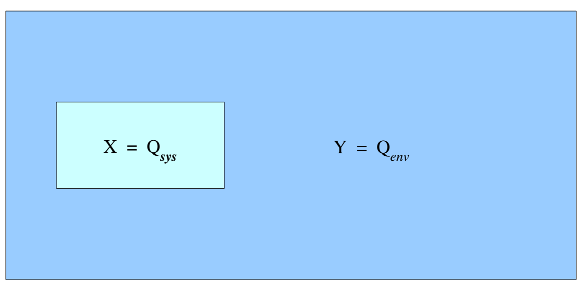

Figure 2: A subsystem of a Bohmian universe

$Q = (Q_{sys}, Q_{env}) = (X, Y)$ of the configuration of the universe into the configurations of system, $Q_{sys} = X$, and environment, $Q_{env} = Y$.

The wave function ψ of the system must be constructed from Ψ, X, and Y, since these provide the complete description of our Bohmian universe (at a given time). The right construction is the following: The wave function ψ of the system, its *conditional wave function* is given by

$$\psi(x) = \Psi(x, Y). \tag{7}$$

Putting in the explicit time dependence, we have that

$$\psi_t(x) = \Psi_t(x, Y_t). \tag{8}$$

Here Y_t is the evolving configuration of the environment, corresponding to the configuration $Q_t = (X_t, Y_t)$, which evolves according to the guiding equation (3) (for the universe, with Ψ instead of ψ).

Note that the conditional wave function, as given in (7) and (8), need not be normalized. In fact these equations should be understood projectively, as defining a ray in the Hilbert space for the system, with wave functions related by a (nonzero) constant factor regarded as equivalent. Of course it is important in probability formulas involving the wave function that it be normalized. In any such formulas it will be assumed that this has been done.

Because of the double time dependence in (8), the conditional wave function ψ_t evolves in a complicated way, and need not obey Schrödinger's equation for the system. Nonetheless, it can be shown [12] that it does evolve according to Schrödinger's equation when the system is suitably decoupled from its environment. While most readers are probably prepared to accept this, since they are quite accustomed to wave functions obeying Schrödinger's equation, that this is so is a bit delicate. What is really easy to see, but what most readers are likely to resist, is the fact, derived in the next subsection, that this wave function collapses according to the usual textbook rules when the system interacts with its environment in the usual measurement-like way.

But before turning to that we should pause to examine the construction (7) of the conditional wave function a little more closely. We would expect a property of a system to correspond to a function of its basic variables—e.g., of its configuration. Note, however, that ψ is a function of the configuration Y of the environment—like a property of the environment! And to the extent that we come to know ψ, that property of the environment can be identified with what we would tend to regard as information about the system—so that it is perhaps only a bit of a stretch to say that ψ represents, or is, our information about the system. (But it is still a stretch.)

4.1 Collapse of the Wave Packet

Consider a quantum observable for the system, given by a self-adjoint operator A on its Hilbert space. For simplicity we assume that A has nondegenerate point spectrum, with normalized eigenstates $\psi_\alpha(x) = |A = \alpha\rangle$, $\|\psi_\alpha\| = 1$,

$$A\psi_\alpha(x) = \alpha\psi_\alpha(x) \tag{9}$$

corresponding to the eigenvalues α. According to standard quantum measurement theory, what is called an ideal measurement of A is implemented by having the system interact with its environment in a suitable way. (To avoid complications we shall assume here that this environment consists of a suitable apparatus, and that the rest of the environment of the system can be ignored—for the the wave function evolution, for the evolution of the configuration of system and apparatus, and for the definition of the conditional wave function of the system. Thus in what follows the configuration of the apparatus will be identified with the configuration Y of the environment of the system.)

The measurement begins, say, at time 0, with the initial ("ready") state of the apparatus given by a wave function $\Phi_0(y)$, and ends at time t. The interaction is such that when the state of the system is initially ψ_α it produces a normalized apparatus state $\Phi_\alpha(y) = |A_{app} = \alpha\rangle$, $\|\Phi_\alpha\| = 1$, that registers that the value found for A is α without having affected the state of the system,

$$\psi_\alpha(x)\Phi_0(y) \xrightarrow{t} \psi_\alpha(x)\Phi_\alpha(y). \tag{10}$$

Here $\xrightarrow{t}$ indicates the unitary evolution induced by the interaction. If the measurement is to provide useful information, the apparatus states must be noticeably different, corresponding, say, to a pointer on the apparatus pointing in different directions. We thus have that the Φ_α have disjoint supports in the configuration space for the environment,

$$\text{supp}(\Phi_\alpha) \cap \text{supp}(\Phi_\beta) = \emptyset, \ \alpha \neq \beta. \tag{11}$$

Now suppose that the system is initially, not in an eigenstate of A, but in a general state, given by a superposition

$$\psi(x) = \sum_\alpha c_\alpha \psi_\alpha(x). \tag{12}$$

We then have, by the linearity of the unitary evolution, that

$$\Psi_0(x,y) = \psi(x)\Phi_0(y) \xrightarrow{t} \Psi_t(x,y) = \sum_\alpha c_\alpha \psi_\alpha(x)\Phi_\alpha(y), \tag{13}$$

so that the final wave function Ψ_t of system and apparatus is itself a superposition. The fact that the pointer ends up pointing in a definite direction, even a random one, is not discernible in this final wave function. Insofar as orthodox quantum theory is concerned, we've arrived at the measurement problem.

However, insofar as Bohmian mechanics is concerned, we have no such problem, because in Bohmian mechanics particles always have positions and pointers, which are made of particles, always point—in a direction determined by the final configuration Y_t of the apparatus. Moreover, in Bohmian mechanics we find that the state of the system is transformed in exactly the manner prescribed by textbook quantum theory.

We have—and this is no surprise—that the initial wave function of the system is

$$\psi_0(x) = \Psi_0(x, Y_0) = \psi(x)\Phi_0(Y_0) \stackrel{p}{=} \psi(x). \tag{14}$$

And for the final wave function of the system we have that

$$\psi_t(x) = \Psi_t(x, Y_t) = \sum_\alpha c_\alpha \psi_\alpha(x)\Phi_\alpha(Y_t) = c_a \psi_a(x)\Phi_a(Y_t) \stackrel{p}{=} \psi_a(x) \tag{15}$$

when $Y_t \in \text{supp}(\Phi_a)$. Here the $\overset{p}{=}$ refers to projective equality, and reminds us that the wave function is to be regarded projectively in Bohmian mechanics.

Thus in Bohmian mechanics the effect of ideal quantum measurement on the wave function of a system is to produce the transition

$$\psi(x) \to \psi_a(x) \quad \text{with probability } p_a, \tag{16}$$

where p_a is the probability that $Y_t \in \text{supp}(\Phi_a)$, i.e., that the value a is registered. Assuming the *quantum equilibrium hypothesis*, that when a system has wave function Ψ its configuration is random, with distribution $|\Psi(q)|^2$, we find, by integrating $|\Psi_t(x,y)|^2$ over $\text{supp}(\Phi_a)$, that $p_a = |c_a|^2$, the usual textbook formula for the probability of the result of the measurement.

4.2 The Fundamental Conditional Probability Formula

The analysis just given suggests—and it is indeed the case [12, 14]—that Bohmian mechanics is empirically equivalent to orthodox quantum theory provided we accept the quantum equilibrium hypothesis. But that the quantum equilibrium hypothesis is true, and even what exactly it means, is a tricky matter, requiring a careful analysis [12] involving typicality that I shall not delve into here. Rather, I shall focus on a simple but important ingredient of that analysis, a probability formula strongly suggesting a connection, if not quite an identification, between the wave function of a system and our information about that system.

This *fundamental conditional probability formula* is the following:

$$P(X_t \in dx \,|\, Y_t) = |\psi_t(x)|^2 dx. \tag{17}$$

Here P is the probability distribution on universal Bohmian trajectories arising from the distribution $|\Psi_0|^2$ on the initial configuration of the universe, with the initial time $t = 0$ the time of the big bang, or shortly thereafter. Of course, by the equivariance of the $|\Psi|^2$ distribution, $|\Psi_t|^2$ at any other time t would define the same distribution on trajectories. The formula says that the conditional distribution of the configuration X_t of the system at time t, given the configuration Y_t of its environment at that time, is determined by the wave function ψ_t of the system in the familiar way.

As a mathematical formula, this is completely straightforward: By equivariance, the joint distribution of X_t and Y_t, i.e., the distribution of $Q_t = (X_t, Y_t)$, is $|\Psi_t(x,y)|^2$. To obtain the conditional probability, y must be replaced by Y_t and the result normalized, yielding $|\psi_t(x)|^2$ with normalized conditional wave function ψ_t.

It is also tempting to read the formula as making genuine probability statements about real-world events, statements that are relevant to expectations about what should actually happen. To do so, as I shall do here, of course goes beyond simple mathematics. At the end of the day, however, such a usage can be entirely justified [12].

I wish to focus a bit more carefully on what is suggested by the fundamental conditional probability formula (17). I shall do so in the next subsection, but before doing so let me rewrite the formula, suppressing the reference to the time t under consideration to obtain

$$P(X \in dx|\,Y) = |\,\psi(x)|^2 dx. \tag{18}$$

It is perhaps worthwhile to compare this with one of the fundamental formulas of statistical mechanics, the Dobrushin-Lanford-Ruelle (DLR) equation

$$P(X \in dx|\,Y) \propto e^{-H(x|Y)/kT} dx \tag{19}$$

for the conditional distribution of the configuration of a classical system given the configuration of its environment, a heat bath at temperature T. Here $H(x|y)$, the energy of the system when its configuration is x, includes the contribution to this energy arising from interaction with the environment. The existence of such a simple formula, which is in fact sometimes used to define the notion of classical equilibrium state, is the main reason that in statistical mechanics, equilibrium is so much easier to deal with than nonequilibrium.

4.3 Quantum Equilibrium and Absolute Uncertainty

There are many ways that we may come to have information about a system. It would be difficult if not impossible to consider all of the possibilities. However, whatever the means by which the information has been obtained, it must be reflected in a correlation between the state of the system and suitable features of the system's environment, such as pointer orientations, ink marks on paper, computer printouts, or the configuration of the brain of the experimenter. All such features are determined by the much more detailed description provided by the complete configuration Y of the environment of the system, which contains much more information than we could hope to have access to.

Nonetheless, the fundamental conditional probability formula (18) says that even this most detailed information can convey no more about the system than knowledge of its wave function ψ, so that in a Bohmian universe the most we could come to know about the configuration of a system is that it has the quantum equilibrium distribution $|\psi|^2$. Thus in a Bohmian universe we have an *absolute uncertainty*, in the sense that the limitations on our possible knowledge of the state of a system expressed by (18) can't be overcome by any clever innovation, regardless of whether it employs current technology or technological breakthroughs of the distant future.

In other words, the fundamental conditional probability formula (18) is a sharp expression of the inaccessibility in a Bohmian universe of micro-reality, of the unattainability of knowledge of the configuration of a system that transcends the limits set by its wave function ψ. This makes it very natural to regard or speak of quantum mechanics, or the wave function, as about information, since the wave function does indeed provide optimal information about a system. At the same time, it seems to me that our best understanding of this informational aspect of the wave function emerges from a theory that is primarily about the very micro-configuration that it shows to be inaccessible!

4.4 Random Systems

While the fundamental conditional probability formula (18) seems very strong, the following stronger version, that applies to random systems, is also true and is often useful, particularly for a careful analysis of the empirical implications of Bohmian mechanics for the results of a sequence of experiments performed at different times [12]:

$$P(X_\sigma \in dx \,|\, Y_\sigma, \sigma) = |\psi_\sigma(x)|^2 dx. \tag{20}$$

In this formula, σ denotes a random system, i.e., a random subsystem with configuration X_σ at a random time T,

$$\sigma = (\pi, T). \tag{21}$$

Here π is a projection, defining a random splitting

$$q = (\pi q, \pi^\perp q) = (x, y). \tag{22}$$

For a given initial universal wave function Ψ_0, σ is determined (like everything else in a Bohmian universe) by the initial universal configuration Q,

$$\sigma = \sigma(Q) = (\pi(Q), T(Q)). \tag{23}$$

Thus

$$X_\sigma = \pi Q_T, \quad Y_\sigma = \pi^\perp Q_T. \tag{24}$$

More explicitly,

$$X_\sigma(Q) = \pi(Q) Q_{T(Q)}, \quad Y_\sigma(Q) = \pi(Q)^\perp Q_{T(Q)}. \tag{25}$$

ψ_σ is defined analogously.

The formula (20) holds provided the random system obeys the measurability condition

$$\{\sigma = \sigma_0\} \in \mathcal{F}(Y_{\sigma_0}), \tag{26}$$

which expresses the requirement that the identity of the random system be determined by its environment. See [12] for details. With this condition, the notion of a random system becomes roughly analogous to that of a *stopping time* in the theory of Markov processes. And the random system fundamental conditional probability formula (20) then becomes analogous to the *strong Markov property,* which plays a crucial role in the rigorous analysis of these processes.

5 The Classical Limit

The classical limit of Bohmian mechanics is reasonably clear [15]; I don't intend to enter into any details here. Rather I wish merely to note that it would be nice to have some rigorous mathematical results in this direction and to make two comments:

- Decoherence plays a controversial role in the classical limit of orthodox quantum theory. It also important for a full appreciation of this limit for Bohmian mechanics, where in fact it is entirely uncontroversial and straightforward. And insofar as decoherence is strongly associated with measurement and observation, Bohmian mechanics provides a natural explanation of the apparent importance of information for the emergence of classical behavior.

- Considerations related to decoherence suggest the following: *In Bohmian mechanics an observed motion, if it seems deterministic, will appear to be classical.* This conjecture provides an ahistorical explanation of why in a Bohmian world classical mechanics would be discovered before Bohmian mechanics: the *observed* deterministic regularities would be classical. (Of course the real explanation, not unrelated, is that we live on the macroscopic level, where objects behave classically.)

6 The Wave Function as Nomological

Perhaps the most significant informational aspect of the wave function is that it is best regarded as fundamentally nomological, as a component of physical law rather than of the physical reality described by the law [17, 18], as I shall now argue.

The wave function in Bohmian mechanics is rather odd in at least two ways—how it behaves and the kind of thing that it is:

- While the wave function is crucially implicated in the motion of the particles, via equation (3), the particles can have no effect whatsoever on the wave function, since Schrödinger's equation is an autonomous equation for ψ, that does not involve the configuration Q.

- For an N-particle system the wave function $\psi(q) = \psi(\mathbf{q}_1, \ldots, \mathbf{q}_N)$ is, unlike the electromagnetic field, not a field on physical space but on configuration space, an abstract space of great dimension.

Though it is possible to perhaps temper these oddities with suitable responses—for example that the action-reaction principle is normally associated with conservation of momentum, which in turn is now taken to be an expression of translation invariance, a feature of Bohmian mechanics—I think we should take them more seriously, and try to come to grips with what they might be telling us.

We are familiar with an object that is somewhat similar to the wave function, namely the Hamiltonian of classical mechanics, a function on a space, phase space, of even higher dimension than configuration space. In fact the classical Hamiltonian is surprisingly analogous to the wave function, or, more precisely, to its logarithm:

$$\log \psi(q) \leftrightarrow H(q,p) = H(\mathscr{X}) \tag{27}$$

where $\mathscr{X} = (q,p) = (\mathbf{q}_1, \ldots, \mathbf{q}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N)$ is the phase space variable. Corresponding to these objects we have the respective equations of motion

$$dQ/dt = der(\log \psi) \leftrightarrow d\mathscr{X}/dt = der H \tag{28}$$

with *der* representing suitable first derivatives.

Note as well that both $\log \psi(q)$ and $H(\mathscr{X})$ are defined only up to an additive constant. For "normalized" choices we further have that

$$\log \text{Prob} \propto \log|\psi| \leftrightarrow \log \text{Prob} \propto -H \tag{29}$$

(This should not be taken too seriously!)

Of course nobody has a problem with the fact that the Hamiltonian is a function on the phase space, since it is not a dynamical variable at all but rather an object that generates the classical Hamiltonian dynamics. As such, it would not be expected to be affected by anything physical either.

But there are some important differences between ψ and H. Unlike H, ψ typically changes with time and serves moreover as (the paradigmatic) initial condition in quantum mechanics:

- ψ_t is dynamical.

- ψ is controllable.

These quite naturally tend to undercut the suggestion that ψ should be regarded as nomological, since, unlike dynamical variables, laws are not supposed to be like that. However, it is important in this regard to bear in mind the distinction:

$$\psi \quad \text{versus} \quad \Psi. \tag{30}$$

6.1 The Universal Level

In Bohmian mechanics the wave function Ψ of the universe is fundamental, while the wave function ψ of a subsystem of the universe is derivative, defined in terms of Ψ by (7). Thus the crucial question about the nature of the wave function in Bohmian mechanics must concern Ψ; once this is settled the nature of ψ will then be determined.

Accordingly, the claim that the wave function in Bohmian mechanics is nomological should be understood as referring primarily to the wave function Ψ of the universe, concerning which it is important to note the following:

- Ψ is not controllable: it is what it is.

- If we are seriously considering the universal or cosmological level, then we should perhaps take the lessons of general relativity into account. Now the significance of being "dynamical," of having an explicit time dependence, is transformed by general relativity, and indeed by special relativity, since the (3,1) splitting of space and time is thereby transformed to a $3 + 1 = 4$ dimensional space-time that admits no special splitting.

- There may well be no "*t*" in Ψ. The Wheeler-DeWitt equation, the most famous equation for the wave function of the universe in quantum gravity, is of the form
$$\mathscr{H}\Psi = 0 \tag{31}$$
with $\mathscr{H}$ a sort of Laplacian on a space of configurations of suitable structures on a 3-dimensional space and with Ψ a function on that configuration space that does not contain a time variable at all. For orthodox quantum theory this is a problem, the *problem of time*: of how change can arise when the wave function does not change. But for Bohmian mechanics, that the wave function does not change is, far from being a problem, just what the doctor ordered for a law, one that governs the changes that really matter in a Bohmian universe: of the variables Q describing the fundamental objects in the theory, including the 3-geometry and matter. The evolution equation should be regarded as more or less of a form
$$dQ/dt = v^{\Psi}(Q) \tag{32}$$
roughly analogous to (3), one that defines an evolution that is natural for the PO of the theory under consideration.

6.2 Schrödinger's Equation as Phenomenological

Of course, accustomed as we are to Schrödinger's equation, we can hardly resist regarding the wave function as time dependent. And it is hard to imagine a simple description of the measurement process in quantum mechanics that does not invoke a time dependent wave function. In this regard, it is important to bear in mind that the fact—if it is a fact—that the wave function Ψ of the universe does not change in no way precludes the wave function ψ of a subsystem from changing. On the contrary, since a solution to the Wheeler-deWitt equation (31) is in fact just a special (time-independent) solution to Schrödinger's equation, it follows, as said earlier in Section 4—assuming that the considerations alluded to earlier for Bohmian mechanics apply to the relevant generalization of Bohmian mechanics—that the conditional wave function

$$\psi_t(x) = \Psi(x, Y_t) \tag{33}$$

will evolve according to Schrödinger's equation when the subsystem is suitably decoupled from its environment (and $\mathscr{H}$ is of the appropriate form).

In this way what is widely taken to be the fundamental equation of quantum mechanics, the time-dependent Schrödinger equation, might turn out to be merely phenomenological: an emergent equation for the wave function of suitable subsystems of a Bohmian universe. Moreover, even the time-independent Schrödinger equation (31) might best be regarded accidental rather than fundamental. What happens in a Bohmian universe with universal wave function Ψ is entirely determined by the equation of motion (32) for the PO of the theory. This theory is then determined by Ψ and the form of v^Ψ. (31) will be fundamental only if it constrains the choice of Ψ, but this need not be so. It might well be that the choice of Ψ is fundamentally constrained by entirely different considerations, such as the desired symmetry properties of the resulting theory, with the fact that Ψ also obeys (31) thus being accidental.

6.3 Two Transitions

Suppose what I've written here about the fundamental Bohmian mechanics, Universal Bohmian Mechanics (UBM), is correct. Then our understanding of the nature of quantum reality is completely transformed, as is the question about the nature of the wave function in quantum mechanics with which we began:

- $$\begin{array}{ccccc} \text{OQT} & & \text{BM} & & \text{UBM} \\ \Psi & \longrightarrow & (\Psi, Q) & \longrightarrow & Q \end{array}$$

- $$\begin{array}{ccc} \begin{array}{ccc} & ? & \\ ? & \psi & ? \\ & ? & \end{array} & \longrightarrow & \begin{array}{ccc} & ? & \\ ? & \Psi & ? \\ & ? & \end{array} \end{array}$$

The first transition is of the basic variables involved as we proceed from orthodox quantum theory, which seems to many to involve as a basic variable only the wave function Ψ—and certainly no hidden variables; to the usual Bohmian mechanics, whose basic variables are Ψ and Q; to UBM, with Q the only fundamental physical variable, the universal wave function Ψ remaining only as a mathematical object convenient for expressing the law of motion (32).

And accordingly, the question about the meaning of the wave function in quantum mechanics is utterly transformed, from something like, What on earth does the wave function ψ of a system physically describe? to, Why on earth should a wave function Ψ play a prominent role in the law of motion (32) defining quantum theory? What's so good about such a motion?

Once we recognize that the wave function is nomological we are confronted with a transformed landscape for understanding why nature should be quantum mechanical. We will fully comprehend this once we understand what is so special and compelling about a motion governed by a wave function in Bohmian way.

6.4 Nomological versus Nonnomological

I can well imagine many physicists, when confronted with the question of whether the wave function should be regarded as nomological or as more concretely physical, responding with a loud, Who cares! What difference does it make? But quite aside from the fact that it is conceptually valuable to understand the nature of the objects we are dealing with in a fundamental physical theory, the question matters in a practical way. It is relevant to our expectations for future theoretical developments.

In particular, laws should be simple, so that if Ψ is nomological, it too—and the law of motion (32) it defines–should somehow be simple as well. The contention that Ψ is nomological would be severely undermined if this were not achievable.

Simplicity of course comes in many varieties. Ψ might be straightforwardly simple, i.e., a simple function of its argument, expressible in a compelling way using the structure at hand. It might be simple because it is a solution, perhaps the unique solution, to a simple equation. Or it might be the case that there is a compelling principle, one that is simple and elegant, that is satisfied, perhaps uniquely, by a law of motion of the form (32) with a specific Ψ and v^{Ψ}. For example, the principle might express a very strong symmetry condition.

6.5 Covariant Geometrodynamics

Stefan Teufel and I have examined such a possibility for quantum gravity [19], with the symmetry principle that of 4-diffeomorphism invariance. Within (an extension of) the framework of the ADM formalism, the dynamical formulation of general relativity of Arnowitt, Deser, and Misner [20], we considered the possibilities for a first-order covariant geometrodynamics.

In the ADM formalism the dynamics corresponds to the change of structures, most importantly a 3-geometry, on a space-like hypersurface as that surface is infinitesimally deformed. In a theory for which there is no special foliation of space-time into hypersurfaces (that might define the notion of simultaneity if it existed), a hypersurface Σ can naturally be deformed in an infinite dimensional variety of ways. These are given by the function $\mathrm{N} = (N, \vec{N})$, where $N = N(x)$, $x \in \Sigma$, is the *lapse* function describing deformations normal to the surface, and $\vec{N} = \vec{N}(x)$ is the *shift* function describing deformations in the surface, i.e., infinitesimal 3-diffeomorphisms. Corresponding to the many possible deformations N, one often speaks here of a *multi-fingered time*.

The deformations N form an algebra, the Dirac Algebra, which is almost a Lie algebra and should be regarded as somehow corresponding to the group of 4-diffeomorphisms of space-time. The Dirac Algebra, with Dirac bracket $[\mathrm{N}, \mathrm{M}]$, is defined, using linearity, by

$$[N, M] = N\vec{\nabla}M - M\vec{\nabla}N; \quad [N, \vec{M}] = \vec{M} \cdot \vec{\nabla}N \tag{34}$$

together with the usual Lie bracket $[\vec{N}, \vec{M}]$ for the Lie algebra of the group of 3-diffeomorphisms.

Within this multi-fingered time framework, a first-order dynamics corresponds, not to a single vector field on the configuration space $\mathscr{Q}$—of decorations of Σ—in which the evolution occurs, but to a choice of vector field $\mathscr{V}(\mathrm{N})$ for each deformation N. (See [21] for the more familiar second-order, phase space, Poisson bracket approach.) Moreover, it seems, at least heuristically, that the dynamics so defined will be covariant precisely in case $\mathscr{V}(\mathrm{N})$ forms a *representation of the Dirac algebra*:

$$[\mathscr{V}(\mathrm{N}), \mathscr{V}(\mathrm{M})] = \mathscr{V}([\mathrm{N}, \mathrm{M}]), \tag{35}$$

where the bracket on the left is the Lie bracket of vector fields.

The claim that such a dynamics is covariant is intended to convey that it defines a 4-diffeomorphism invariant law for a decoration of space-time; a crucial ingredient in this is that the dynamics be path-independent: that two different foliations that connect the same pair Σ_i and Σ_f of hypersurfaces, corresponding to two different paths through the multi-fingered time $\{\mathrm{N}\}$, yield the same evolution map connecting decorations of Σ_i to decorations of Σ_f.

The requirement that $\mathscr{V}(\mathrm{N})$ form a representation of the Dirac Algebra is a very strong symmetry condition. Our hope was that it was so strong that it would force the dynamics to be quantum mechanical: $\mathscr{V}(\mathrm{N}) = \mathscr{V}^{\Psi}(\mathrm{N})$ where $\mathscr{V}^{\Psi}$ is a suitable functional of Ψ, with Ψ obeying an equation of the form (31). It seems, however, for pure quantum gravity, with $\mathscr{Q}$ the space of 3-geometries (super-space), that any covariant dynamics is classical, yielding 4-geometries that obey the Einstein equations, with a possible cosmological constant, and with no genuinely quantum mechanical possibilities arising.

When, in addition to geometry, structures corresponding to matter are included in $\mathscr{Q}$, it is not at all clear what the possibilities are for the representations of the Dirac algebra. It seems a long shot that a quantum mechanical dynamics could be selected in this way as the only possibility, let alone one that corresponds to a more or less unique Ψ. But since a positive result in this direction would be so exciting, this program seems well worth pursuing further—even if only to establish its impossibility.

6.6 The Value of Principle

It is often suggested that what is unsatisfactory about orthodox quantum theory is that it was not formulated as a theory based on a compelling principle, an information theoretic principle or whatever. Often such a derivation is then supplied. If, as is usually the case, what we then arrive at is—as presumably intended—plain old orthodox quantum theory, I find myself unsatisfied by the accomplishment.

The reason is this. The problem with orthodox quantum theory is not that the principles from which it might be derived are unclear or absent, but that the theory itself is, in the words of Bell [2, page 173], "unprofessionally vague and ambiguous." Thus if derivation from a principle only yields orthodox quantum theory, how has the problem of understanding what quantum mechanics actually says been at all addressed? Of course, if the derivation yields, not orthodox quantum theory, but an improved formulation of quantum mechanics, then the problem may well have been alleviated. But this rarely happens.

It is fine and good to want to understand why a theory should hold. But before worrying about this we should first get clear about what the theory in fact says. The crucial distinction is between the question, Why? and the question, What?: Why should quantum theory hold? versus What does quantum theory say? A derivation of quantum theory will address the real problem with quantum mechanics if it provides answer to What? and not just an answer to Why? The sorts of derivation from a principle contemplated in Sections 6.4 and 6.5 are of this form.

7 Quantum Rationality

I conclude with two quotations. The first addresses the question, if Bohmian mechanics is so simple and elegant, and accounts for quantum phenomena is such a straightforward way, why is this not recognized by the physics community?

> I know that most men, including those at ease with problems of the highest complexity, can seldom accept even the simplest and most obvious truth if it be such as would oblige them to admit the falsity of conclusions which they have delighted in explaining to colleagues, which they have proudly taught to others, and which they have woven, thread by thread, into the fabric of their lives. (*Leo Tolstoy*)

I have another reason for quoting Tolstoy here: I would like to know where he said this. If any reader knows, I would be very grateful if he contacted me with the information.

The Tolstoy is of course a bit depressing. So I will conclude on a more optimistic note [22, page 145], from the philosopher of science Imre Lakatos, who was an early teacher of Jeffrey's.

> In the new, post-1925 quantum theory the 'anarchist' position became dominant and modern quantum physics, in its 'Copenhagen interpretation', became one of the main standard bearers of philosophical obscurantism. In the *new* theory Bohr's notorious 'complementarity principle' enthroned [weak] inconsistency as a basic ultimate feature of nature, and merged subjectivist positivism and antilogical dialectic and even ordinary language philosophy into one unholy alliance. After 1925 Bohr and his associates introduced a new and unprecedented lowering of critical standards for scientific theories. This led to a defeat of reason within modern physics and to an anarchist cult of incomprehensible chaos. (*1965*)

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