NOTES TO LECTURE 9 Math 621, Fall 2011

I Second order Taylor polynomials and remainder

A. Functions of a scalar variable.

Let f be a function defined on $\mathbb{R}$ and let $a \in \mathbb{R}$. The second order Taylor polynomial of f at a is

$$P_{2,a}(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2} f''(a)(x-a)^2$$

It is the unique quadratic function satisfying $P_{2,a}(a) = f(a)$ $P_{2,a}'(a) = f'(a)$, and $P_{2,a}''(a) = f''(a)$. The remainder $R_a(x)$ is defined by

$$\begin{aligned} f(x) &= P_{2,a}(x) + R_a(x) \\ &= f(a) + f'(a)(x-a) + \tfrac{1}{2} f''(a)(x-a)^2 + R_a(x) \end{aligned}$$

$P_{2,a}$ should be thought of as a quadratic approximation to f near a; $R_a(x)$ is the difference between f and $P_{2,a}$. How good is it?

<u>Theorem 1</u> If f, f', and f'' exist and are continuous everywhere

$$\lim_{x \to a} \frac{|R_a(x)|}{(x-a)^2} = 0 \qquad (1)$$

Thus $|R_a(x)|$ is an order of magnitude smaller than the quadratic terms in $P_{2,a}(x)$ as $x \to a$.

B. Functions $f(x_1, \ldots, x_n)$ on $\mathbb{R}^n$.

In this section $x = (x_1, \ldots, x_n)$.

<u>Def</u>: We say $f \in C^2$ if f_{x_i}, $f_{x_i x_j}$ exist and are continuous for all i and j, $1 \le i, j \le n$. Here

$$f_{x_i}(x) = \frac{\partial f}{\partial x_i}(x) \quad , \quad f_{x_i x_j}(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

The second order Taylor polynomial of f at $a = (a_1, \dots, a_n)$ is

$$P_{2,a}(x) = f(a) + \sum_{i=1}^{n} f_{x_i}(a)(x_i - a_i)$$

(2)
$$+ \frac{1}{2} \sum_{i,j=1}^{n} f_{x_i x_j}(a)(x_i - a_i)(x_j - a_j)$$

In the case $n = 2$, $P_{2,a}(x)$ written out is

$$f(a) + f_{x_1}(a)(x_1 - a_1) + f_{x_2}(a)(x_2 - a_2)$$

(3)
$$+ \frac{1}{2} f_{x_1 x_1}(a)(x_1 - a_1)^2 + f_{x_1 x_2}(a)(x_1 - a_1)(x_2 - a_2) + \frac{1}{2} f_{x_2 x_2}(a)(x_2 - a_2)^2$$

Theorem I $R_{2,a}(x)$ is defined as before by

$$f(x) = P_{2,a}(x) + R_a(x)$$

Theorem 2 If $f \in C^2$, $\lim_{x \to a} \frac{|R_a(x)|}{\sum_1^n (x_i - a_i)^2} = 0$.

Again, $R_a(x)$ is of an order smaller than the terms in $P_{2,a}(x)$ as $x \to a$.

C. Convenient form (intuitive) to represent the Taylor polynomial approximation to second order

(i) For functions of a scalar variable

$$f(y + \Delta y) - f(y) = f'(y)\Delta y + \frac{1}{2} f''(y) \Delta y^2 + o((\Delta y)^2) \qquad (4)$$

(Here $o((\Delta y)^2)$ indicates a term such that $\lim_{\Delta y \to 0} \frac{o((\Delta y)^2)}{(\Delta y)^2} = 0$)

We get this by replacing a by y and x by $y + \Delta y$, in the definitions of section I.A.

(ii) Functions of n-variables

Replacing $a = (a_1, \dots, a_n)$ by $y = (y_1, \dots, y_n)$ in II.B. and x by $x = y + \Delta y$, where $\Delta y = (\Delta y_1, \dots, \Delta y_n)$,

$$f(y+\Delta y) - f(y) = \sum_1^n f_{x_i}(y)\Delta y_i + \frac{1}{2}\sum_{i,j=1}^n f_{x_i x_j}(y)\Delta y_i \Delta y_j + o\left(\sum_1^n (\Delta y_i)^2\right) \qquad (5).$$

II Stochastic Calculus

Throughout, W is a Brownian motion and $\{\mathcal{F}(t)\}_{t\geq 0}$ is a filtration for W.

A. Itô processes; differential notation

DEFINITION A stochastic process of the form

$$X(t) = X(0) + \int_0^t \beta(s)ds + \int_0^t \alpha(s)dW(s), \quad t \leq T, \qquad (6)$$

where i) $\{\beta(s)\}_{0\leq s\leq T}$, $\{\alpha(s)\}_{0\leq s\leq T}$ are stochastic processes adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$; ii) the integrals in (6) are defined; and iii) $X(0)$ is $\mathcal{F}(0)$-measurable, is called an Itô process.

Another notation used to express (6) is

$$dX(t) = \beta(t)dt + \alpha(t)dW(t), \quad t \leq T. \qquad (7)$$

The expressions '$dX(t)$', 'dt' and '$dW(t)$' are called differentials. We do not try to assign rigorous mathematical definitions of differentials. One should consider (7) to be a formal way to express (7).

But why bother with this formal, differential notation? Because differentials can be given intuitive meanings that help one understand Itô calculus conceptually and that help guide modeling and computation using Itô processes. As in ordinary calculus, 'dt' should be thought of as an infinitesimally small increment in the t variable. Differentials of a stochastic process should be thought of as <u>forward</u> increments of the process over a time interval of duration dt; thus

$$\text{“}dX(t) = X(t+dt) - X(t)\text{”}, \quad \text{“}dW(t) = W(t+dt) - W(t)\text{”}$$

Here, <u>forward</u> means that the differential at t, is the increment over the time interval $[t, t+dt]$ going forward into the future. The identities are put in quotes to indicate their formal nature. Then (7) can be interpreted as saying

$$\text{“}X(t+dt) - X(t) = \beta(t)\,dt + \alpha(t)\left[W(t+dt) - W(t)\right]\text{”}. \tag{8}$$

This expression makes clear that the change in X due to W over an infinitesimally small interval is the product of an $\mathcal{F}(t)$-measurable random variable $\alpha(t)$ --- remember, we assume $\alpha(\cdot)$ is adapted to the filtration --- times the <u>forward</u> increment $W(t+dt) - W(t)$, which is <u>independent</u> of $\alpha(t)$. That Itô integrals are built by adding up such products is a central concept of stochastic calculus. An integral is a limit of sums of increments, so formally $X(t) - X(0) = \int_0^t dX(s)$. By replacing $dX(s)$ by $\beta(s)\,ds + \alpha(s)\,dW(s)$, one is led from the formal expression (7) back to (6).

If $\alpha \equiv 0$, the differential notation $dX(t) = \beta(t)\,dt$ gives a different way to express the ordinary integral $X(t) - X(0) = \int_0^t \beta(s)\,ds$.

B. Differentials and Modeling

Usually one constructs Itô process models starting from a differential view-point. To illustrate, we write down a generalized Black-Scholes-Merton model for the price of a risky asset. Let $\{S(t),\ 0 \le t \le T\}$ denote the price process. The return on owning one share over the time interval $[t, t+dt]$ is

$$\frac{S(t+dt) - S(t)}{S(t)}.$$

Think of this as a random value that consists of a known expected <u>rate</u> of return

$$\alpha(t) = \frac{E\left[\left.\frac{S(t+dt)-S(t)}{S(t)}\right| \mathcal{F}(t)\right]}{dt}$$

(We condition on $\mathcal{F}(t)$ because the rate is known given the information in $\mathcal{F}(t)$)

plus an random fluction of zero mean around this rate One way to model this is

$$\frac{dS(t)}{S(t)} = \frac{S(t+dt)-S(t)}{S(t)} = \alpha(t)dt + \sigma(t)dW(t) \qquad (9)$$

where $\sigma(\cdot)$ is an $\{\mathcal{F}(t)\}$-adapted <u>volatility</u> process. The fluctuation $\sigma(t)dW(t) = \sigma(t)[W(t+dt)-W(t)]$ has variance $\sigma^2(t)dt$ and, since $W(t+dt)-W(t)$ is independent of the past $\mathcal{F}(t)$, has zero mean (assuming $E|\sigma(t)| < \infty$) because $E[\sigma(t)[W(t+dt)-W(t)]|\mathcal{F}(t)] = \sigma(t)\, E[W(t+dt)-W(t)|\mathcal{F}(t)] = 0$

The random input $W(t+dt)-W(t)$ 'driving' the fluctuation in $dS(t)$ is independent of $\mathcal{F}(t)$ and hence is not predictable in any way from past information. The model one gets from (9) is thus

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t), \qquad (10)$$

where $\alpha(\cdot)$ and $\sigma(\cdot)$ are $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted. This equation is an example of a stochastic differential equation. Written in integral form, it is equivalent to

$$S(t) = S(0) + \int_0^t \alpha(s)S(s)\,ds + \int_0^t \sigma(s)S(s)\,dW(s). \qquad (11).$$

These equations do not give explicit formulas for $S(t)$, $t\geq 0$. Rather they specify a condition that $S(t)$ should satisfy. Whether, a solution $S(t)$, $t\geq 0$, to those equations exists is a different matter that will be treated later.

When α and σ are non-random and constant, we get the standard Black-Scholes price model

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t). \qquad (12)$$

C. Products of differentials

Since integrals are limits of sums over finer and finer partititions it makes intuitive sense to think of $\int_0^t (dW(s))^2 = \lim_{\|\Pi\|\to 0} \sum_1^n [W(t_{i+1}) - W(t_i)]^2$ where Π denotes a partition of $[0,t]$. But we know this limit is quadratic variation of Brownian motion leading to the formal identity

$$\text{"} \int_0^t (dW(s))^2 = [W,W](t) = t, \quad 0 < t \text{"} \qquad (13)$$

Again, one should think of this identity in a purely formal way. By considering it's differential form, (13) suggests the formal identity

$$\text{"} \left(dW(t)\right)^2 = dt \text{"} \qquad (14).$$

To generalize this idea, we will express rigorous identities for quadratic variation and quadratic cross variation using products of differentials; that is

$$\text{"} dX(t)\,dY(t) = d\,[X,Y](t) \text{"} \qquad (15)$$

where $$[X,Y] = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} \left[X(t_{i+1}) - X(t_i)\right]\left[Y(t_{i+1}) - Y(t_i)\right].$$

Hence, since we have shown

$$\lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} \left[t_{i+1} - t_i\right]^2 = 0$$

$$\lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} \left[t_{i+1} - t_i\right]\left[W(t_{i+1}) - W(t_i)\right] = 0$$

we will write

$$(dt)^2 = d\,[t,t] = 0$$

$$dt\,dW(t) = d\,[t,W] = 0 \qquad (16)$$

At the formal level it is valid to apply the usual algebraic rules to products of differentials. Thus, if $dX(t) = \beta(t)dt + \alpha(t)dW(t)$

$$dX(t)\,dt = \beta(t)(dt)^2 + \alpha(t)\,dW(t)\,dt = 0$$

$$dX(t)\,dW(t) = \beta(t)\,dt\,dW(t) + \alpha(t)(dW)^2(t) = \alpha(t)\,dt$$

and

$$\left(dX(t)\right)^2 = \left[\beta(t)dt + \alpha(t)dW(t)\right]^2$$

$$= \beta^2(t)(dt)^2 + 2\beta(t)\alpha(t)\,dt\,dW(t) + \alpha^2(t)\left(dW(t)\right)^2$$

$$= \alpha^2(t)\,dt \quad . \qquad (17).$$

D. <u>Itô's rule</u>

Let $X(t)$, $0 \le t \le T$, be an Itô process, and let f be a function of a real variable.

<u>Question</u> Is $f(X(t))$ and Itô process. If so, what is $d\left[f(X(t))\right]$?

Itô's rule addresses this question.

(i) <u>The chain rule case</u>.

When $dX(t) = \beta(t)dt$, that is, when $X(t) = X(0) + \int_0^t \beta(s)ds$, our question above is answered by the chain rule from calculus.

<u>Proposition</u> If f' exists everywhere and is continuous. and if $X(t) = X(0) + \int_0^t \beta(s)ds$, then

$$df(X(t)) = f'(X(t))dX(t) = f'(X(t))\beta(t)dt.$$

<u>Proof</u>. By the fundamental theorem of calculus and the chain rule

$$f(X(t)) - f(X(0)) = \int_0^t \frac{d}{ds} f(X(s))\,ds = \int_0^t f'(X(s))X'(s)\,ds$$

so $f(X(t)) - f(X(0)) = \int_0^t f'(X(s))\beta(s)ds$, which, in differential notation is $d[f(X(t))] = f'(X(t))\beta(t)dt$. ∎

(ii) The stochastic case; formal derivation.

Assume now that

$$dX(t) = \beta(t)dt + \alpha(t)dW(t) \qquad (18)$$

and that $f \in C^2$. We will assume that $f(X(t))$ is an Itô process. To formally compute $d[f(X(t))]$ we will formally approximate

$$f(X(t+dt)) - f(X(t))$$

and keep only terms of order dt; that is, any terms which go to zero faster than dt will be discarded. Write

$$X(t+dt) = X(t) + dX(t)$$

By the Taylor polynomial approximation (4), with $X(t)$ in place of y and $dX(t)$ in place of Δy,

$$d[f(X(t))] = f(X(t+dt)) - f(X(t))$$

$$= f'(X(t))dX(t) + \tfrac{1}{2}f''(X(t))(dX(t))^2 + o((dX(t))^2)$$

But by (17), $(dX(t))^2 = \alpha^2(t)dt$ and so the $o((dX(t))^2)$ term goes to zero faster than dt and may be neglected. Thus using (17) and (18)

$$d[f(X(t))] = f'(X(t))\beta(t)dt + \tfrac{1}{2}f''(X(t))\alpha^2(t)dt$$
$$+ f'(X(t))\alpha(t)dW(t).$$

Although this calculation was purely formal, it gives the correct answer, which we state as a theorem (but we do not prove the theorem.)

<u>Theorem</u> (Itô's rule, case a).

Let $dX(t) = \beta(t)dt + \alpha(t)dW(t)$ be an Itô process. Let $f \in C^2$. Then $f(X(t))$ is an Itô process and.

$$d\left[f(X(t))\right] = \left[f'(X(t))\beta(t) + \tfrac{1}{2}f''(X(t))\alpha^2(t)\right]dt + f'(X(t))\alpha(t)\,dW(t) \qquad (19)$$

In integral form

$$f(X(t)) = f(X(0)) + \int_0^t \left[f'(X(s))\beta(s) + \tfrac{1}{2}f''(X(s))\alpha^2(s)\right]ds + \int_0^t f'(X(s))\alpha(s)\,dW(s). \qquad (20).$$

<u>Remarks</u>

1. The novel feature appearing in (19) and (20) that does not appear in the chain rule for functions of ordinary integrals is the term

$$\tfrac{1}{2}f''(X(t))\alpha^2(t)\,dt.$$

The fact that this term appears is due entirely to the fact that the quadratic variation of Brownian motion up to time t equals t for all t. This term is called the <u>Itô correction</u>

term.

2. It can happen that $f \in C^2$ and

$$E\left[\int_0^T \left[f'(X(t))\,\alpha(t)\right]^2 dt\right] = \infty \qquad (21)$$

However we required this expectation to be finite to define the stochastic integral term

$$\int_0^t f'(X(s))\,\alpha(s)\,dW(s), \quad \text{for } t \le T.$$

But, there is really no problem. One can extend the stochastic integral and define $\int_0^t \gamma(s)\,dW(s)$, $t \le T$, assuming only that γ is adapted and

$$\mathbb{P}\left(\int_0^T \gamma^2(s)\,ds < \infty\right) = 1$$

This condition <u>will</u> be satisfied for $\gamma(s) = f'(X(s))\,\alpha(s)$ if $P\left(\int_0^T \alpha^2(s)\,ds < \infty\right) = 1$. However, if (21) is true

$\int_0^t f'(X(s))\,\alpha(s)\,dW(s)$ may not be a martingale.

E. <u>Examples</u>

Many examples are worked out in the text and in problem assignments. Here we concentrate on one, which is also in the text.

<u>Claim:</u> $S(t) = S(0)\exp\left\{ \int_0^t \sigma(s)\,dW(s) + \int_0^t [\alpha(s) - \tfrac{1}{2}\sigma^2(s)]\,ds \right\}$

solves the price model (10):

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t).$$

To show this let

$$f(x) = S(0)e^x$$

and

$$X(t) = \int_0^t \sigma(s)\,dW(s) + \int_0^t [\alpha(s) - \tfrac{1}{2}\sigma^2(s)]\,ds$$

Then $S(t) = f(X(t))$ and $f'(x) = f''(x) = f(x)$.

Hence, by Itô's rule

$$dS(t) = \left[f'(X(t))\left[\alpha(t) - \tfrac{1}{2}\sigma^2(t)\right] + \tfrac{1}{2} f''(X(t))\sigma^2(t) \right] dt$$

$$+ f'(X(t))\,\sigma(t)\,dW(t)$$

$$= S(t)\left[\alpha(t) - \tfrac{1}{2}\sigma^2(t) + \tfrac{1}{2}\sigma^2(t) \right] dt + S(t)\sigma(t)\,dW(t)$$

$$= S(t)\alpha(t)\,dt + S(t)\sigma(t)\,dW(t).$$

F. Other variants on Itô's rule.

See the text and Assignment 8.