II. Martingales and risk-neutral measures.

A. Martingales; Definition

Martingales are models of gambling games that are fair to risk-neutral players.

Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \cdots$ be a filtration; $\mathcal{F}_k$ represents the information available to the player after the k^{th} play. Let $X_1, X_2, \ldots$ represent the player's fortune; X_k is what the player has after the k^{th} play. The conditional expectation

$$E[X_{k+1} | \mathcal{F}_k]$$

is what the player expects to have after the next play given his position and knowledge at the end of play k. He considers the game fair if

$$E[X_{k+1} | \mathcal{F}_k] = X_k.$$

This is the martingale property.

Definition $\{X_n\}_{n \geq 0}$ is a martingale w.r.t. the filtration $\{\mathcal{F}_n\}_{n \geq 0}$ if

(i) X_n is $\mathcal{F}_n$-measurable for all n;

(ii) $E|X_n| < \infty$ for all n;

(iii)

$$E[X_{n+1} | \mathcal{F}_n] = X_n \quad \text{for all } n. \qquad (12)$$

(actually (iii) implies X_n is $\mathcal{F}_n$-meas. but we include this as condition (i) for emphasis)

B. Martingales; Examples

Example 1. Let $\xi_1, \xi_2, \ldots$ be independent and identically distributed with $\mathbb{P}(\xi_i = 1) = \frac{1}{2}$, $\mathbb{P}(\xi_i = -1) = \frac{1}{2}$.
Let $W_n = \sum_{i=1}^{n} \xi_i$ and let $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$. Let $W_0 = 0$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Observe that $E[\xi_i] = 0$ for all i.
Then $\{W_n\}_{n \geq 0}$ is a martingale w.r.t. $\{\mathcal{F}_n\}_{n \geq 0}$, because W_n is $\mathcal{F}_n$-measurable for every n, and

$$
\begin{aligned}
E[W_{n+1} \mid \mathcal{F}_n] &= E[W_n + \xi_{n+1} \mid \mathcal{F}_n] \\
&= E[W_n \mid \mathcal{F}_n] + E[\xi_{n+1} \mid \mathcal{F}_n] && (\text{Thm } 2.3.2\ (i)) \\
&= W_n + E[\xi_{n+1}] && (\text{Thm } 2.3.2\ (ii) \text{ and } (iv)) \\
&= W_n, && (\text{since } E[\xi_{n+1}] = 0)
\end{aligned}
$$

(Think of a game in which, on every toss you lose a dollar with probability and win a dollar with probability $1/2$; W_n is then your total winnings after n plays) This example is also called symmetric random walk

Example 2 Betting on a martingale

Let $\{W_n\}$ be the martingale of Example 1. Suppose now that you can bet any amount Δ_{n-1} on play n (Δ_{n-1} is the amount you bet after completing play 1) with the restrictions that for every $n \geq 1$ Δ_{n-1} must be $\mathcal{F}_{n-1}$-measurable (so that you can only use the information known up to time $n-1$ in deciding the amount Δ_{n-1}) and that $E|\Delta_k| < \infty$ for all $k \geq 0$.
The amount you win on play n is therefore $\Delta_{n-1}\xi_n = \Delta_{n-1}(W_n - W_{n-1})$
Your total winnings after n plays is

$$X_n = \sum_{k=1}^{n} \Delta_{k-1}\xi_k = \sum_{k=1}^{n} \Delta_{k-1}(W_k - W_{k-1})$$

Set $X_0 = 0$.

Then $\{X_n\}_{n\geq 0}$ is a martingale w.r.t. $\{\mathcal{F}_n\}_{n\geq 0}$.

Since

$$X_n = \sum_{k=1}^{n} \Delta_{k-1}\xi_k$$

contains only terms that depend on $\xi_1, \ldots, \xi_n$, it is $\mathcal{F}_n$-measurable. Also

$$E\left[X_{n+1} \middle| \mathcal{F}_n\right] = E\left[X_n + \Delta_n \xi_{n+1} \middle| \mathcal{F}_n\right]$$

$$= E\left[X_n \middle| \mathcal{F}_n\right] + E\left[\Delta_n \xi_{n+1} \middle| \mathcal{F}_n\right]$$

$$= X_n + \Delta_n E\left[\xi_{n+1} \middle| \mathcal{F}_n\right] \quad \left(\begin{matrix}\text{Thm } 2.3.2 \\ \text{(ii)}\end{matrix}\right)$$

$$= X_n + \Delta_n E\left[\xi_{n+1}\right]^{\nearrow 0} \quad \left(\begin{matrix}\text{since } \xi_{n+1} \text{ is} \\ \text{independent of } \mathcal{F}_n\end{matrix}\right)$$

$$= X_n,$$

which confirms the martingale property.

<u>Example 3</u> <u>Claim 2</u> above, formula (5) with $j = k+1$, shows that using the measure $\tilde{\mathbb{P}}$,

$$\left\{ \frac{1}{(1+rh)^k} S(t_k) \right\} \text{ is a martingale w.r.t } \{\mathcal{F}(t_k)\}$$

To emphasize that the probability measure $\tilde{\mathbb{P}}$ with $\tilde{p}$ and $\tilde{q}$ defined by $\tilde{p} = \dfrac{1+rh-d}{u-d}$ $\tilde{q} = \dfrac{u-(1+rh)}{u-d}$ is being used; we will say $\left\{ \dfrac{1}{(1+rh)^k} S(t_k) \right\}$ (the discounted price process) is a martingale with respect to $\{\mathcal{F}(t_k)\}$ <u>and</u> $\tilde{\mathbb{P}}$.

Example 4. Consider again the arbitrage free multi-period model. Let $\{C(t_k, S(t_k))\}$ be the price process of a contingent claim with payoff $C(T,S(T))$ at $T=t_n$. It was shown in the proof of Theorem 2 that

$$C(t_k, S(t_k)) = \frac{1}{1+rh} \tilde{E}\left[C(t_{k+1}, S(t_{k+1})) \,\middle|\, \mathcal{F}(t_k)\right]$$

It follows that

$$\left\{ \frac{1}{(1+rh)^k} C(t_k, S(t_k)) \right\}_{k\geq 0} \text{ is a martingale w.r.t } \{\mathcal{F}(t_k)\}_{k\geq 0} \text{ and } \tilde{\mathbb{P}}$$

C. Simple properties of martingales.

Let $\{X_n\}_{n\geq 0}$ be a random process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $\{X_n\}_{n\geq 0}$ is a martingale w.r.t $\{\mathcal{F}_n\}_{n\geq 0}$. Then

$$E[X_n] = E[X_0] \quad \text{for all } n. \qquad (13)$$

$$\text{whenever } m > n \quad E[X_m | \mathcal{F}_n] = X_n. \qquad (14).$$

(13) is true because

$$E[X_n] = E[E[X_n | \mathcal{F}_{n-1}]] = E[X_{n-1}] \quad \text{for all } n \geq 1.$$

(14) is true by iterated conditioning: if $m > n+1$

$$\begin{aligned} E[X_m | \mathcal{F}_n] &= E[E[X_m | \mathcal{F}_{m-1}] | \mathcal{F}_n] = E[X_{m-1} | \mathcal{F}_n] \\ &= E[E[X_{m-1} | \mathcal{F}_{m-2}] | \mathcal{F}_n] = E[X_{m-2} | \mathcal{F}_n] \\ &= \cdots\cdots = E[X_{n+1} | \mathcal{F}_n] = X_n. \end{aligned}$$

if $m-1 > n+1$

continue in this fashion

The converse to (14) is also true. Let $\{\mathcal{F}_n\}_{n\geq 0}$ be a filtration. Fix $N>1$ and suppose Z is $\mathcal{F}_N$-measurable. Define

$$X_n = E[Z|\mathcal{F}_n] \qquad n=0,1,\dots,N$$

Then $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}_{n\geq 0}$ and $X_N=Z$. Indeed, since Z is $\mathcal{F}_N$-measurable, $X_N = E[Z|\mathcal{F}_N]=Z$. and if $0\leq k<N$

$$\begin{aligned} E[X_{k+1}|\mathcal{F}_k] &= E\left[E[Z|\mathcal{F}_{k+1}]\,\middle|\,\mathcal{F}_k\right] \\ &= E[Z|\mathcal{F}_k] \quad \text{(iterated conditioning)} \\ &= X_k \quad \text{(by definition).} \end{aligned}$$

<u>Example 5</u>. Consider an arbitrage free multi-period binomial market model. We saw in problem 7 of Assignment 4 that a contingent claim that pays $H(\omega_1,\dots\omega_n)$ at $T=t_n=nh$ has price

$$V(t_k) = \frac{1}{(1+rh)^{n-k}}\tilde{E}\left[H|\mathcal{F}(t_k)\right]$$

Multiplying by $\frac{1}{(1+rh)^k}$,

$$\frac{V(t_k)}{(1+rh)^k} = \tilde{E}\left[\frac{1}{(1+rh)^n}H|\mathcal{F}(t_k)\right]$$

Thus $\left\{\frac{V(t_k)}{(1+rh)^k}\right\}_{k\geq 0}$ is a martingale w.r.t $\{\mathcal{F}(t_k)\}_{k\geq 0}$ and $\tilde{\mathbb{P}}$. This result generalizes the result of Example 4 on the previous page.

III Risk-Neutral Measures for multi-period models.

Look at the results of Examples 3, 4, and 5. They all say that the discounted price process, whether it be for the underlying price or for a contingent claim, is a martingale under the probability measure $\tilde{\mathbb{P}}$.

Notice also that the value $B(t_k)$ of a unit of money market account is $B(t_k) = (1+rh)^k$, and if this is discounted

$$\frac{B(t_k)}{(1+rh)^k} = 1 \quad \text{for all } k$$

A process which is just constant is trivially a martingale.

<u>Definition</u> A general multi-period market model (not necessarily binomial) is defined by a risk-free rate r, period h and risky asset processes $\{S_1(t_k)\}_{0\le k\le n}$, $\ldots$ $\{S_p(t_k)\}_{0\le k\le n}$, on $(\Omega, \mathcal{F})$.

Let
$$\mathcal{F}(t_k) = \sigma\left(S_1(t_j), \ldots, S_p(t_j),\ 0 \le j \le k\right)$$

A probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ is said to be risk-neutral if

$$\left\{\frac{1}{(1+rh)^k} S_i(t_k)\right\}_{k\ge 0}$$

is a martingale w.r.t. $\{\mathcal{F}(t_k)\}$ and $\tilde{\mathbb{P}}$ <u>for</u> <u>each</u> i, $1\le i\le p$.

Example 6. The measure $\tilde{\mathbb{P}}$ defined on the multi-asset, binomial model when $d < 1+rh < u$ is risk-neutral. This follows from Example 3.

Theorem 3. Assume Ω is finite. A multi-period market is arbitrage free if and only if there exists a risk-neutral probability measure $\tilde{\mathbb{P}}$ for the model.

In this case, any attainable contingent claim has a unique price process $\{V(t_k)\}$ given by

$$V(t_k) = \frac{1}{(1+rh)^{n-k}} \tilde{E}\left[H \,\middle|\, \mathcal{F}(t_k) \right] \qquad (15)$$

where H is the payoff of the claim at $T = t_n$.

The problem with this statement is that we have not yet given a definition of 'attainable' for a contingent claim in a multi-period model --- we will come to this. Meanwhile, note that (15) looks exactly like the formula in Example 5, but the setting is now more general -- the model is not necessarily binomial and there may be many risky assets.

Theorem 3 is the first fundamental theorem of asset pricing for a discrete time - discrete outcome multi-period model.