Math 621 Lecture 5 Notes Fall 2011

I Properties of conditional expectation

A. Theorem 2.3.2, Shreve, Volume II.

Theorem 1. Let X and Y be r.v.'s on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a sub-sigma-algebra of $\mathcal{F}$. Assume $E[|X|], E[|Y|] < \infty$

(i) For any constants c_1, c_2, $E[c_1 X + c_2 Y \mid \mathcal{G}] = c_1 E[X \mid \mathcal{G}] + c_2 E[Y \mid \mathcal{G}]$.

(ii) If $E[|XY|] < \infty$ also and if X is $\mathcal{G}$-measurable

$$E[XY \mid \mathcal{G}] = X E[Y \mid \mathcal{G}]$$

(iii) (Tower property or property of iterated conditioning).
If $\mathcal{H}$ is a sub-σ-algebra and $\mathcal{H} \subseteq \mathcal{G}$ then

$$E\big[E[X \mid \mathcal{G}] \mid \mathcal{H}\big] = E[X \mid \mathcal{H}]$$

(iv) If X is independent of $\mathcal{G}$ (every $A \in \mathcal{G}$ is independent of every B in $\sigma(X)$), then $E[X \mid \mathcal{G}] = E[X]$.

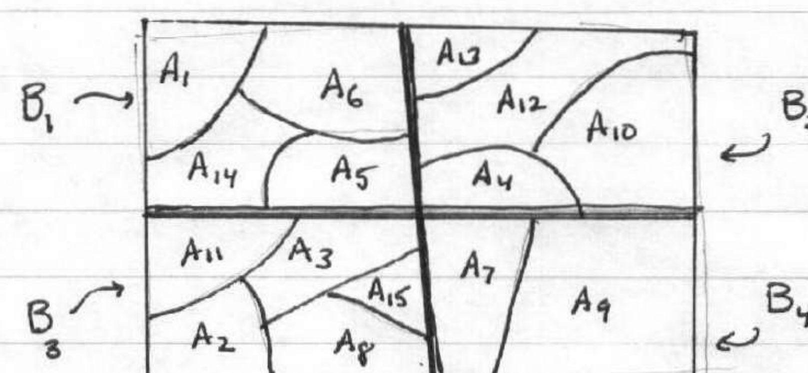
These properties of conditional expectation are all proved in Shreve by appeal to the measurability and partial averaging properties defining conditional expectation.

Remarks to aid understanding of this theorem.

a) Property (ii) is easily explained at the intuitive level.
If X is $\mathcal{G}$-measurable, and if we observe $\mathcal{G}$, then we know the value of $X(\omega)$. Thus, given $\mathcal{G}$, $X(\omega)$ is a known constant and it factors out from under the conditional expectation.

b). If X is independent of $\mathcal{G}$, knowing $\mathcal{G}$ does not affect the probabilities we assign to outcomes concerning X. This is the intuition behind property (iv).

c) To understand (iii) it helps to look at the case when $\mathcal{G}$ is the σ-algebra induced by a finite disjoint partition $\{A_1, \dots, A_K\}$ of Ω. If $\mathcal{H} \subseteq \mathcal{G}$, $\mathcal{H}$ has a finite number of events in it, each of which is a union of a sub-family of the collection $\{A_1, \dots, A_K\}$. Thus $\mathcal{H}$ is also induced by a partition $\{B_1, \dots, B_L\}$ of Ω where each B_i is a union of sets from $\{A_1, \dots, A_K\}$. This is illustrated in the figure



Ω

Assume all events in the partition $\{A_1, \dots, A_K\}$ have positive probability, and recall the formula

$$E[W \mid \mathcal{H}] = \sum_{i=1}^{L} \frac{E[1_{B_i} W]}{\mathbb{P}(B_i)} 1_{B_i}$$

By applying this formula with $W = E[X \mid \mathcal{G}]$ and recalling that $E[1_U E[X \mid \mathcal{G}]] = E[1_U X]$ for any $U \in \mathcal{G}$ -- in particular for $U = B_i$,

$$E[E[X \mid \mathcal{G}] \mid \mathcal{H}] = \sum_{i=1}^{L} \frac{E[1_{B_i} E[X \mid \mathcal{G}]]}{\mathbb{P}(B_i)} 1_{B_i} = \sum_{i=1}^{L} \frac{E[1_{B_i} X]}{\mathbb{P}(B_i)} = E[X \mid \mathcal{H}].$$

B. Application to the multi-period, binomial model

Consider the multi-period binomial model with the probability measure

$$\tilde{P}(\{(\omega_1,..,\omega_n)\}) = \tilde{p}^{N_n(\omega)}\tilde{q}^{n-N_n(\omega)},$$

$$N_n(\omega) = \text{number of } 1\text{'s in } (\omega_1,..,\omega_n),$$

$$\tilde{p} = \frac{1+rh-d}{u-d} \quad \tilde{q} = 1-\tilde{p} = \frac{u-(1+rh)}{u-d}.$$

(no arbitrage is assumed)

The notation for this model is from previous lectures and lecture slides. In particular, recall

$$\xi(t_1)(\omega) = \omega_1, \ .., \ \xi(t_n)(\omega) = \omega_n$$

$$\mathcal{F}(t_k) = \sigma(\xi(t_1), .., \xi(t_k))$$

Note that the price of the risky asset is

$$S(t_k)(\omega) = u^{\frac{1}{2}\sum_1^k (1+\omega_i)} d^{\frac{1}{2}\sum_1^k (1-\omega_i)} S(0)$$

$$= u^{\frac{1}{2}\sum_1^k (1+\xi(t_i)(\omega))} d^{\frac{1}{2}\sum_1^k (1-\xi(t_i)(\omega))} S(0)$$

Since this depends only on $\xi(t_1), .., \xi(t_k)$, $S(t_k)$ is $\mathcal{F}(t_k)$-measurable (this is discussed also in Lecture 4).

Furthermore

(1) $$S(t_{k+1})(\omega) = S(t_k)(\omega)\, u^{\frac{1}{2}(1+\xi(t_{k+1})(\omega))} d^{\frac{1}{2}(1-\xi(t_{k+1})(\omega))}$$

(2) $\xi(t_1), ..., \xi(t_n)$ are independent when $\tilde{P}$ is the probability measure, and $\tilde{P}(\xi(t_i)=1) = \tilde{p}$, $\tilde{P}(\xi(t_i)=-1) = \tilde{q}$ for each i, $1 \le i \le n$.

Using these facts, we derive some important identities using the properties of conditional expectation from Theorem 2.3.2

<u>Claim 1</u> $$\tilde{E}\left[S(t_{k+1})\,\middle|\,\mathcal{F}(t_k)\right] = (1+rh)\,S(t_k) \qquad (3)$$

(i.e., the expected growth in the value of the risky asset, using prob. measure $\tilde{\mathbb{P}}$, over period $k+1$, given observation of $\mathcal{F}(t_k)$, which is the market history up to t_k, is the same as what the risk free rate gives)

<u>Derivation</u>

$$\tilde{E}\left[S(t_{k+1})\,\middle|\,\mathcal{F}(t_k)\right] = \tilde{E}\left[S(t_k)\,u^{\frac{1}{2}(1+\xi(t_{k+1}))}d^{\frac{1}{2}(1-\xi(t_{k+1}))}\,\middle|\,\mathcal{F}(t_k)\right]$$

(by (1))

$$= S(t_k)\,\tilde{E}\left[u^{\frac{1}{2}(1+\xi(t_{k+1}))}d^{\frac{1}{2}(1-\xi(t_{k+1}))}\,\middle|\,\mathcal{F}(t_k)\right]$$

(by Theorem 2.3.2 (ii), since $S(t_k)$ is $\mathcal{F}(t_k)$-measurable)

$$= S(t_k)\,\tilde{E}\left[u^{\frac{1}{2}(1+\xi(t_{k+1}))}d^{\frac{1}{2}(1-\xi(t_{k+1}))}\right]$$

(by Theorem 2.3.2 (iv) since $\xi(t_{k+1})$ is independent of $\mathcal{F}(t_k) = \sigma(\xi(t_1), .., \xi(t_k))$)

$$= S(t_k)\left[d\,\tilde{\mathbb{P}}(\xi(t_{k+1}) = -1) + u\,\tilde{\mathbb{P}}(\xi(t_{k+1}) = 1)\right]$$

$$= S(t_k)\left[d\left[\frac{u-(1+rh)}{u-d}\right] + \frac{u(1+rh-d)}{u-d}\right]$$

$$= (1+rh)\,S(t_k) \qquad \square$$

<u>Claim 2</u> (An extension of Claim 1)
If $0 \le k < j \le n$

$$S(t_k) = \frac{1}{(1+rh)^{j-k}} \tilde{E}\left[S(t_j) \middle| \mathcal{F}(t_k) \right] \quad (4)$$

which implies

$$\frac{1}{(1+rh)^k} S(t_k) = \tilde{E}\left[\frac{1}{(1+rh)^j} S(t_j) \middle| \mathcal{F}(t_k) \right]. \quad (5)$$

<u>Derivation</u>. Consider first the case $j = k+2$. Because $\mathcal{F}(t_k) \subseteq \mathcal{F}(t_{k+1})$, the iterated conditioning property (iii) of Theorem 2.3.2 implies

$$\tilde{E}\left[S(t_{k+2}) \middle| \mathcal{F}(t_k) \right] = \tilde{E}\left[\tilde{E}\left[S(t_{k+2}) \middle| \mathcal{F}(t_{k+1}) \right] \middle| \mathcal{F}(t_k) \right]$$

But Claim 1 applied to the inner term yields

$$\tilde{E}\left[S(t_{k+2}) \middle| \mathcal{F}(t_{k+1}) \right] = (1+rh) S(t_{k+1})$$

and so, from a second application of Claim 1,

$$\begin{aligned} \tilde{E}\left[S(t_{k+2}) \middle| \mathcal{F}(t_k) \right] &= \tilde{E}\left[(1+rh) S(t_{k+1}) \middle| \mathcal{F}(t_k) \right] \\ &= (1+rh)^2 S(t_k), \end{aligned}$$

which proves (4) when $j = k+2$.

The proof of (4) for any $k < j$ proceeds by induction. If we have established (4) for j, then

$$\frac{1}{(1+rh)^{j+1-k}} \tilde{E}\left[S(t_{j+1}) \middle| \mathcal{F}(t_k)\right]$$

$$= \frac{1}{(1+rh)^{j-k}} \tilde{E}\left[\frac{1}{1+rh} \tilde{E}\left[S(t_{j+1}) \middle| \mathcal{F}(t_j)\right] \middle| \mathcal{F}(t_k)\right]$$

(using Theorem 2.3.2 (iii))

$$= \frac{1}{(1+rh)^{j-k}} \tilde{E}\left[S(t_j) \middle| \mathcal{F}(t_k)\right] \quad \text{(using Claim 1)}$$

$$= S(t_k) \quad \text{(using the induction assumption that (4) is valid for } j.)$$

Thus validity of (4) for $j > k$ implies validity of (4) for $j+1$. By induction (4) is valid for all $j > k$.

C. Lemma 2.3.4 of Shreve

This is a kind of generalization of properties (ii) and (iv) of Theorem 2.3.2. Let $f(x_1, .., x_K, y_1, .. y_L)$ be a function and let $X_1, .., X_K, Y_1, .., Y_L$ be r.v.'s. Define

$$g(x_1, .., x_K) = E\left[f(x_1, .., x_K, Y_1, .., Y_L)\right]$$

Suppose $Y_1, .., Y_L$ are independent of $\mathcal{G}$ and $X_1 ... X_K$ are $\mathcal{G}$-measurable. Thus if we observe $\mathcal{G}$, we know the values of $X_1(\omega), .. X_K(\omega)$. Since observing $\mathcal{G}$ does not affect the probabilities we assign to outcomes involving $Y_1 ... Y_L$

$$E\left[f(X_1, .., X_K, Y_1, .., Y_L) \middle| \mathcal{G}\right] = g(X_1, .., X_K)$$

D. Example in the multi-period binomial market.

Let $B(x)$ be a function of x. Then

$$B(S(t_{k+1})) = B\left(S(t_k)\, u^{\frac{1}{2}(1+\xi(t_{k+1}))}\, d^{\frac{1}{2}(1-\xi(t_{k+1}))}\right)$$
$$= f(s, y)$$

where
$$f(s,y) = B\left(s\, u^{\frac{1}{2}(1+y)}\, d^{\frac{1}{2}(1-y)}\right)$$

Since $S(t_k)$ is $\mathcal{F}(t_k)$-measurable and $\xi(t_{k+1})$ is independent of $\mathcal{F}(t_k)$ under $\tilde{\mathbb{P}}$, Lemma 2.3.4 implies

$$\tilde{E}\left[B(S(t_{k+1})) \middle| \mathcal{F}(t_k)\right] = G\left(S(t_k)\right)$$

where
$$G(s) = \tilde{E}\left[B\left(s\, u^{\frac{1}{2}(1+\xi(t_{k+1}))}\, d^{\frac{1}{2}(1-\xi(t_{k+1}))}\right)\right]$$
$$= \tilde{q}\, B(sd) + \tilde{p}\, B(su) \qquad (6)$$

This leads to the following result on pricing.

Theorem 2. Consider an arbitrage-free, n period binomial market model. Consider a contingent claim that pays $C(T, S(T))$ at time $T = nh$. The price of the claim at t_k is

$$V(t_k) = \frac{1}{(1+rh)^{n-k}}\, \tilde{E}\left[C(T, S(T)) \middle| \mathcal{F}(t_k)\right] \qquad (7)$$
$$= C(t_k, S(t_k))$$

where

$$(8) \quad C(t_k, s) = \frac{1}{(1+rh)^{n-k}} \tilde{E}\left[C\left(T, s\, u^{\frac{1}{2}\sum_{k+1}^{n}(1+\xi(t_i))} d^{\frac{1}{2}\sum_{k+1}^{n}(1-\xi(t_i))}\right)\right]$$

This theorem is a consequence of the backward recursion derived for the price of a contingent claim with payoff $C(T, S(T))$ derived in Theorem 1 of the Notes to Lecture 2. There it was shown that

$$V(t_k, S(t_k)) = C(t, S(t_k))$$

where $C(t_k, s)$ satisfies

$$C(t_k, s) = \frac{1}{1+rh}\left[\tilde{q}\, C(t_{k+1}, ds) + \tilde{p}\, C(t_{k+1}, us)\right] \tag{9}$$

By comparing this to formula (6) we see that it implies

$$C(t_k, S(t_k)) = \frac{1}{1+rh} \tilde{E}\left[C(t_{k+1}, S(t_{k+1})) \,\middle|\, \mathcal{F}(t_k)\right] \tag{10}$$

Equation (7) is derived using this and iterated conditioning. By applying (10) with $k = n-1$ and recalling $T = t_n$,

$$V(t_{n-1}) = C(t_{n-1}, S(t_{n-1})) = \frac{1}{1+rh} \tilde{E}\left[C(T, S(T)) \,\middle|\, \mathcal{F}(t_{n-1})\right]$$

which establishes (7) for $k = n-1$.

Now apply (10) with $k=n-2$ and use the result just derived

$$V(t_{n-2}) = C(t_{n-2}, S(t_{n-2})) = \frac{1}{1+rh}\tilde{E}\left[C(t_{n-1}, S(t_{n-1}))\,\middle|\,\mathcal{F}(t_{n-2})\right]$$

$$= \frac{1}{1+rh}\tilde{E}\left[\frac{1}{1+rh}\tilde{E}\left[C(T,S(T))\,\middle|\,\mathcal{F}(t_{n-1})\right]\,\middle|\,\mathcal{F}(t_{n-2})\right]$$

$$= \frac{1}{(1+rh)^2}\tilde{E}\left[C(T,S(T))\,\middle|\,\mathcal{F}(t_{n-2})\right].$$

where in the last equality we used Theorem 2.3.2 (iii), which ~~we~~ is valid here because $\mathcal{F}(t_{n-2}) \subseteq \mathcal{F}(t_{n-1})$. By continuing in this fashion one proves (7) for all $k<n$. For example

$$V(t_{n-3}) = C(t_{n-3}, S(t_{n-3})) = \frac{1}{1+rh}\tilde{E}\left[C(t_{n-2}, S(t_{n-2})\,\middle|\,\mathcal{F}(t_{n-3})\right]$$

$$= \frac{1}{(1+rh)^3}\tilde{E}\left[\tilde{E}\left[C(T,S(T))\,\middle|\,\mathcal{F}(t_{n-2})\right]\,\middle|\,\mathcal{F}(t_{n-3})\right]$$

$$= \frac{1}{(1+rh)^3}\tilde{E}\left[C(T,S(T))\,\middle|\,\mathcal{F}(t_{n-3})\right],$$

and so on.

Formula (8) is derived from (7) and from

$$S(T) = S(t_n) = S(t_k)\, u^{\frac{1}{2}\sum_{k+1}^{n}(1+\xi(t_i))}\, d^{\frac{1}{2}\sum_{k+1}^{n}(1-\xi(t_i))}$$

using Lemma 2.3.4. Thus

$$V(t_k) = \frac{1}{(1+rh)^{n-k}} \tilde{E}\left[C(T, S(T)) \,\middle|\, \mathcal{F}(t_k) \right]$$

$$= \frac{1}{(1+rh)^{n-k}} \tilde{E}\left[C\left(T, S(t_k)\, u^{\frac{1}{2}\sum_{k+1}^{n}(1+\xi(t_i))}\, d^{\frac{1}{2}\sum_{k+1}^{n}(1-\xi(t_i))}\right) \,\middle|\, \mathcal{F}(t_k) \right]$$

$$= C(t_k, s)$$

where

$C(t_k, s)$ is given as in (8),

because $S(t_k)$ is $\mathcal{F}(t_k)$-measurable and

$\xi(t_{k+1}), \dots, \xi(t_n)$ are independent of $\mathcal{F}(t_k)$. $\boxed{\equiv}$

Theorem 2 can also be derived from the following result, proved in problem 7 of Assignment 4, again using Lemma 2.3.4 (see the solutions)

<u>Theorem 3</u> In an arbitrage-free, n-period binomial model the price of a contingent claim paying $H(\omega_1, \dots \omega_n)$ at time $T = t_n$ is

$$V(t_k) = \frac{1}{(1+rh)^{n-k}} \tilde{E}\left[H \,\middle|\, \mathcal{F}(t_k) \right] \qquad (11).$$

Equation (7) is a direct consequence of (11).