642:621: Mathematical Finance: Lecture 1 Notes; Professor Ocone

I. Goal of course; overview. Math 621 treats *the mathematical tools and economic principles for analyzing and pricing* **financial derivatives** *using* **continuous-time models**.

A. Explanation. To explain this summary statement, we define what is meant by a 'financial derivative' and a 'continuous-time model.'

1. *Financial derivatives.* The word 'contingent' is important to understanding what a financial derivative is. It has several related meanings, the ones most relevant to us being: "(a) happening by chance or unforeseen causes; (b) dependent on or conditioned by something else."

A financial derivative is a contract between two parties for the exchange of money and/or assets at a future date. The date of exchange is called the exercise time. What is special about financial derivatives is that the dollar value of the exchange is *derived* from the values, at exercise time, of *underlying variables*, according to a formula specified in the contract. The underlying variables are usually prices of assets traded in a market. The key element is that they fluctuate as time progresses in unpredictable, essentially random ways. Since the value at exercise time of a derivative contract is contingent on underlying variables, financial derivatives are also called *contingent claims.*

The simplest example, the forward contract, illustrates all the basic features of financial derivatives. In this contract, party A agrees to purchase an asset from party B, at a future time T, for an agreed-upon price K. The underlying variable in this case is the price of the asset. If we denote this price at time t by $S(t)$, then the value of the contract to A at the time of exercise is $S(T)-K$, because A pays K at time T for what is really worth $S(T)$. The value $S(T)-K$ is contingent in both senses (a) and (b) above: it depends on the future value $S(T)$, which is subject to unforeseen fluctuations due to changing economic or historical circumstances.

In principle, the underlyings of a derivative could be anything, including other financial derivatives, or even the weather. Typically though, financial derivatives ultimately depend on the values of basic assets traded in a financial market. By 'basic', we mean assets which involve ownership of real wealth (stocks, foreign currency, commodities), or the contractual promise of future earnings (bonds), and which are actively traded on open markets. In this course, it is always assumed that underlyings are assets in a financial market.

The defining feature of financial markets is risk; we do not know and cannot predict exactly what future prices will be. Thus, returns on investments in financial assets are uncertain. We refer to this uncertainty as 'risk,' a term employed here only in a general sense. There are serious theories attempting to define quantitative measures of risk and to analyze their behavior; these will not be treated in the course.

Some lingo: If S denotes an underlying variable, or set of variables, a contingent claim depending wholly on the value of S is said to be *written on* S. The term 'the underlying variable' is often shortened to 'the underlying' .

2. *Continuous-time models.* Analyzing a financial derivative requires having a mathematical model for the financial market on which it is based. A model is simply a way to generate what one believes are the possible future behaviors of the market, so that one can assess risk quantitatively. In a *continuous-time* model, asset prices are random functions $S(t)$ of a time parameter t that varies continuously in an interval domain $t_0 \leq t \leq T$. This is equivalent to assuming that trades can occur at any time in $[t_0, T]$, which is approximately true, at least during the hours that a market is open.

In discrete-time models, market variables are modeled by discrete-time stochastic processes, $S(0), S(h), S(2h), \ldots$. A discrete-time model is appropriate when trading or exercising the derivative is allowed only at a discrete set of times. For instance, since a forward contract, entered at time $t = 0$, must be exercised at the exercise time T, a model containing only prices $S(0)$ and $S(T)$ suffices. However, to treat common financial derivatives, such as options or futures, which can be traded at any time up to the exercise time, continuous-time models capture market behavior more accurately. Understanding continuous-time models, the mathematics for working with them, and how they are applied to derivative pricing is the heart of this course.

3. *The basic principle of derivative pricing.* Much derivative pricing theory is based on the principle that markets should not admit *arbitrage.* Loosely speaking, an arbitrage is a strategy for allocating investments that produces money from nothing. Thus, if you take advantage of an arbitrage, money flows to you from other investors just because of they way prices are structured, not because your investments are producing wealth. For example, suppose you can find a bank willing to loan you money at 5% interest and another bank offering 6% interest on your savings account. By borrowing from the first bank and depositing in the second you will make money with no initial investment of your own. Obviously such a situation is not stable and cannot last. Investors, called arbitrageurs, are constantly on the lookout for such arbitrage opportunities, and as they begin to exploit these opportunities, their actions will raise and lower demand in such a way to push markets back toward no-arbitrage conditions. At least that is the belief! So while it is not true that there are no arbitrage opportunities in real markets, one expects them to be small and not last too long, leading to markets in which the no-arbitrage assumption is approximately correct. Of course, the reasoning here involves a bit of paradox. We postulate there is no arbitrage, because, in fact, it exists, but arbitrageurs rapidly take advantage of it!

B. Overview

Continuous-time models are powerful, but the mathematical theory needed to handle them is sophisticated. We will need to learn: an abstract formulation of probability theory; a general formulation of conditional probability and expectation; the concept of a martingale, the Markov property, Brownian motion, stochastic integrals constructed on Brownian motion, and the Itô calculus of stochastic integrals. It will be much easier to learn the material if we first work out some of the concepts for discrete-time models. Thus, we will spend the first few lectures discussing discrete-time pricing theory—which is also very important in practice—and only then move on to the continuous-time generalization, as covered in the course text. The plan is to cover most of Chapters 1-6 of the text *Stochastic Calculus for Finance II: Continuous-Time Models*, by Steven E. Shreve.

C. Background on Financial Derivatives

Knowledge of the following terminology and derivative types will be assumed of all students in 621. Even if you know this, it is advisable to read Hull, *Options, Futures, and Other Derivatives*(the chapter numbers here are for the sixth edition, but may differ for newer editions): Chapter 1, Introduction; Chapter 4, Interest Rates; Chapter 10, Trading Strategies involving options.

In each derivative contract there is a buyer and seller. The buyer is said to have the long side of the derivative, or to be *long* in the derivative, or to be the holder; the seller is said to be *short* the derivative.

A *forward contract* is an agreement to transact a purchase in the future, and was define briefly above. Here we state the definition in more detail. The long party to a forward contract agrees to pay a stipulated sum of money for a stipulated quantity of an asset, T units of time into the future. T is called the time to maturity; the day on which the exchange takes place is called the *delivery date.* No money exchanges hands at the time the future contract is entered into, and both parties are obligated to fulfill the terms of the contract. The *payoff function* of the forward contract is the monetary value of the exchange to the long party at the delivery date. If $S(T)$ denotes the price per unit of asset, and if K is the price per unit agreed to in the forward contract, the payoff (per unit) is $S(T) - K$, since the buyer is paying K for a unit worth $S(T)$. Thus the payoff function is $V(s) = s - K$ as a function of price s.

An *option* is a contract which gives it holder the right, but not the obligation, to exercise the contract. Of course, the holder will not exercise if doing so would incur a loss, so an option contract is equivalent to the long and short parties agreeing to exchange zero goods and money if exercising would mean a loss for the long party. In an option contract, the last date on which the option can be exercised is called the expiration date or expiry date. An option is of *European* type if it can only be exercised at expiry; it is of *American* type if the buyer is free to choose any exercise time before expiry. This choice does not have to be announced in advance.

The basic (also called ‘vanilla”) options are European and American calls and puts written on a single underlying. Typical underlyings of vanilla options are stock prices and market indices. A call option gives its holder the right, but not the obligation, to buy a unit of the underlying asset at specified price K, called the strike price. If T is the expiry date and $S(T)$ is the price of the underlying at T, the payoff per unit of asset, of a European call is

$$(S(T) - K)^+ := \max\{S(T) - K, 0\}.$$

(The notation $A := B$ means we are defining A by B; to be more explicit, we use the notation $(x)^+$ to denote the function $(x)^+ = \max\{0, x\}$). Indeed, if the price $S(T) < K$, the holder can buy the asset more cheaply in the market and will not exercise; if $S(T)$, the holder can exercise the option, buy a unit for price K, and immediately sell it in the market at price $S(T)$ for a profit of $S(T) - K$. For the case of an American option, define τ to be the exercise time at which the holder chooses to exercise, assuming he or she does, and to be T if not. Then the payoff is $(S(\tau) - K)^+$.

A put option gives the holder the right, but not the obligation to sell a unit of asset for the strike price K. For a European put, the payoff is $(K - S(T))^+$, and a similar formula obtains for the American put.

The New York Stock Exchange (NYSE) runs the Amex Options Market, which trades American calls and puts on equities (stocks), market indices, and exchange traded index funds and trusts. You can find price quotes on their web site.

Another important financial derivative is the *future*, which is basically a forward that can be traded. Information about futures markets and futures pricing can be found in Hull’s book, *Options, Futures, and Other Derivatives*, Chapters 2, 3, 5, and 6. We will not discuss futures in this course; our focus will be on options.

II. Discrete models; definitions and examples

A. Mathematical elements of a discrete-time market model.

A discrete-time model is one in which trading can only take place at discrete times. By a *market* we have in mind a set of financial instruments that can be liquidly traded by all market participants. A model will essentially be a hypothetical list of all the ways the market can evolve in the future.

A discrete-time model for a market of p risky financial assets is composed of the following elements:

1. A set $\mathcal{T}$ of trading times $t_0 < t_1 < t_2 < \ldots < t_n$.
2. A set Ω, called the outcome space. The elements ω of Ω label the possible future states of the market admitted by the model.
3. For each risky asset i, $1 \leq i \leq p$, and each $\omega \in \Omega$, a function $S_i(t)(\omega)$, $t \in \mathcal{T}$, called the *price path*, that gives the price of asset i at time t if the market is in state ω.

   Since we think of time $t = 0$ as the present, when all prices are known, it is always assumed that $S_i(0)$ is deterministic, in the sense that it does not depend on ω.
4. A risk-free asset, called the money market account.

Comments:

1. On the trading times $t_0 < t_1 < t_2 < \ldots < t_n$.

   By convention we always take $t_0 = 0$ and usually we denote t_n by T, which represents the *time horizon* of the model. Very often, we think of $t_0 = 0$ as representing today, the present. For convenience, time is always time in years. For example, a model with $t_0 = 0$, $t_1 = 0.25$, $t_2 = 0.5\, t_3 = 0.75, t_4 = T = 1$ is a 4-period model that allows quarterly trading.
2. About Ω.

   The elements ω of this set label all the possible future states of the market admitted by the model. It is at this point that we incorporate risk. We do not know what the future state of the market—its prices and factors affecting its prices—will be; but it is the job of the model to list the possibilities. Connoisseurs of probability theory, seeing Ω, will look next for a probability measure on subsets of Ω, but we do not introduce that yet! We will begin the course with the study of models for which Ω is a finite set $\{\omega_1, \omega_2, \ldots, \omega_m\}$. As I said, we do not yet put probabilities on these outcomes as part of the model, but there is a probabilistic idea lurking in the background; for the finite case, it is assumed that every one of the outcomes in Ω has a positive probability of occurring—it does not make sense to put a state in the model if it can't possibly happen! (Uncountably infinite Ω will be necessary later on to support continuous-time models, and then, for reasons we hope will become apparent, we will need to introduce probability.)

4. About the risk-free asset (money market).

   This is an asset that earns a certain interest rate, risk-free, and is liquid in the sense that one can invest and borrow money from it at will. (So for example, cash deposits available at banks, which require you to invest your money for stipulated periods of time or pay a penalty, do not qualify.) In the real world, this is an instrument with virtually zero probability of default. For further discussion, see the book of Hull cited above, Chapter 4.

   Suppose we invest \$1 in the money market at time $t_0 = 0$. The value accrued by compound interest on this account by time $t > 0$ is denoted by $B(t)$. The rate at which $B(t)$ increases is called the risk-free rate, and, if it is constant in time, is denoted by r. There are two possibilities. Sometimes r will be interpreted as a *simple* rate of interest per annum, compounded only at trading times. Then,

$$B(t_1) = (1+rt_1), \quad B(t_2) = (1+r(t_2-t_1))B(t_1) = (1+rt_1)(1+r(t_2-t_1)), \quad \text{etc.}$$

   If the trading occurs at regular intervals of length h (that is $t_0 = 0$, $t_1 = h$, $t_2 = 2h, \ldots$, then $B(t_k) = B(kh) = (1+rh)^k$. Other times, r will be interpreted as the rate of interest per annum, continuously compounded, in which case $B(t) = e^{rt}$.

Examples.

(i). *One period binomial model model with one risky asset.*

One period means that the only trading times are $t = 0$, the beginning of the period, and T, the end of the period. Binomial means that the future economy has two states, which are labeled 1 and -1; hence $\Omega = \{-1, 1\}$. There is one risky asset, whose price at t is denoted by $S(t)(\omega)$. The model is specified by four parameters: $S(0)$ (the price at $t = 0$); r, the risk free interest rate; and *return parameters* $0 < d < u$. If the future state of the economy is 1 (the 'up' state), $S(T)(1) = uS(0)$; if it is -1 ('down' state), $S(T)(-1) = dS(0)$

(ii). *One period, multiple risky assets, general outcome space.*

Again, the only trading times are 0 and T, and there is a money market with risk free rate r. But now we let Ω be arbitrary, and suppose that there are p risky assets. These are defined by p constants, $S_1(0), \ldots, S_p(0)$, the asset prices at time 0, and p functions on Ω, $S_1(T)(\omega), \ldots, S_p(T)(\omega)$, representing asset prices at time T.

Here is a particular example with $p = 1$. Let $\Omega = [a, b]$, where $0 < a < b$, and suppose $S_1(T)(\omega) = \omega S_1(0)$. This model allows for a continuous range of possible asset returns. (Here Ω is not discrete.)

Suppose $p = 2$ and the two risky assets, S_1 and S_2, we want to model can fluctuate independently of one another. The binomial model for Ω is not rich enough to support a model of this situation. Indeed, suppose price process S_1 and S_2 are both defined on $\Omega = \{1, -1\}$ as in (i). If we observe $S_1(T)(\omega)$, then we know whether $\omega = 1$ or $\omega = -1$, and hence we know $S_2(T)(\omega)$. Thus the two prices are perfectly correlated. Either they both move in the same direction or in opposite directions, when in real markets either situation may be possible. The simplest model with two risky assets fixing this problem would be: $\Omega = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$ with $S_1(T)(1,1) = S_1(T)(1,-1) = u_1 S_1(0)$, $S_1(T)(-1,1) = S_1(T)(-1,-1) = d_1 S(0)$, $S_2(T)(1,1) = S_2(T)(-1,1) = u_2 S_2(0)$, and $S_2(1,-1) = S_2(T)(-1,-1) = d_2 S_2(0)$, where $0 < d_1 < u_1$ and $0 < d_2 < u_2$.

(iii). *Multi-period, binomial (recombining) tree for one risky asset.*

This model extends the binomial model through multiple periods. For simplicity, assume, n equal length periods, $t_0 = 0$, $t_1 = h, \ldots, t_k = kh, \ldots, t_n = nh = T$.

In this model a constant, simple rate of interest r is assumed, and is compounded at trading times only. Thus $B(t_k) = (1 + rh)^k$.

Let

$$\Omega := \Big\{\omega = (\omega_1, \ldots, \omega_n) : \omega_i \in \{-1, 1\} \;\text{ for each } i\Big\}.$$

The idea is that in each period, the economy can move up or down. If the history of the economy is given by $\omega = (\omega_1, \ldots, \omega_n)$, then the economy moves up in period i, if $\omega_i = 1$, it moves down if $\omega_i = -1$. When the economy moves up in period i, the return on the asset is u, when it moves down, the return is d, where $0 < d < u$. Thus, the asset price from period to period is governed by the equation.

$$S(t_{i+1})(\omega) = \begin{cases} uS(t_i)(\omega), & \text{if } \omega_{i+1} = 1; \\ dS(t_i)(\omega), & \text{if } \omega_{i+1} = -1. \end{cases} \tag{1}$$

You may check easily that this can be written in the form

$$S(t_{i+1})(\omega) = u^{(\omega_{i+1}+1)/2} d^{(1-\omega_{i+1})/2} S(t_i)(\omega) = (ud)^{1/2} \left(\frac{u}{d}\right)^{\omega_{i+1}/2} S(t_i)(\omega). \tag{2}$$

For example, if $\omega_{i+1} = 1$, this formula gives $S(t_{i+1}) = (ud)^{1/2}(u/d)^{1/2} S(t_i)(\omega) = uS(t_i)(\omega)$, which matches the first option in equation (1).

Now suppose we start with $k = i + 1$ in (2), so that $i = k - 1$, and then apply formula (2) twice, the second time starting from $k - 1$,

$$\begin{aligned} S(t_k)(\omega) &= (ud)^{1/2} \left(\frac{u}{d}\right)^{\omega_k/2} S(t_{k-1})(\omega) = (ud)^{1/2} \left(\frac{u}{d}\right)^{\omega_k/2} (ud)^{1/2} \left(\frac{u}{d}\right)^{\omega_{k-1}/2} S(t_{k-2})(\omega) \\ &= (ud) \left(\frac{u}{d}\right)^{(\omega_{k-1}+\omega_k)/2} S(t_{k-2})(\omega). \end{aligned}$$

By continuing this procedure back to time $t_0 = 0$, we find the formula,

$$S(t_k)(\omega) = (ud)^{i/2} \left(\frac{u}{d}\right)^{(1/2)\sum_1^k \omega_i} S(0). \quad \text{for any } k \geq 0 \text{ and any } \omega. \tag{3}$$

Notice that, as it should, $S(t_k)(\omega_1, \ldots, \omega_n)$ depends only on the market movements $\omega_1, \ldots, \omega_k$ in the first k periods.

Figure 1 on the following page shows a typical representation of a binomial tree model. The elements of Ω, the future histories of the economy, are the possible paths through this tree, starting at the leftmost vertex and ending up at the rightmost set of nodes. The nodes in the k^{th} column label the possible asset prices at time t_{k-1}. Note that there are multiple paths leading to each internal node (nodes not on the upper or lower edge). In fact, the price at nodes corresponding to t_k, after k periods have elapsed, will have the form $u^\ell d^{k-\ell} S(0)$ for some ℓ, $0 \leq \ell \leq k$. The number of paths to the node for $u^\ell d^{k-\ell} S(0)$ is the binomial coefficient $\binom{k}{\ell}$, since to get to this node the economy must experience exactly ℓ 'up' movements in k periods. This explains the terminology, 'binomial tree.' Because multiple paths end up at the same node, the graph in Figure 1 is sometimes said to be recombining. (The lattice in this picture is not truly a tree in the graph-theoretic sense.)

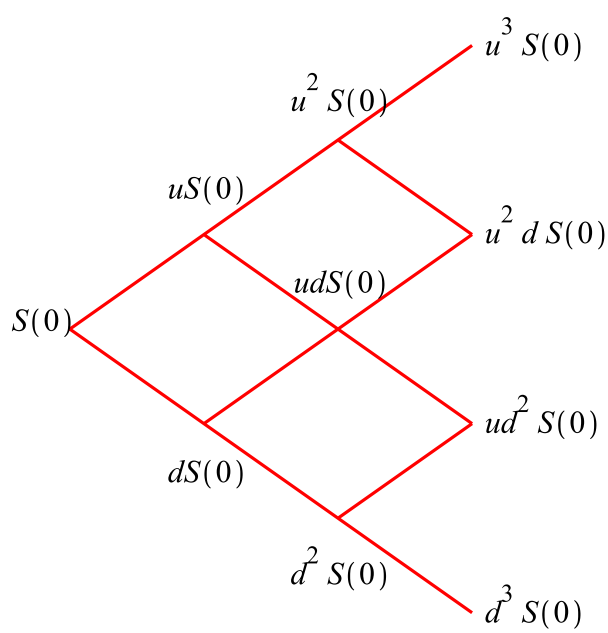

Figure 1: Binomial tree, three periods.

B. Portfolios and portfolio processes. Consider an economy at time t with p risky assets, whose prices are $S_1(t)(\omega), \dots, S_p(t)(\omega)$, and with a risk free money market account of value $B(t)$. Let

$$A(t)(\omega) := (B(t), S_1(t)(\omega), \dots, S_p(t)(\omega)).$$

This is called the *asset price vector* at (t, ω). A portfolio is a list of the amounts of each asset held by an investor. It may be represented as a vector in $\mathbb{R}^{p+1}$

$$\triangle = \begin{pmatrix} \triangle_0 \\ \triangle_1 \\ \vdots \\ \triangle_p \end{pmatrix},$$

where $\triangle_0$ is the number of shares of the money market account, and, for $i \geq 1$, $\triangle_i$ being the number of units of asset i, held in the portfolio. (One share of a money market account is purchased for \$1 at time 0 and is worth $B(t)$ at time t; hence $\triangle_0 B(t)$ is the dollar amount of the portfolio in the money market at time t. If the the risky assets are stocks, $\triangle_i$ is the number of shares held in stock i.) We shall assume that any vector in $\mathbb{R}^p$ qualifies as a possible portfolio. This means first that it is possible to hold fractional units of any asset. Second, it means one can hold negative amounts of an asset; a negative value of $\triangle_i$ is interpreted to mean that the investor has borrowed $|\triangle_i|$ units of asset i. (One says that the investor is short $|\triangle_i|$ units of asset.) If one of the assets on the market is the risk free bond we discussed above, this assumption on portfolios means one can both invest and borrow money at the risk free interest rate.

The monetary value of a portfolio $\triangle$ at time t, when the history of the economy is ω, is

$$\triangle_0 B(t) + \sum_{i=1}^{p} \triangle_i S_i(t)(\omega) = A(t)(\omega) \cdot \triangle,$$

where $z \cdot y$ denotes the vector inner product.

In multi-period models, one can update the portfolio at each trading time and adjust the portfolio according to what happens in the market. Suppose the trading times are $\{t_0, t_1, \dots, t_n\}$. A *portfolio process* is a function which assigns to $\omega \in \Omega$ and trading times t_i for $i < n$, a vector

$$\triangle(t_i)(\omega) = \begin{pmatrix} \triangle_0(t_i)(\omega) \\ \triangle_1(t_i)(\omega) \\ \vdots \\ \triangle_p(t_i)(\omega) \end{pmatrix},$$

This vector represents the portfolio the investor chooses to hold over the time period from t_i to t_{i+1}—this is the $(i+1)^{\text{st}}$ time period. Let the monetary value of the portfolio $\triangle(t_i)(\omega)$ at time t_i be denoted by $X(t_i)(\omega)$; X is called the wealth process. Then

$$X(t_i)(\omega) = \triangle(t_i)(\omega) \cdot A(t_i)(\omega)$$

At the end of the period over which the portfolio is held, it is worth

$$\triangle(t_i) \cdot A(t_{i+1})(\omega)$$

The portfolio process $(\triangle(t_0), \cdots, \triangle(t_n))$ is called *self-financing* if for each i, $0 \leq i \leq n-1$,

$$X(t_{i+1}) = \triangle(t_i) \cdot A(t_{i+1})(\omega).$$

This means the amount you have to invest for the period from t_{i+1} to t_{i+2} is what you have at t_{i+1} from the portfolio you invested at time t_i. Thus, there is no infusion of new cash to invest at times $t_1, \cdots, t_n$. All the wealth in a self-financing portfolio results from investment earnings on the original amount of money $X(t_0) = \triangle(t_0) \cdot A(t_0)$ in the portfolio. $X(t_0)$ is called the *initial endowment*.

We assume that investors do not have inside information, nor can they see into the future. (A fancy English word–it is actually French in origin—for seeing into the future is *clairvoyance*). Clairvoyance will be ruled out by a condition on portfolio processes called *admissibility.* A portfolio process is *admissible* if it does not allow the investor to make portfolio decisions at time t_k based on what the economy does in future periods. The precise, mathematical definition of admissibility depends on the market model. For the multi-period, binomial model defined in Example (iii) of section II.A (see page 6), this definition is as follows: $\triangle(0)$ may not depend on ω; and for all i, $1 \leq i \leq n$, $\triangle(t_i)(\omega_1, \ldots, \omega_n)$ may depend only on $(\omega_1, \ldots, \omega_i)$, that is, it is really a function $\triangle(t_i)(\omega_1, \ldots, \omega_i)$. The reason for this is that at time t_i the investor knows $\omega_1, \ldots, \omega_i$, which completely describe the market history up to time i, but does not know $\omega_{i+1}, \ldots, \omega_n$, which represent future market movements.

C. **Frictions:** *Friction* is the term for impediments entering into financial transactions or contracts, or preventing the flow of information about assets. It is a term, in short, for reality, especially any features of reality that have the bad taste to contradict the assumptions of our models! For example, investors cannot generally both invest and borrow at the same risk free interest rate, as we are assuming. This is a friction. Other frictions that we will assume away, even if we do not always mention it explicitly, will be: transaction costs and tax costs, restrictions on borrowing, liquidity constraints, bid-ask spreads. We will also assume: all investors are price takers, meaning each individual's purchases or sales are not large enough, by themselves, to move prices; an investor can borrow any amount of any asset, sell it, and use the proceeds; and also, all market information is available to all investors.

D. Arbitrage: Loosely speaking, an arbitrage is a portfolio that earns a riskless profit simply by trading several assets. Indeed, economists define arbitrage as simultaneous transactions in different markets that lead to a guaranteed profit, whatever happens to the economy. This definition reflects the belief that arbitrage can only arise from imbalances between different markets, between which information and trades do not normally pass. In a single market, it is assumed that arbitrage does not exist, and this the basic assumption of derivative pricing; *no financial derivative should be priced in a way that creates an arbitrage opportunity.*

For discrete-time models with discrete outcome space Ω, initial time $t_0 = 0$ and final time $t_n = T$, the precise definition is as follows. An arbitrage, is an admissible, self-financing portfolio process, whose wealth process $X(t)$ satisfies one of the following properties:

(a) $X(0) < 0$ and $X(T)(\omega) \geq 0$ for all $\omega \in \Omega$; (here T is the final time); or

(b) $X(0) = 0$, and $X(T)(\omega) \geq 0$ for all $\omega \in \Omega$ and there is at least one $\omega' \in \Omega$, such that $X(T)(\omega') > 0$.

(In case (a), we start by borrowing and end up with at least 0 no matter what happens in the economy; in case (b), we start with nothing and, at the end, we are sure not to have lost anything and for at least one possible outcome of the future, we end up with a profit.)

It is easy to see that (a) implies (b), when the market contains a risk-free money market account, or even if you can assume that the currency of your market will never become worthless. In fact, under this assumption, if (a) is true, you can start with zero wealth and end up with positive wealth for every possible outcome ω. (This is left as an exercise.)

Example (i): Condition for arbitrage in the one-period binomial model. We work with the model defined in Example (i) of section II.A above.

For this model, we claim:

$$\text{there is no-arbitrage if and only if } \; d < 1 + rT < u. \tag{4}$$

It is easy to see intuitively why arbitrage is possible if these inequalities are violated. For instance, suppose $rT \leq d < u$; then the risky asset, the stock, returns at least as much or more than the money market no matter what happens. Hence arbitrage is possible by borrowing at money at the risk free interest rate and investing it in the stock. It is left as an exercise to write down arbitrage portfolios explicitly.

To finish justifying the claim 4 we need to show that arbitrage is *not* possible if $d < 1 + rT < u$. To do this, we first express very explicitly what must occur mathematically for an arbitrage opportunity to exist. In the one period model, we get to choose a portfolio at $t = 0$ only. Call this portfolio

$$\triangle = \begin{pmatrix} \triangle_0 \\ \triangle_1 \end{pmatrix},$$

where, in conformity with the previous definitions, $\triangle_0$ represents the amount invested in the money market and $\triangle_1$ is the number of units held in the risky asset. At $t_0 = 0$, the beginning of the period, the monetary value of this portolio is $\triangle_0 + S(0)\triangle_1$. At the end of the period, it is worth $X(T)(\omega) = (1+rT)\triangle_0 + S(T)(\omega)\triangle_1$. There are just two possibilities: $X(T)(1) = (1+rT)\triangle_0 + uS(0)\triangle_1$ and $X(T)(-1) = (1+rT)\triangle_0 + dS(0)\triangle_1$. Hence, $\triangle$ is an arbitrage if either

$$\begin{aligned} \triangle_0 + S(0)\triangle_1 &< 0 \text{ and} \\ (1+rT)\triangle_0 + dS(0)\triangle_1 \geq 0 & \quad \text{and} (1+rT)\triangle_0 + uS(0)\triangle_1 \geq 0 \end{aligned}$$

or

$$\begin{aligned} \triangle_0 + S(0)\triangle_1 &= 0 \text{ and} \\ (1+rT)\triangle_0 + dS(0)\triangle_1, (1+rT)\triangle_0 + uS(0)\triangle_1 & \quad \text{are both} \geq 0 \text{ and at least one is} > 0. \end{aligned}$$

The figure below shows the points in the $(\triangle_0, \triangle_1)$-plane such that the initial wealth $\triangle_0 + S(0)\triangle_1 \leq 0$; it is the region below the line $\triangle_1 = -\triangle_0/S(0)$.

The next figure shows the region of the $(\triangle_0, \triangle_1)$-plane for which both $(1+rT)\triangle_0 + dS(0)\triangle_1 \geq 0$ and $(1+rT)\triangle_0 + uS(0)\triangle_1 \geq 0$. In this figure, the line marked L_d is the graph of $(1+rT)\triangle_0 + dS(0)\triangle_1 = 0$, and the line marked L_u is the graph of $(1+rT)\triangle_0 + uS(0)\triangle_1 = 0$.

At every point of the shaded region in figure 3, including the boundaries, except for the origin $(0,0)$, at least one of $(1+rT)\triangle_0 + dS(0)\triangle_1$ and $(1+rT)\triangle_0 + uS(0)\triangle_1$ is strictly positive. The origin $(0,0)$ does not correspond to an arbitrage. Thus the one-period, binomial model admits arbitrage if and only if the shaded regions of Figures 2 and 3 intersect in a point other than the origin. For example, the graphs of Figures 2 and 3 were plotted using the parameters, $T = 1$, $r = 0.5$, $S(0) = 3$, $u = 1$, and $d = 0.5$. If the graphs are superimposed, the result is Figure 4 on the next page. (The line denoted L_0 in this figure is the boundary $\triangle_0 + S(0)\triangle_1 = 0$ of the region of Figure 1 corresponding to portfolios built from a non-positive initial endowment.) One sees here that the regions of Figures 2 and Figure 3 intersect in a wedge, every point of which represents an arbitrage portfolio. Even the boundary points of this intersection region, except for the origin, represent arbitrage possibilities. A point anywhere in the intersection, except on the upper boundary L_0, is an arbitrage of

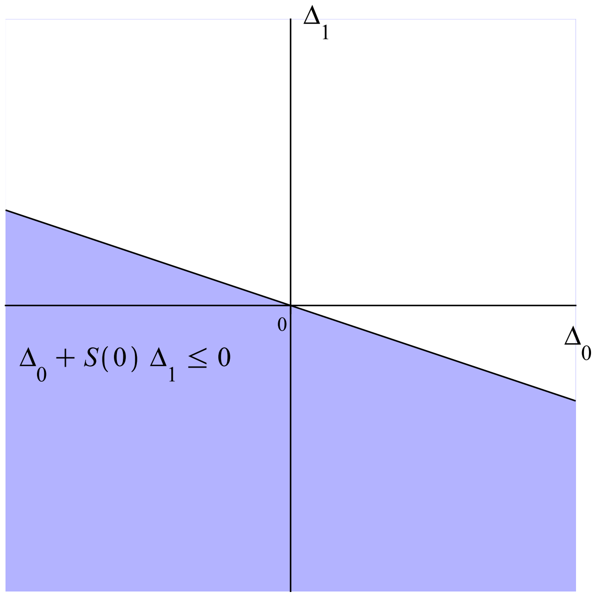

Figure 2: Region of portfolios corresponding to non-positive initial endowment.

type (a); an intersection point on L_0 is a an arbitrage of type (b) (see the definition above).

It is clear then that there is no arbitrage if and only if

$$\mathrm{slope}(L_u) > \mathrm{slope}(L_0) > \mathrm{slope}(L_d),$$

as in Figure 5 (see below). This is equivalent to

$$-\frac{1+rT}{uS(0)} > -\frac{1}{S(0)} > -\frac{1+rT}{dS(0)}$$

which is equivalent to

$$d < 1 + rT < u.$$

This proves the claim (4).

Example. Multi-period, binomial model. Consider the multi-period model defined in Example (iii) of section II.A (see Figure 1). We claim that there is no arbitrage in this model if and only if $d < 1 + rh < u$. Indeed if this condition is violated, we can achieve an arbitrage in the first period, by the analysis we just gave of the one period model. By then investing in the money market, we will preserve this arbitrage out to the last period.

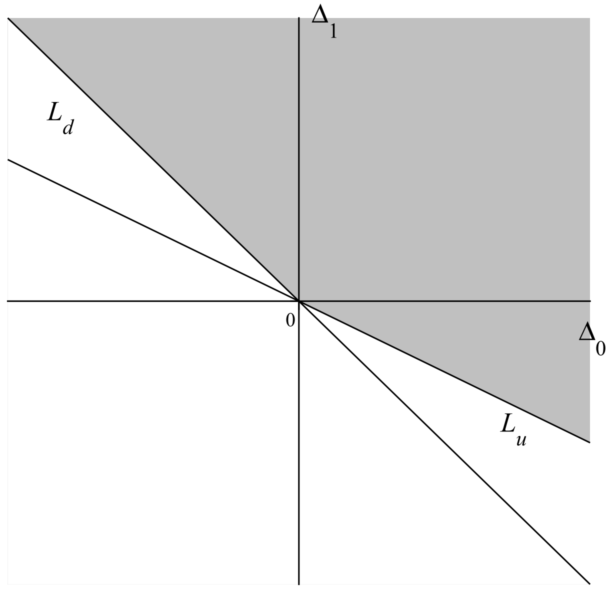

Figure 3: Portfolios always resulting in non-negative wealth at T.

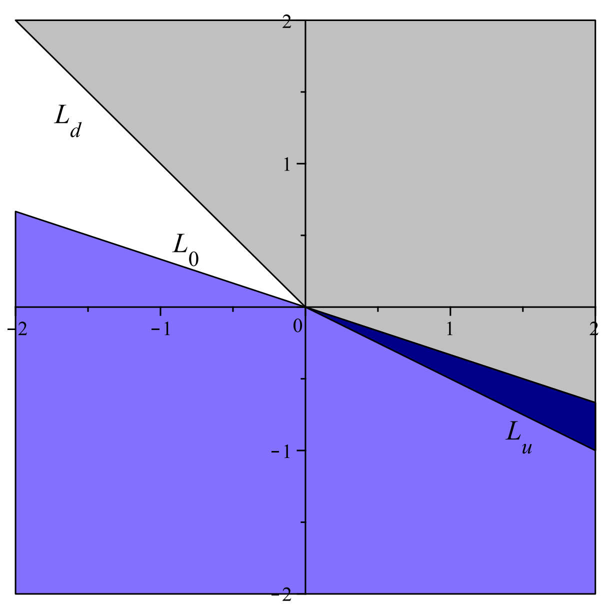

Figure 4: Example in which arbitrage exists in the one-period, binomial model.

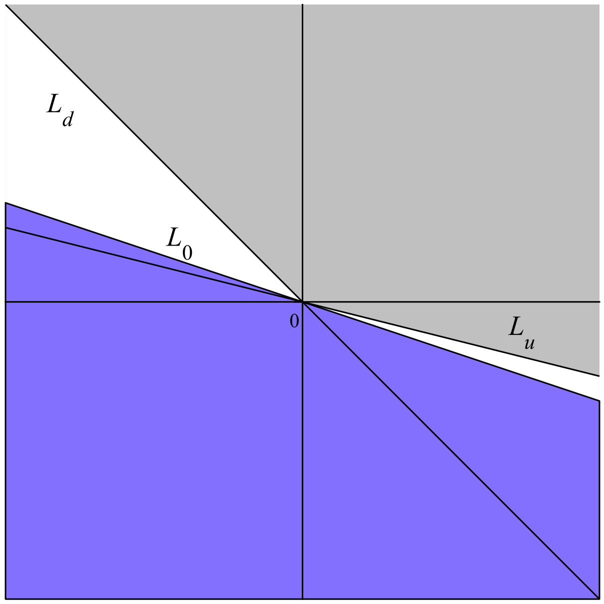

Figure 5: Situation allowing no arbitrage; binomial model

To prove that if $d < 1 + rh < u$ there cannot be an arbitrage is a little trickier. Later we will give a proof based on probabilistic argument. For now, a more informal explanation must suffice. The argument is a proof by contradiction. Suppose there is an arbitrage portfolio process. By definition it must be self-financing. Let $X(t_i)(\omega)$ be the corresponding wealth process. One possibility is that $X(0) < 0$ and $X(t_n)(\omega) \geq 0$ for all ω. Pick any $\omega = (\omega_1, \ldots, \omega_k)$, and consider the sequence $X(0), X(t_1)(\omega), X(t_2)(\omega), \ldots$. Since one cannot look into the future (admissibility), it must be true that each $X(t_i)$ really only depends on $(\omega_1, \ldots, \omega_i)$, so it can be written $X(t_i)(\omega_1, \ldots, \omega_i)$. Since this is the wealth process of an arbitrage portfolio, by following the wealth process along the nodes of the binomial tree there must be some k, $0 \leq k < n$ such that $X(t_k)(\omega_1, \ldots, \omega_k) < 0$ but $X(t_{k+1})(\omega_1, \ldots, \omega_k, \eta) \geq 0$, whether $\eta = 1$ or $\eta = -1$. This means we can achieve an arbitrage in the single period, binomial model starting at the node corresponding to $(\omega_1, \ldots, \omega_k)$ in the binomial tree. But, we know this cannot happen if $d < 1 + rh < u$. Hence an arbitrage portfolio starting from $X(0) < 0$ is not possible. A similar argument rules out arbitrage starting from $X(0) = 0$.

III. Pricing contingent claims; elementary theory for one-period models.

A. Contingent claims.

Suppose a financial market model is given that admits no arbitrage. As usual, T will denote the end of the final period; in an n-period model, $T = t_n$.

A function V on Ω is called a *contingent claim.* For example, the payoff $(S(T)(\omega) - K)^+$ of a European call at strike K and expiry T, as a function on Ω, is a contingent claim. Mathematically, any function on Ω is a contingent claim, but in practice the term is applied mostly to recognized financial derivatives. The interpretation is that V represents a derivative that pays V to the holder at time T. The question that is addressed here is: what is a fair price to charge to someone who wishes to buy this derivative at time $t = 0$?

When $\Omega = \{\omega_1, \ldots, \omega_m\}$, it will be convenient to represent a contingent claim V by the m-dimensional vector,

$$\begin{pmatrix} V(\omega_1) \\ \vdots \\ V(\omega_m) \end{pmatrix}.$$

B. Replicating portfolios and complete markets. Let $V(T)$ be a contingent claim. (The parameter T is inserted here to indicate the payoff is at T.) An admissible, self-financing portfolio $\triangle$, with associated wealth process $X_\triangle(t)(\omega)$ is said to *replicate* $V(T)$ if

$$X_\triangle(T)(\omega) = V(T)(\omega) \quad \text{for all } \omega \in \Omega.$$

A contingent claim for which there exists a replicating portfolio is said to be *attainable.* A market model is said to be *complete* if every contingent claim is attainable.

Attainability in the one-period model. When $\Omega = \{\omega_1, \ldots, \omega_m\}$, one can write down the condition for attainability in the one-period model using linear algebra. Let there be p risky assets and a money market Recall the asset price vector at (T, ω): $A(T)(\omega) = (B(t), S_1(T)(\omega), \ldots, S_p(T)(\omega))$. List these for all ω in the *asset price matrix,*

$$\mathcal{A}(T) = \begin{pmatrix} B(T) & S_1(T)(\omega_1) & \ldots & S_p(T)(\omega_1) \\ \vdots & \vdots & \vdots & \vdots \\ B(T) & S_1(T)(\omega_m) & \ldots & S_p(T)(\omega_m) \end{pmatrix}.$$

A portfolio $\triangle = (\triangle_0, \triangle_1, \ldots, \triangle_p)'$ (if v is a vector v' denotes its transpose) held over the period from 0 to T will, for each ω, have the value $A(T)(\omega) \cdot \triangle$. Thus $V(T)$ will be attainable if there exists a vector $\triangle$ such that

$$\mathcal{A}(T) \cdot \begin{pmatrix} \triangle_0 \\ \triangle_1 \\ \vdots \\ \triangle_p \end{pmatrix} = \begin{pmatrix} V(\omega_1) \\ \vdots \\ V(\omega_m) \end{pmatrix}.$$

Thus the set of attainable contingent claims is equal to the column space of the matrix $\mathcal{A}(T)$, and the market is complete if the rank of $\mathcal{A}(T)$ is m.

Examples:

(i) Consider a market with a money market account at rate r, continuously compounded, and an asset with price $S(t)(\omega)$. Let $V = S(T)(\omega) - K$ be the payoff(per unit) of a forward contract maturing at T with delivery price K. The portfolio $\triangle_0 = -e^{-rT}K$, $\triangle_1 = 1$ replicates this forward payoff. It requires borrowing $e^{-rT}K$ at the risk-free rate and purchasing one share of stock. At time T the owner of this portfolio with owe $e^{rT}e^{-rT}K = K$ to the lender and own a share of asset worth $S(T)(\omega)$. Hence $X(T)(\omega) = S(T)(\Omega) - K$. This is true whatever no matter how $S(T)(\omega)$ is defined. The value of this replicating portfolio at time $t = 0$ is

$$X(0) = S(0) - e^{-rt}K. \tag{5}$$

(ii) Consider a one-period, binomial model that satisfies $d < 1+rT < u$ and hence admits no arbitrage.

Claim: There is a unique replicating portfolio for each contingent claim in this market.

Indeed, if V is a contingent claim, $(\triangle_0, \triangle_1)$ is a replicating portfolio if and only if

$$\begin{aligned} (1+rT)\triangle_0 + dS(0)\triangle_1 &= V(-1) \\ (1+rT)\triangle_0 + uS(0)\triangle_1 &= V(1) \end{aligned} \tag{6}$$

This has the unique solution

$$\begin{pmatrix} \triangle_0 \\ \triangle_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{(1+rT)(u-d)}\left(uV(-1) - dV(1)\right) \\ \frac{1}{S(0)(u-d)}\left(V(1) - V(-1)\right) \end{pmatrix} \tag{7}$$

The value of this portfolio at time $t = 0$ is

$$X(0) = \triangle_0 + S(0)\triangle_1 = \frac{1}{1+rT}\left(\frac{u-(1+rT)}{u-d}V(-1) + \frac{(1+rT)-d}{u-d}V(1)\right). \tag{8}$$

C. Pricing by no arbitrage.

General principle. Consider a financial market that does not admit arbitrage. Now introduce a contingent claim $V(T)$ for which there is a replicating portfolio with wealth process X. Assume that all replicating portfolios must start with the same initial endowment $X(0)$. Recall that $B(T)$ is the amount earned by time T of one dollar invested at the risk free rate. Let $V(0)$ denote the price at time 0 for the contingent claim. The unique value of $V(0)$ for which there is no arbitrage is

$$V(0) = X(0). \tag{9}$$

This is easy to see: if $X(0) < V(0)$, you can sell a contingent claim for $V(0)$, invest $X(0)$ of this according to the replicating portfolio, and invest $V(0)-X(0)$ at the risk-free rate. At time T, the replicating portfolio will yield $V(T)$ and allow you to pay the contingent claim to the holder, leaving a guaranteed profit of $(V(0)-X(0))B(T) > 0$. If instead $V(0) < X(0)$, buy the contingent claim and borrow a replicating portfolio. Your financial position is $V(0) - X(0) < 0$, since you own a claim worth $V(0)$ but have a debt of $X(0)$. At time T, you will owe the value $X(T)$ to the lender of the replicating portfolio. But the contingent claim you own provides you with exactly $V(T) = X(T)$ to pay off this debt.

Examples. Forward price.

(i) *Forward price.* We saw in Example (i) of the previous section that a forward contract maturing in T years for a delivery price of K can be replicated by a portfolio with initial value worth $e^{-rt}K - S(0)$. Hence, by the no-arbitrage principle, this is what the forward contract is worth at time $t = 0$. Forward contracts are set so that there is no initial exchange of money. This means that K should be set so that

$$K = e^{rT}S(0). \tag{10}$$

This is called the forward price, and is sometimes denoted $F_0 = e^{rT}S(0)$.

(ii) *The one-period binomial model.* By equation (8) and the no-arbitrage principle, the price at time $t = 0$ of contingent claim V in the one-period binomial model is

$$\begin{aligned} V(0) &= \frac{1}{1+rT}\Big(\frac{u-(1+rT)}{u-d}V(-1) + \frac{(1+rT)-d}{u-d}\Big)V(1) \\ &= \frac{1}{1+rT}\Big(\frac{u-(1+rT)}{u-d}, \frac{(1+rT)-d}{u-d}\Big) \cdot (V(-1), V(1)). \end{aligned}$$

The vector $\frac{1}{1+rT}(\frac{u-(1+rT)}{u-d}, \frac{(1+rT)-d}{u-d})$ is called a state-price vector. Notice that its components are positive and sum to $\frac{1}{1+rT}$, which is the discount factor for computing the present value, at $t = 0$, of future cash payments at $t = T$.

D. The Fundamental Theorem of Asset Pricing for One-period, Finite Models

Consider a one-period model in which Ω is finite and there are p risky assets with price processes $S_1, \ldots, S_p$.

First we explain the general concept of a state-price vector. We saw in Example (ii) of the previous section that for the one-period binomial model there is a vector

$$\mathbf{p} = (p_1, p_2) = \frac{1}{1+rT}\left(\frac{u-(1+rT)}{u-d}, \frac{(1+rT)-d}{u-d}\right)$$

so that the no-arbitrage price of any contingent claim defined by $(V(T)(-1), V(T)(1))$ is $(p_1 V(T)(-1) + p_2 V(T)(1) = \mathbf{p} \cdot (V(-1), V(1))$. In particular, a money market investment of $B(0) = 1$ at $t = 0$ pays $1 + rT$ at time T, so in this case $V(T)(-1) = V(T)(1) = 1 + rT$. Indeed

$$(1 + rT, 1 + rT) \cdot (p_1, p_2) = (1 + rT)(p_1 + p_2) = 1 = B(0).$$

Likewise, an investment in one unit of stock pays out $V(T)(-1) = dS(0)$, $V(T)(1) = uS(0)$ and we have

$$(dS(0), uS(0)) \cdot (p_1, p_2) = S(0)\frac{d(u-(1+rT)) + u((1+rT) - d)}{(1+rT)(u-d)} = S(0).$$

These are two linear equations in two unknowns and hence they determine the price vector. A state-price vector for a more general, one-period model is defined by generalizing these conditions on the assets of a model.

Consider a one period model with $\Omega = \{\omega_1, \ldots, \omega_m\}$, one risk-free money market with rate r, and p risky assets. In this model, the value of the money market is given by $B(0) = 1$ and $B(T)(\omega) = 1 + rT$ for all ω. The ratio $1/(1 + rT)$ is called the discount factor.

A *state-price vector* is a vector $\mathbf{p} = (p_1, \ldots, p_m)$ is a vector of positive components such that

$$\sum_1^m p_i = \frac{1}{1+rT} \tag{11}$$

and

$$S_i(0) = \Big(S_i(T)(\omega_1), \ldots, S_i(T)(\omega_m)\Big) \cdot \mathbf{p} \quad \text{for } 1 \le i \le p. \tag{12}$$

In vector-matrix notation, a state-price vector $\mathbf{p}$ is any solution, whose components are all positive, to

$$\mathbf{p} \cdot \mathcal{A}(\mathcal{T}) = (1, S_1(0), \ldots, S_p(0)). \tag{13}$$

If $\mathbf{p}$ is a state-price vector,

$$\mathbf{q} = (q_1, \ldots, q_m) := (1 + rT)(p_1, \ldots, p_m)$$

is a vector with positive components that solves,

$$\sum_{1}^{m} q_i = 1, \tag{14}$$

and

$$S_i(0) = \frac{1}{1+rT}\Big(S_i(T)(\omega_1), \ldots, S_i(T)(\omega_m)\Big) \cdot \mathbf{q} \quad \text{for } 1 \leq i \leq p. \tag{15}$$

Conversely, if $\mathbf{q}$ solves these equations, $\frac{1}{1+rT}\mathbf{q}$ is a state-price vector. It will be convenient to always represent state-price vectors in this form.

Now we come to a major result.

Theorem 1 *(*The Fundamental Theorem of Asset Pricing for the One-period Model*)* *A one-period model with $\Omega = \{\omega_1, \ldots, \omega_m\}$, p risky assets, and a risk-free money market with simple interest rate r is arbitrage-free if and only if there exists a state-price vector*

$$\frac{1}{1+rT}(q_1, \ldots, q_m).$$

In this case the no-arbitrage price at time $t = 0$ of an attainable contingent claim with payoff $V(T)(\omega)$ at time T is

$$V(0) = \frac{1}{1+rT}\mathbf{q} \cdot V(T) = \frac{1}{1+rT}\sum_{1}^{m} q_i V(T)(\omega_i). \tag{16}$$

We will not give a full proof. It is not hard to see that if a state-price vector exists, there cannot be arbitrage. Indeed, suppose $\triangle = (\triangle_0, \ldots, \triangle_p)'$ is any portfolio. Let $X(T)(\omega_i) = A(T)(\omega_i) \cdot \triangle$ be the value of this portfolio at (T, ω_i). Then

$$\begin{pmatrix} X(T)(\omega_1) \\ \vdots \\ X(T)(\omega_m) \end{pmatrix} = \mathcal{A}(T) \cdot \begin{pmatrix} \triangle_0 \\ \vdots \\ \triangle_p \end{pmatrix}$$

Thus

$$\begin{aligned}
\mathbf{p} \cdot \begin{pmatrix} X(T)(\omega_1) \\ \vdots \\ X(T)(\omega_m) \end{pmatrix} &= \mathbf{p} \cdot \mathcal{A}(T) \cdot \begin{pmatrix} \triangle_0 \\ \vdots \\ \triangle_p \end{pmatrix} \\
&= (1, S_1(0), \ldots, S_p(0)) \cdot \begin{pmatrix} \triangle_0 \\ \vdots \\ \triangle_p \end{pmatrix} \\
&= \triangle_0 + \sum_{1}^{p} \triangle_i S_i(0) = X(0).
\end{aligned}$$

The last expression is the value of the portfolio at time 0. Now suppose that $X(T)(\omega_i) \geq 0$ for all ω_i, then the first term,

$$\mathbf{p} \cdot \begin{pmatrix} X(T)(\omega_1) \\ \vdots \\ X(T)(\omega_m) \end{pmatrix} \geq 0$$

since all the components of $\mathbf{p}$ are positive, and so it follows that $X(0) \geq 0$. Therefore there is no portfolio that gives an arbitrage of type (a). A similar argument shows there can be no arbitrage of type (b).

It is more subtle to show that if there is no arbitrage, a state-price vector must exist. This requires some theory of convex sets and will be omitted. The interested reader is referred to Chapter 1, section A, of D. Duffie, *Dynamic Asset Pricing Theory*, Princeton University Press.

Once we have a state-price vector, it is also relatively straightforward to derive the pricing formula (16). Let $\triangle$ replicate the contingent claim $V(T)$, and let $X(0)$ and $X(T)(\omega)$ be the values of this replicating portfolio at times 0 and T. Of course, $X(t)(\omega) = V(T)(\omega)$ for all ω. By the no-arbitrage principle, $V(0) = X(0)$ is the no-arbitrage price of the cliam. Working the equations, taken in the reverse direction,

$$X(0) = \mathbf{p} \cdot X(T) = \frac{1}{1+rT}\mathbf{q} \cdot V(T).$$

This completes the proof.