Tensor Products, Positive Linear Operators, and Delay-Differential Equations

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Abstract

To be written.

1 Introduction

In this paper we develop the theory of compound functional differential equations. This follows in the spirit of compound ordinary differential equations and dynamical systems as developed by J. Muldowney [6] and Q. Wang [8]. Broadly, this topic concerns tensor products and exterior products of linear nonautonomous evolutionary systems. We also explore positivity issues connected with compound systems for a class of linear scalar delay-differential equations (1.1) with a single delay and a signed feedback.

Abstractly, a linear (evolutionary) process $U(t,\tau) : X \to X$ on a Banach space X is a collection of bounded linear operators $U(t,\tau)$, for $t \geq \tau$, for which $U(\tau,\tau) = I$ and $U(t,\sigma)U(\sigma,\tau) = U(t,\tau)$ whenever $t \geq \sigma \geq \tau$, with $U(t,\tau)x$ varying continuously in (t,τ) for each fixed x. Linear processes occur as solution maps of a wide variety of nonautonomous linear equations, including of course the finite-dimensional case $\dot{x} = A(t)x$ of an ordinary differential equation. Our interest in a large part is with linear processes generated by the delay-differential equation (int1)

$$\dot{x}(t) = -\alpha(t)x(t) - \beta(t)x(t-1), \tag{1.1}$$

where $\alpha : \mathbf{R} \to \mathbf{R}$ and $\beta : \mathbf{R} \to \mathbf{R}$ are locally integrable functions and where generally $\beta(t)$ is of constant sign, either positive or negative, for almost every t. Typically, the underlying Banach space for a system such as (1.1) is $X = C([-1,0])$.

Given an abstract linear process $U(t,\tau)$ as above, and given an integer $m \geq 1$, one obtains the co-called compound processes

$$\mathbf{U}(t,\tau) = U(t,\tau)^{\otimes m}, \qquad \mathbf{W}(t,\tau) = U(t,\tau)^{\wedge m},$$

by taking the m-fold tensor product and m-fold wedge product, respectively, of the operator $U(t,\tau)$. These compound processes are themselves linear processes on the tensor and wedge products $X^{\otimes m}$ and $X^{\wedge m}$ of the space X. In general, if X_j for $1 \leq j \leq m$ are Banach spaces, then one may consider the tensor product

$$X_0 = X_1 \otimes X_2 \otimes \cdots \otimes X_m$$

of these spaces. For infinite dimensional spaces, there are typically many inequivalent norms for X_0 arising from the norms on the X_j. For our purposes, the so-called injective cross norm is the suitable choice for X_0, and it is used throughout this paper. In a natural way, if A_j are bounded linear operators on X_j for $1 \leq j \leq m$, one obtains a bounded linear operator $A_0 = A_1 \otimes A_2 \otimes \cdots \otimes A_m$ on X_0. In the case all $X_j = X$ are the same space one writes $X_0 = X^{\otimes m}$, and also $A_0 = A^{\otimes m}$ if all operators $A_j = A$ are the same. The wedge product, or exterior product $X^{\wedge m} \subseteq X^{\otimes m}$, is a subspace of $X^{\otimes m}$ of elements which satisfy an anti-symmetry property in a fashion analogous to the well-known finite-dimensional case. The subspace $X^{\wedge m}$ is invariant for the operator $A^{\otimes m}$, and one denotes by $A^{\wedge m} = A^{\otimes m}|_{X^{\wedge m}}$ the restriction of this operator to this subspace.

A key point connected with tensor and wedge products of operators is the behavior of their spectra. Suppose that the essential spectral radius $\rho(A)$ of A satisfies $\rho(A) = 0$; this is the case if either A or some power A^n of A is compact, which is the case for $t > \tau$ for the operators $U(t, \tau)$ associated to the delay equation (1.1). Then for any $m \geq 1$ the spectrum of $A^{\wedge m}$ consists of all products $\lambda_1 \lambda_2 \cdots \lambda_m$ where the λ_j are elements of the spectrum of A, and where the number of repetitions of a given λ_j in this product cannot exceed the multiplicity of λ_j as an element of the spectrum of A. (For a precise statement of this result, including a formula for the multiplicity of $\lambda_1 \lambda_2 \cdots \lambda_m$ as an element of $\operatorname{spec}(A^{\wedge m})$ and a description of the eigenspace, see Proposition 2.3 below, along with Corollary 2.2.) As we point out, this fact has ramifications for the stability of periodic orbits of nonlinear systems as it is applied to the Floquet ananalysis of the linearized system. **MENTION MULDOWNEY/WANG?**

A surprising aspect of compound systems for equation (1.1) relates to positivity properties when the feedback coefficient $\beta(t)$ is of constant sign. A main result of this paper is Theorem 4.1, the Positivity Theorem. This states that if $(-1)^m \beta(t) \geq 0$ almost everywhere, then for any t and τ with $t \geq \tau$, the operator $\mathbf{W}(t, \tau) = U(t, \tau)^{\wedge m}$ associated to equation (1.1) is a positive operator with respect to the appropriate cone in $X^{\wedge m} = C([-1, 0])^{\wedge m}$.

If additionally the coefficients in (1.1) are periodic, say if $\alpha(t + \gamma) \equiv \alpha(t)$ and $\beta(t + \gamma) \equiv \beta(t)$ hold identically for some $\gamma > 0$, and if also for the second coefficient there is a uniform positive lower bound $(-1)^m \beta(t) \geq (-1)^m \beta_0 > 0$ for almost every t and some integer m, then computable lower bounds on the norms $|\lambda|$ of the Floquet multipliers (characteristic multipliers) can be obtained. More precisely, the set of nonzero Floquet multipliers $\{\lambda_k\}_{k=1}^{\infty}$ is a countably infinite set. If it is ordered so that

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$$

with repetitions according to algebraic multiplicity, then an explicit lower bound for each $|\lambda_k|$ can be given. Further, the strict inequality

$$|\lambda_k| > |\lambda_{k+1}|$$

holds for each k for which $k - m$ is even (that is, for a particular parity class, odd or even, for k). For each such m, the monodromy operator $U(\tau + \gamma, \tau)^{\wedge m}$ possesses a positive eigenvector, where here positive is interpreted in the sense of a particular cone to be described below. **MORE ABOUT u_0-POSITIVITY. k OR m?**

2 Tensor Products of Banach Spaces

In what follows we let $\mathcal{L}(X, Y)$ denote the space of bounded linear operators between Banach spaces X and Y. We also denote $\mathcal{L}(X) = \mathcal{L}(X, X)$. For any operator $A \in \mathcal{L}(X)$, we let spec(A) and ess spec(A) denote the spectrum and the essential spectrum of A, and we let

$$r(A) = \sup\{|\lambda| \mid \lambda \in \mathrm{spec}(A)\}, \qquad \rho(A) = \sup\{|\lambda| \mid \lambda \in \mathrm{ess\ spec}(A)\},$$

denote the spectral radius and essential spectral radius of A, respectively.

To begin our discussion of tensor products, let X and Y be Banach spaces, and let $X \odot Y$ denote their algebraic tensor product. Then $X \odot Y$ is the vector space consisting of equivalence classes of elements of the form (01)

$$z = \sum_{i=1}^{n} a_i (x_i \otimes y_i) \tag{2.1}$$

with $x_i \in X$ and $y_i \in Y$, and $a_i \in \mathbf{C}$, under the equivalence relation generated by all identities of the form

$$(x + x') \otimes y = x \otimes y + x' \otimes y, \qquad x \otimes (y + y') = x \otimes y + x \otimes y',$$

$$a(x \otimes y) = (ax) \otimes y = x \otimes (ay),$$

and only those identities. There are various possible (generally inequivalent) norms for $X \odot Y$, among which are the so-called **cross norms**, namely norms for which $\|x \otimes y\| = \|x\|\|y\|$ holds for every x and y, and with the corresponding equation holding with the dual norms. In particular, the norm defined by (02)

$$\|z\| = \sup_{(\xi,\eta)\in\mathcal{B}} \left| \sum_{i=1}^{n} a_i \xi(x_i)\eta(y_i) \right|, \qquad \mathcal{B} = \{(\xi, \eta) \in X^* \times Y^* \mid \|\xi\| = \|\eta\| = 1\}, \tag{2.2}$$

for z as in (2.1), where X^* and Y^* are the dual spaces to X and Y, is a cross norm, called the **injective cross norm**. One easily checks that $\|z\|$ is well-defined, that is, it is independent of the representation (2.1) of z, and that the formula (2.2) does indeed define a norm on $X \odot Y$. Now define $X \otimes Y$ to be the Banach space which is the completion of $X \odot Y$ with respect to this norm. Throughout this paper, we shall always take the injective cross norm when considering tensor products of Banach spaces.

If E and G are two other Banach spaces, and $A \in \mathcal{L}(X, E)$ and $B \in \mathcal{L}(Y, G)$ are bounded linear operators, then one defines the tensor product $A \otimes B$ of these operators by $(A \otimes B)(x \otimes y) = (Ax) \otimes (By)$, and extends this by linearity first to $X \odot Y$, and then continuously to all of $X \otimes Y$. It is easily checked that this construction determines a unique bounded linear operator

$$A \otimes B \in \mathcal{L}(X \otimes Y, E \otimes G), \qquad \|A \otimes B\| = \|A\|\|B\|, \tag{2.3}$$

with norm as indicated. One also sees that

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2) \tag{2.4}$$

for operators defined on appropriate spaces.

If we have a direct sum decomposition $X = X_1 \oplus X_2$ for X, where $X_1, X_2 \subseteq X$ are closed subspaces, then there is a direct sum decomposition

$$X \otimes Y = (X_1 \otimes Y) \oplus (X_2 \otimes Y). \tag{2.5}$$

We note that *a priori* there are two possible definitions for $X_j \otimes Y$. Namely, $X_j \otimes Y$ can be defined either (a) directly, by considering X_j as a Banach space in its own right and taking the tensor product with Y, or (b) by taking the closure in $X \otimes Y$ of the subspace spanned by elements $x' \otimes y$ with $x' \in X_j$ and $y \in Y$. That these two constructions yield the same result, namely isometric Banach spaces, follows from the identity

$$\sup_{(\xi', \eta) \in \mathcal{B}'} \left| \sum_{i=1}^{n} \xi'(x_i')\eta(y_i) \right| = \sup_{(\xi, \eta) \in \mathcal{B}} \left| \sum_{i=1}^{n} \xi(x_i')\eta(y_i) \right|, \qquad \mathcal{B}' = \{(\xi', \eta) \in X_j^* \times Y^* \mid \|\xi'\| = \|\eta\| = 1\},$$

with $x_i' \in X_j$ and $y \in Y$, and $\mathcal{B}$ as in (2.2), which is an immediate consequence of the Hahn-Banach theorem. (For the norm in X_j we always take the norm inherited as a subspace of X.) In a similar fashion, if $Y = Y_1 \oplus Y_2$ then $X \otimes Y = (X \otimes Y_1) \oplus (X \otimes Y_2)$.

The above constructions extend in the obvious way to products and sums of several Banach spaces. In particular, if X, Y, and Z are Banach spaces, then $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ are naturally isometrically isomorphic. If X_j are Banach spaces for $1 \leq j \leq m$ then one can define $X_1 \otimes X_2 \otimes \cdots \otimes X_m$ in a natural fashion, along with products $A_1 \otimes A_2 \otimes \cdots \otimes A_m$ of operators $A_j \in \mathcal{L}(X_j, E_j)$ where the E_j are Banach spaces, with the obvious generalization of (2.3). Similarly, (2.5) generalizes to the case of multiple summands and multiple factors. We also note that for spaces X_j of either finite or infinite dimension, we have that

$$\dim(X_1 \otimes X_2 \otimes \cdots \otimes X_m) = \prod_{j=1}^{m} \dim X_j,$$

with the convention that $0 \times \infty = 0$.

The following result will play an important role.

Theorem 2.1 (Ichinose [3, Theorem 4.3]; see also [4] and Schechter [7]). *Let X_j be a Banach space and $A_j \in \mathcal{L}(X_j)$ for $1 \leq j \leq m$. Then* (14)

$$\operatorname{spec}(A_1 \otimes A_2 \otimes \cdots \otimes A_m) = \{\lambda_1 \lambda_2 \cdots \lambda_m \mid \lambda_j \in \operatorname{spec}(A_j) \textit{ for every } 1 \leq j \leq m\} \tag{2.6}$$

for the spectrum of the tensor product.

The above theorem can be generalized to count multiplicities, at least of isolated spectral points, as follows. **ICHINOSE REFERENCES AND REMARKS.**

IS THE FOLLOWING RESULT IN ICHINOSE?

Corollary 2.2. *Let A_j and X_j for $1 \leq j \leq m$ be as in Theorem 2.1. Denote $A_0 = A_1 \otimes A_2 \otimes \cdots \otimes A_m$ and take any $\lambda_0 \in \operatorname{spec}(A_0)$ with $\lambda_0 \neq 0$ for which λ_0 is an isolated point of $\operatorname{spec}(A_0)$. Then there are finitely many distinct m-tuples* (13)

$$(\lambda_1^k, \lambda_2^k, \ldots, \lambda_m^k) \in \mathbf{C}^m, \tag{2.7}$$

for $1 \leq k \leq p$, such that (17a)

$$\lambda_0 = \lambda_1^k \lambda_2^k \cdots \lambda_m^k, \qquad \lambda_j^k \in \operatorname{spec}(A_j), \tag{2.8}$$

for $1 \leq j \leq m$ and $1 \leq k \leq p$. Moreover, each such λ_j^k is an isolated point of $\operatorname{spec}(A_j)$. Let $G_j^k \subseteq X_j$ denote the spectral subspace of A_j corresponding to λ_j^k, let $\nu_j^k = \dim G_j^k$, so $1 \leq \nu_j^k \leq \infty$, and let (18a)

$$\nu_0 = \sum_{k=1}^{p} \nu_1^k \nu_2^k \cdots \nu_m^k. \tag{2.9}$$

Then the spectral subspace $G_0 \subseteq X_0$ *of* A_0 *corresponding to* λ_0 *is given by* (18b)

$$G_0 = \bigoplus_{k=1}^{p} G_1^k \otimes G_2^k \otimes \cdots \otimes G_m^k, \tag{2.10}$$

where $\dim G_0 = \nu_0$*, and where each subspace* $G_1^k \otimes G_2^k \otimes \cdots \otimes G_m^k$ *is invariant under* A_0*.*

Remark. We are assuming that every possible m-tuple (2.7) satisfying (2.8) has been enumerated and thus occurs for some k. Also, the m-tuples (2.7) are geometrically distinct points in $\mathbf{C}^m$, with no repetitions for multiplicity as elements of a spectrum, that is, $(\lambda_1^k, \lambda_2^k, \ldots, \lambda_m^k) = (\lambda_1^{k'}, \lambda_2^{k'}, \ldots, \lambda_m^{k'})$ as points in $\mathbf{C}^m$ if and only if $k = k'$. But note it can still happen that for some j, there may be repetitions among the quantities $\lambda_j^1, \lambda_j^2, \ldots, \lambda_j^p$, say $\lambda_j^k = \lambda_j^{k'}$ and thus $G_j^k = G_j^{k'}$, even if $k \neq k'$.

Remark. A sufficient condition for λ_0 to be an isolated point of $\operatorname{spec}(A_0)$, as in the statement of Corollary 2.2, is easily given. Namely, assume that $\lambda_0 \in \operatorname{spec}(A_0)$ satisfies

$$|\lambda_0| > \max_{1 \leq j \leq m} \{\rho_j r_j^{-1}\} r_1 r_2 \cdots r_m,$$

where $r_j = r(A_j)$ and $\rho_j = \rho(A_j)$ are the spectral radii and essential spectral radii, respectively, of these operators, and where we assume that $r_j > 0$ for each j. To prove that λ_0 is an isolated point of $\operatorname{spec}(A_0)$, it is enough to prove that for every representation $\lambda_0 = \lambda_1 \lambda_2 \cdots \lambda_m$ where $\lambda_j \in \operatorname{spec}(A_j)$, that each λ_j is an isolated point of $\operatorname{spec}(A_j)$. To this end, it is enough to prove that $|\lambda_j| > \rho_j$ for each j. Thus assume that $|\lambda_{j_0}| \leq \rho_{j_0}$ for some j_0. Then as $|\lambda_j| \leq r_j$ for each j, it follows that

$$|\lambda_0| = |\lambda_1 \lambda_2 \cdots \lambda_m| \leq (\rho_{j_0} r_{j_0}^{-1}) r_1 r_2 \cdots r_m,$$

which is a contradiction, and thus λ_0 is isolated.

We remark that in our analysis of delay-differential equaions below, it is the case that $\rho_j = 0$ for each j.

Proof of Corollary 2.2. The fact that λ_0 is a nonzero isolated point of $\operatorname{spec}(A_0)$, along with (2.6) from Theorem 2.1, implies that if $\lambda_0 = \lambda_1 \lambda_2 \cdots \lambda_m$ with $\lambda_j \in \operatorname{spec}(A_j)$, then each λ_j is an isolated point of $\operatorname{spec}(A_j)$. This in turn implies that there is a finite number p of such representations of λ_0 as a product. Let us enumerate all such representations, as in (2.8) in the statement of the corollary.

For every j satisfying $1 \leq j \leq m$, let q_j be the number of distinct quantities λ_j^k for $1 \leq k \leq p$. Here we mean numerically distinct quantities, that is, without repetitions for multiplicity as an element of spec(A_j). Let $\widetilde{\lambda}_j^i$ for $1 \le i \le q_j$ be a renumbering of these quantities where each occurs only once, and so we have

$$\{\lambda_j^1, \lambda_j^2, \ldots, \lambda_j^p\} = \{\widetilde{\lambda}_j^1, \widetilde{\lambda}_j^2, \ldots, \widetilde{\lambda}_j^{q_j}\},$$

with equality as unordered sets. Let $\widetilde{G}_j^i \subseteq X_j$ denote the spectral subspace of A_j corresponding to $\widetilde{\lambda}_j^i$. Then for each j we have a direct sum decomposition

$$X_j = \widetilde{G}_j^0 \oplus (\widetilde{G}_j^1 \oplus \widetilde{G}_j^2 \oplus \cdots \oplus \widetilde{G}_j^{q_j}),$$

where $\widetilde{G}_j^0$ is the spectral subspace of A_j corresponding to $\operatorname{spec}(A_j) \setminus \{\widetilde{\lambda}_j^1, \widetilde{\lambda}_j^2, \ldots, \widetilde{\lambda}_j^{q_j}\}$.

Now consider all m-tuples $\iota = (i_1, i_2, \ldots, i_m) \in \mathcal{I}$ where

$$\mathcal{I} = \{(i_1, i_2, \ldots, i_m) \in \mathbf{Z}^m \mid 0 \le i_j \le q_j \text{ for every } j \text{ satisfying } 1 \le j \le m\}$$

and for each such $\iota \in \mathcal{I}$ let (16)

$$\Gamma^\iota = \widetilde{G}_1^{i_1} \otimes \widetilde{G}_2^{i_2} \otimes \cdots \otimes \widetilde{G}_m^{i_m} \subseteq X_1 \otimes X_2 \otimes \cdots \otimes X_m. \tag{2.11}$$

Then

$$X_1 \otimes X_2 \otimes \cdots \otimes X_m = \bigoplus_{\iota \in \mathcal{I}} \Gamma^\iota.$$

By construction, each subspace $\widetilde{G}_j^i \subseteq X_j$ is invariant for the operator A_j, and thus each subspace Γ^ι is invariant for A_0. Thus the multiplicity of λ_0 as a point in the spectrum of spec(A_0), namely the dimension of the corresponding spectral subspace, equals the sum of the multiplicities of λ_0 as a point in the spectrum of $A_0|_{\Gamma^\iota}$ for the various Γ^ι, where $A_0|_{\Gamma^\iota}$ is the restriction of A_0 to Γ^ι.

Note that not every $A_0|_{\Gamma^\iota}$ need have λ_0 in its spectrum. In fact, there are precisely p of the m-tuples $\iota \in \mathcal{I}$ for which (15)

$$\lambda_0 \in \operatorname{spec}(A_0|_{\Gamma^\iota}) \tag{2.12}$$

holds, with these corresponding to the p different m-tuples in (2.7), (2.8). Moreover, it is the case that $i_j \ne 0$ for each i_j occurring in such an m-tuple ι, that is, the associated subspace $\widetilde{G}_j^{i_j}$ is a spectral subspace of A_j corresponding to $\widetilde{\lambda}_j^{i_j}$, and not the complementary space $\widetilde{G}_j^0$. The spectral subspace $G_0 \subseteq X_0$ of A_0 corresponding to λ_0 is thus the direct sum of those $\Gamma^\iota \subseteq X_0$ satisfying (2.12). These facts are direct consequences of the definition of the quantities λ_j^k, along with the re-labeling of the λ_j^k as $\widetilde{\lambda}_j^i$ and the construction of the set $\mathcal{I}$. Let us denote by

$$\iota^k = (i_1^k, i_2^k, \ldots, i_m^k), \qquad 1 \le k \le p,$$

those $\iota \in \mathcal{I}$ for which (2.12) holds. We may assume these m-tuples are labeled to correspond with the m-tuples in (2.7), (2.8), namely that (17)

$$(\widetilde{\lambda}_1^{i_1^k}, \widetilde{\lambda}_2^{i_2^k}, \ldots, \widetilde{\lambda}_m^{i_m^k}) = (\lambda_1^k, \lambda_2^k, \ldots, \lambda_m^k), \qquad 1 \leq k \leq p. \tag{2.13}$$

Thus the spectral subspace of A_0 for λ_0 is the direct sum (19)

$$G_0 = \Gamma^{\iota^1} \oplus \Gamma^{\iota^2} \oplus \cdots \oplus \Gamma^{\iota^p} \tag{2.14}$$

in this notation. Then from (2.11) and using (2.13), (20)

$$\Gamma^{\iota^k} = \widetilde{G}_1^{i_1^k} \otimes \widetilde{G}_2^{i_2^k} \otimes \cdots \otimes \widetilde{G}_m^{i_m^k} = G_1^k \otimes G_2^k \otimes \cdots \otimes G_m^k. \tag{2.15}$$

Combining (2.14) and (2.15) gives the desired formula in (2.10) for G_0. Furthermore, as we have defined $\nu_j^k = \dim G_j^k$, we obtain the formula in (2.9) for $\nu_0 = \dim G_0$. ■

Now take any Banach space X and consider the m-fold tensor product, denoted

$$X^{\otimes m} = X \otimes X \otimes \cdots \otimes X,$$

with m identical factors on the right-hand side. Let $\mathcal{S}_m$ denote the symmetric group on m elements, namely the set of all maps $\sigma : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$ which are one-to-one and therefore onto. Taking any $\sigma \in \mathcal{S}_m$, we define a linear operator $S_\sigma \in \mathcal{L}(X^{\otimes m})$ as follows. Let (cals)

$$S_\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(m)}, \tag{2.16}$$

then extend S_σ to all of the algebraic tensor product $X^{\odot m} = X \odot X \odot \cdots \odot X$ by linearity, and finally extend S_σ to all of $X^{\otimes m}$ by continuity. One checks that S_σ is well-defined, and is an isometry, $\|S_\sigma z\| = \|z\|$ for every $z \in X^{\otimes m}$. Clearly, $S_{\sigma_1} S_{\sigma_2} = S_{\sigma_1 \sigma_2}$ and $S_\sigma^{-1} = S_{\sigma^{-1}}$. We now define the m-fold exterior product $X^{\wedge m}$ to be

$$X^{\wedge m} = \{z \in X^{\otimes m} \mid S_\sigma z = \operatorname{sgn}(\sigma) z \text{ for every } \sigma \in \mathcal{S}_m\},$$

which is a closed subspace of $X^{\otimes m}$. Here $\operatorname{sgn}(\sigma) = \pm 1$ is the sign of the permutation σ. Equivalently, we may define $P \in \mathcal{L}(X^{\otimes m})$ by (pdef)

$$P = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} \operatorname{sgn}(\sigma) S_\sigma, \tag{2.17}$$

which is easily seen to be a projection, $P^2 = P$. Then $X^{\wedge m} = PX^{\otimes m}$ is the range of P, and we generally denote

$$x_1 \wedge x_2 \wedge \cdots \wedge x_m = P(x_1 \otimes x_2 \otimes \cdots \otimes x_m).$$

We note here, for future use, that (psig)

$$PS_\sigma = \operatorname{sgn}(\sigma)P \tag{2.18}$$

for every $\sigma \in \mathcal{S}_m$. Let us remark also that (wdim)

$$\dim X^{\wedge m} = \binom{\dim X}{m}, \tag{2.19}$$

where $\binom{a}{b}$ denotes the binomial coefficient for $1 \leq a \leq \infty$ and $1 \leq b < \infty$, with $\binom{a}{b} = 0$ if $b > a$ and with $\binom{\infty}{b} = \infty$. One easily checks (2.19), at least if $\dim X = n < \infty$, by noting that if $e_1, e_2, \ldots, e_n \in X$ is a basis for X, then the set of elements $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_m}$ for $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ is a basis for $X^{\wedge m}$.

Now denoting

$$A^{\otimes m} = A \otimes A \otimes \cdots \otimes A \in \mathcal{L}(X^{\otimes m})$$

for the m-fold product of any operator $A \in \mathcal{L}(X)$ on X, we observe that $S_\sigma A^{\otimes m} = A^{\otimes m} S_\sigma$ for every $\sigma \in \mathcal{S}_m$, and thus $PA^{\otimes m} = A^{\otimes m}P$. It follows that $X^{\wedge m}$ is an invariant subspace of $X^{\otimes m}$ for $A^{\otimes m}$. With this, it makes sense to study the spectrum of $A^{\otimes m}$ restricted to $X^{\wedge m}$. Let us denote

$$A^{\wedge m} = A^{\otimes m}|_{X^{\wedge m}} \in \mathcal{L}(X^{\wedge m})$$

for this operator so restricted.

Proposition 2.3. *Let X be a Banach space and $A \in \mathcal{L}(X)$. Then for every $m \geq 1$*

$$\operatorname{spec}(A^{\wedge m}) \subseteq \operatorname{spec}(A^{\otimes m}) = \{\lambda_1\lambda_2 \cdots \lambda_m \mid \lambda_j \in \operatorname{spec}(A) \text{ for every } j \text{ satisfying } 1 \leq j \leq m\}$$

for the operators $A^{\wedge m} \in \mathcal{L}(X^{\wedge m})$ and $A^{\otimes m} \in \mathcal{L}(X^{\otimes m})$. Suppose further that $\lambda_0 \in \operatorname{spec}(A^{\otimes m})$ is a nonzero isolated point of $\operatorname{spec}(A^{\otimes m})$ with spectral subspace $G_0 \subseteq X^{\otimes m}$. Then (21)

$$PG_0 = G_0 \cap X^{\wedge m} \tag{2.20}$$

for the image of this space under P. Moreover, $PG_0 \neq \{0\}$ if and only if $\lambda_0 \in \operatorname{spec}(A^{\wedge m})$, in which case λ_0 is an isolated point of $\operatorname{spec}(A^{\wedge m})$ with PG_0 as its spectral subspace.

Proof. The fact that P commutes with $A^{\otimes m}$ implies that G_0 is invariant under P, which in turn implies the equality in (2.20). The remaining claims are elementary. ■

Assuming the setting of Proposition 2.3, we may use Corollary 2.2 to obtain detailed information about the spectrum and spectral subspaces of $A^{\wedge m}$. In this case each subspace $G_j^k \subseteq X$ in (2.10) is a spectral subspace of A, and it may happen for a given k that there are repetitions among these spaces, namely that $G_j^k = G_{j'}^k$ and so $\lambda_j^k = \lambda_{j'}^k$, for some $j \neq j'$. It is also the case that for every subspace $G_1^k \otimes G_2^k \otimes \cdots \otimes G_m^k$ occurring as a summand in (2.10), and for every permutation $\sigma \in \mathcal{S}_m$, the space obtained by permuting the factors G_j^k using σ must also appear as a summand in (2.10). That is, there exists k' such that

$$G_1^{k'} \otimes G_2^{k'} \otimes \cdots \otimes G_m^{k'} = S_\sigma(G_1^k \otimes G_2^k \otimes \cdots \otimes G_m^k) = G_{\sigma(1)}^k \otimes G_{\sigma(2)}^k \otimes \cdots \otimes G_{\sigma(m)}^k.$$

Of course, it may be the case that $k' = k$ even if σ is not the identity permutation, due to repetitions among the G_j^k.

The following result determines the multiplicity of a point λ_0 in the spectrum of $A^{\wedge m}$, namely the quantity $\dim(PG_0)$ as in the statement of Proposition 2.3. Note that $\dim(PG_0) = 0$ is possible, that is, it is possible that $\lambda_0 \in \operatorname{spec}(A^{\otimes m})$ but $\lambda_0 \notin \operatorname{spec}(A^{\wedge m})$.

Proposition 2.4. *Let X be a Banach space and $A \in \mathcal{L}(X)$. Fix $m \geq 1$ and let $\lambda_0 \in \operatorname{spec}(A^{\otimes m})$ be a nonzero isolated point of $\operatorname{spec}(A^{\otimes m})$. For $1 \leq k \leq p$ denote*

$$H^k = G_1^k \otimes G_2^k \otimes \cdots \otimes G_m^k,$$

where we use the notation in the statement of Corollary 2.2. Define an equivalence relation $\sim$ on the set $\{1, 2, \ldots, p\}$ by letting $k \sim k'$ if and only if there exists $\sigma \in \mathcal{S}_m$ such that

$$H^{k'} = S_\sigma H^k, \qquad \text{that is,} \qquad G_j^{k'} = G_{\sigma(j)}^k \text{ for } 1 \leq j \leq m.$$

(Equivalently, $k \sim k'$ if and only if the two m-tuples in (2.7) corresponding to k and k' are obtained from one another by permuting the entries.) Let $\mathcal{E}^1, \mathcal{E}^2, \ldots, \mathcal{E}^r \subseteq \{1, 2, \ldots, p\}$ denote the corresponding equivalence classes of $\sim$ and let (omq)

$$\Omega^q = \bigoplus_{k \in \mathcal{E}^q} H^k \tag{2.21}$$

for each equivalence class, that is, for $1 \leq q \leq r$. Then (pg0)

$$PG_0 = \bigoplus_{q=1}^{r} P\Omega^q, \qquad \dim(PG_0) = \sum_{q=1}^{r} \dim(P\Omega^q), \tag{2.22}$$

where P is as in (2.17) and $G_0 \subseteq X^{\otimes m}$ is the spectral subspace of λ_0 for $A^{\otimes m}$, as in the statement of Corollary 2.2.

Now fix any q in the range $1 \leq q \leq r$ and select an index $k_* \in \mathcal{E}^q$ such that H^{k_*} has the form (hkap)

$$H^{k_*} = C_1^{\otimes \kappa_1} \otimes C_2^{\otimes \kappa_2} \otimes \cdots \otimes C_d^{\otimes \kappa_d}, \tag{2.23}$$

where for each i we have that $C_i = G_j^{k_*}$ for some j, and where $C_i \neq C_{i'}$ and thus $C_i \cap C_{i'} = \{0\}$ if $i \neq i'$. The integers $\kappa_i \geq 1$ are thus precisely the number times that C_i occurs as a factor in this product. (We remark that for any q such k_* exists, and that k_* and d, and each κ_i and C_i, of course depend on q.) Then (dform2)

$$\dim(P\Omega^q) = \prod_{i=1}^{d} \binom{\dim C_i}{\kappa_i}, \tag{2.24}$$

with the convention in the above product that $0 \times \infty = 0$.

Remark. If, in the setting of Proposition 2.4, every nonzero point of $\operatorname{spec}(A)$ is an isolated point of $\operatorname{spec}(A)$, then the same is true for $\operatorname{spec}(A^{\otimes m})$. In this case the nonzero points in the spectrum of $\operatorname{spec}(A^{\wedge m})$ are precisely those points λ_0 of the form (lpr)

$$\lambda_0 = \lambda_1 \lambda_2 \cdots \lambda_m, \tag{2.25}$$

where each $\lambda_j \in \operatorname{spec}(A)$ with possible repetitions, but where the number of repetitions of each $\lambda \in \operatorname{spec}(A)$ in the product (2.25) is less than or equal to the multiplicity of λ (the dimension of the spectral subspace) as an element of $\operatorname{spec}(A)$. This means that in the formula (2.24), one requires that $\kappa_i \leq \dim C_i$ for each i.

Remark. Suppose, in the setting of Proposition 2.4, that every nonzero point of $\operatorname{spec}(A)$ is an isolated point of simple multiplicity, that is, an element of the point spectrum of algebraic multiplicity one. Let λ_j for $j \geq 1$ denote the distinct nonzero elements of $\operatorname{spec}(A)$. Then every nonzero $\lambda_0 \in \operatorname{spec}(A^{\wedge m})$ has the form

$$\lambda_0 = \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_m}$$

for distinct integers j_i satisfying $1 \leq j_1 < j_2 < \cdots < j_m$. Moreover, the multiplicity of λ_0 as an element of $\operatorname{spec}(A^{\wedge m})$ is precisely the number of possible ways of expressing λ_0 as such a product in this fashion. One sees this easily from Proposition 2.4, in particular, upon noting that in order for the quantity in (2.24) to be positive one must have each $\kappa_i = 1$, as $\dim C_i = 1$ for each i. Thus the space H^{k_*} is a tensor product of m spectral subspaces C_i corresponding to distinct points of $\mathrm{spec}(A)$ whose product is λ_0.

Remark. Suppose, again in the setting of Proposition 2.4, that every nonzero point of $\mathrm{spec}(A)$ is isolated. Suppose further there exists $r > 0$ such that there are exactly m points $\lambda \in \mathrm{spec}(A)$ satisfying $|\lambda| > r$, and where here we count multiplicity. That is, the spectral subspace corresponding to all elements of $\mathrm{spec}(A)$ with $|\lambda| > r$ has dimension exactly m. Denote these elements of $\mathrm{spec}(A)$ by λ_j, for $j = 1, 2, \ldots, m$, listed with repetition in the case of multiplicity. Then $\lambda_0 = \lambda_1\lambda_2\cdots\lambda_m$ is an isolated point of $\mathrm{spec}(A^{\wedge m})$ of simple multiplicity, namely its spectral subspace has dimension $+1$. Further, there exists $\varepsilon > 0$ such that every other $\lambda \in \mathrm{spec}(A^{\wedge m})$ satisfies $|\lambda| < |\lambda_0| - \varepsilon$. Again, these facts follow easily from Proposition 2.4, where the spaces C_i are the spectral subspaces of the various λ_i, with dimension equal to the multiplicity of λ_i, and where $\kappa_i = \dim C_i$.

Proof of Proposition 2.4. It is clear from (2.10) and from (2.21), and the fact that $\sim$ is an equivalence relation, that (gom)

$$G_0 = \bigoplus_{q=1}^{r} \Omega^q. \tag{2.26}$$

Further, it is clear using the definition of $\sim$ that $S_\sigma \Omega^q = \Omega^q$ for every $\sigma \in \mathcal{S}_m$ and $1 \le q \le r$, and so (pomq)

$$P\Omega^q \subseteq \Omega^q \tag{2.27}$$

holds. Thus (2.22) follows from (2.26) and (2.27).

Now let q be fixed, along with k_*, and κ_i and C_i, as in the statement of the proposition. For any $k \in \mathcal{E}^q$ there exists $\pi \in \mathcal{S}_m$ such that (pi)

$$S_\pi H^k = H^{k_*}, \tag{2.28}$$

and thus from (2.18) we see that $PH^k = PS_\pi H^k = PH^{k_*}$. It follows directly from this, and from the definition (2.21) of Ω^q, that (phom)

$$P\Omega^q = PH^{k_*}. \tag{2.29}$$

Let us further denote $\Pi \in \mathcal{L}(\Omega^q, H^{k_*})$ to be the canonical projection of Ω^q onto H^{k_*} associated to the decomposition (2.21). Also define the isotropy group

$$\Psi = \{\sigma \in \mathcal{S}_m \mid S_\sigma H^{k_*} = H^{k_*}\}$$

associated to the subspace H^{k_*}. We claim that (pqp)

$$P\Pi P = \frac{|\Psi|}{m!} P \qquad \text{on } \Omega^q, \tag{2.30}$$

where $|\Psi|$ denotes the cardinality of Ψ. To prove (2.30), it is enough to verify that it holds on each subspace $H^k \subseteq \Omega^q$ for $k \in \mathcal{E}^q$. Fixing such k, and with $\pi \in \mathcal{S}_m$ satisfying (2.28), take any $x \in H^k$ and denote $y = S_\pi x \in H^{k_*}$. Then using (2.18) we have that

$$Px = \operatorname{sgn}(\pi) P S_\pi x = \operatorname{sgn}(\pi) P y = \frac{\operatorname{sgn}(\pi)}{m!} \sum_{\sigma \in \mathcal{S}_m} \operatorname{sgn}(\sigma) S_\sigma y.$$

Upon applying the operator Π, we retain only those terms in the above sum which lie in H^{k_*}, namely, the terms for which $\sigma \in \Psi$. Thus

$$\Pi P x = \frac{\operatorname{sgn}(\pi)}{m!} \sum_{\sigma \in \Psi} \operatorname{sgn}(\sigma) S_\sigma y.$$

Applying P, where we again use (2.18), now gives

$$P \Pi P x = \frac{\operatorname{sgn}(\pi)}{m!} \sum_{\sigma \in \Psi} P y = \frac{\operatorname{sgn}(\pi) |\Psi|}{m!} P y = \frac{|\Psi|}{m!} P x.$$

From this we conclude (2.30), as desired. It follows directly from (2.30) that the map Π is one-to-one on the space $P\Omega^q$. Thus with (2.29) we conclude that (dform)

$$\dim(P\Omega^q) = \dim(\Pi P H^{k_*}). \tag{2.31}$$

Let us now examine the isotropy group Ψ more closely. As the spaces C_i in the product (2.23) are distinct, it follows that $\sigma \in \Psi$ if and only if σ permutes only those indices common to each given term $C_i^{\otimes \kappa_i}$ among themselves without involving other indices. More precisely, define sets $\mathcal{K}_i \subseteq \{1, 2, \ldots, m\}$ for $1 \le i \le d$ by

$$\mathcal{K}_i = \{ n \in \mathbf{Z} \mid \widetilde{\kappa}_{i-1} < n \le \widetilde{\kappa}_i \}, \qquad \widetilde{\kappa}_i = \sum_{j=1}^{i} \kappa_j, \qquad \widetilde{\kappa}_0 = 0,$$

and so $\mathcal{K}_i$ is the set of indices associated with the factor $C_i^{\otimes \kappa_i}$ in (2.23), and each n in the range $1 \le n \le m$ belongs to exactly one $\mathcal{K}_i$. Define subgroups $\Psi_i \subseteq \mathcal{S}_m$, for $1 \le i \le d$, by

$$\Psi_i = \{ \sigma \in \mathcal{S}_m \mid \sigma(n) \in \mathcal{K}_i \text{ if } n \in \mathcal{K}_i, \text{ and } \sigma(n) = n \text{ if } n \in \mathcal{K}_j \text{ for some } j \ne i \},$$

consisting of those σ which permute only the indices in $\mathcal{K}_i$, leaving all other indices fixed. Also define operators

$$P_i = \frac{1}{\kappa_i!} \sum_{\sigma \in \Psi_i} \operatorname{sgn}(\sigma) S_\sigma, \qquad P_0 = P_1 P_2 \cdots P_d,$$

for $1 \le i \le d$. The one easily sees that Ψ is precisely the set of elements of the form (sigi)

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_d \tag{2.32}$$

with $\sigma_i \in \Psi_i$ for $1 \le i \le d$, and that the decomposition in (2.32) is unique for each $\sigma \in \Psi$. Note the commutativity property, that $\sigma_i \sigma_{i'} = \sigma_{i'} \sigma_i$ if $\sigma_i \in \Psi_i$ and $\sigma_{i'} \in \Psi_{i'}$ with $i \ne i'$. One sees that operator $P_i^2 = P_i$ is a projection on H^{k_*} whose range is the space

$$C_1^{\otimes \kappa_1} \otimes \cdots \otimes C_{i-1}^{\otimes \kappa_{i-1}} \otimes C_i^{\wedge \kappa_i} \otimes C_{i+1}^{\otimes \kappa_{i+1}} \otimes \cdots \otimes C_d^{\otimes \kappa_d},$$

and using the above-mentioned commutativity, one sees that $P_0^2 = P_0$ is also a projection on H^{k_*} whose range is the subspace

$$C_1^{\wedge \kappa_1} \otimes C_2^{\wedge \kappa_2} \otimes \cdots \otimes C_d^{\wedge \kappa_d}.$$

We claim that (clm)

$$\Pi P = \frac{\kappa_1! \kappa_2! \cdots \kappa_d!}{m!} P_0 \qquad \text{on } H^{k_*}, \tag{2.33}$$

from which it follows directly, with the above remarks, that (qph)

$$\Pi P H^{k_*} = C_1^{\wedge \kappa_1} \otimes C_2^{\wedge \kappa_2} \otimes \cdots \otimes C_d^{\wedge \kappa_d}. \tag{2.34}$$

Note that (2.34), along with (2.19) and (2.31), implies our desired result (2.24). To prove (2.33), first observe that for every $x \in H^{k_*}$ we have that (pip)

$$\Pi P x = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} \operatorname{sgn}(\sigma) \Pi S_\sigma x = \frac{1}{m!} \sum_{\sigma \in \Psi} \operatorname{sgn}(\sigma) S_\sigma x. \tag{2.35}$$

Now decomposing $\sigma \in \Psi$ as in (2.32), we have that

$$\begin{aligned} \sum_{\sigma \in \Psi} \operatorname{sgn}(\sigma) S_\sigma x &= \sum_{\sigma_1 \in \Psi_1} \sum_{\sigma_2 \in \Psi_2} \cdots \sum_{\sigma_d \in \Psi_d} \operatorname{sgn}(\sigma_1) \operatorname{sgn}(\sigma_2) \cdots \operatorname{sgn}(\sigma_d) S_{\sigma_1} S_{\sigma_2} \cdots S_{\sigma_d} x \\ &= (\kappa_1! \kappa_2! \cdots \kappa_d!) P_1 P_2 \cdots P_d x = (\kappa_1! \kappa_2! \cdots \kappa_d!) P_0 x, \end{aligned}$$

which with (2.35), proves the claim (2.33). With this, the proposition is proved. ∎

We now consider the specific case of Banach spaces

$$X_j = C(\Theta_j) = \{\varphi : \Theta_j \to \mathbf{R} \mid \varphi \text{ is continuous}\}$$

for $1 \le j \le m$, where each Θ_j is a compact Hausdorff space and where the supremum norm is taken for $C(\Theta_j)$. As described in [1, Chapter I, Section 4], one may regard (05)

$$C(\Theta_0) = C(\Theta_1) \otimes C(\Theta_2) \otimes \cdots \otimes C(\Theta_m), \qquad \Theta_0 = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_m, \tag{2.36}$$

as follows. First, taking any $\varphi_j \in C(\Theta_j)$ for $1 \le j \le m$, define $\varphi \in C(\Theta_0)$ by (phi)

$$\varphi(\theta_1, \theta_2, \ldots, \theta_m) = \varphi_1(\theta_1)\varphi_2(\theta_2)\cdots\varphi_m(\theta_m), \tag{2.37}$$

and identify $\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_m$ with φ. More generally, identify any finite sum

$$\sum_{i=1}^{n} \varphi_{1,i} \otimes \varphi_{2,i} \otimes \cdots \otimes \varphi_{m,i} \in C(\Theta_1) \otimes C(\Theta_2) \otimes \cdots \otimes C(\Theta_m)$$

where $\varphi_{j,i} \in C(\Theta_j)$ for $1 \le i \le n$ and $1 \le j \le m$, with $\varphi \in C(\Theta_0)$ given by (06)

$$\varphi(\theta_1, \theta_2, \ldots, \theta_m) = \sum_{i=1}^{n} \varphi_{1,i}(\theta_1)\varphi_{2,i}(\theta_2)\cdots\varphi_{m,i}(\theta_m). \tag{2.38}$$

One sees that this identification is an isometry, that is, (04)

$$\|\varphi\| = \left\| \sum_{i=1}^{n} \varphi_{1,i} \otimes \varphi_{2,i} \otimes \cdots \otimes \varphi_{m,i} \right\|, \tag{2.39}$$

where the norms in (2.39) are those in $C(\Theta_0)$ and $C(\Theta_1) \otimes C(\Theta_2) \otimes \cdots \otimes C(\Theta_m)$, respectively. To prove (2.39), first take elements $\xi_j \in C(\Theta_j)^*$ of the dual spaces, with $\|\xi_j\| = 1$, for $1 \le j \le m$. Each ξ_j is given by integration with respect to a Borel measure $d\mu_j(\theta_j)$ on Θ_j with total variation $|\mu_j|(\Theta_j) = 1$. Then with (2.38) we have, following (2.2), that (meas)

$$\begin{aligned} &\left| \sum_{i=1}^{n} \xi_1(\varphi_{1,i})\xi_2(\varphi_{2,i})\cdots\xi_m(\varphi_{m,i}) \right| \\ &\qquad = \left| \int_{\Theta_1}\int_{\Theta_2}\cdots\int_{\Theta_m} \varphi(\theta_1, \theta_2, \ldots, \theta_m)\, d\mu_m(\theta_m)\cdots d\mu_2(\theta_2)\, d\mu_1(\theta_1) \right| \le \|\varphi\|. \end{aligned} \tag{2.40}$$

Upon taking the supremum over all such ξ_j, we have that (measinq)

$$\left\| \sum_{i=1}^{n} \varphi_{1,i} \otimes \varphi_{2,i} \otimes \cdots \otimes \varphi_{m,i} \right\| \leq \|\varphi\|. \tag{2.41}$$

To obtain equality in (2.41), take any point $(\theta_1^*, \theta_2^*, \ldots, \theta_m^*) \in \Theta_1 \times \Theta_2 \times \cdots \times \Theta_m$ at which the maximum of $|\varphi(\theta_1, \theta_2, \ldots, \theta_m)|$ is achieved, where without loss, by multiplying φ by a scalar of norm $+1$, we may assume that $\varphi(\theta_1^*, \theta_2^*, \ldots, \theta_m^*) = \|\varphi\| \geq 0$. Then letting $d\mu_j(\theta_j)$ be the unit point mass at θ_j^*, we see that the integral expression (2.40) equals $\|\varphi\|$, and thus equality holds in (2.41). This establishes (2.39). With (2.38) and (2.39), it follows that the space $C(\Theta_1) \otimes C(\Theta_2) \otimes \cdots \otimes C(\Theta_m)$ is isometrically embedded as a subspace of $C(\Theta_0)$. In fact this subspace is all of $C(\Theta_0)$, that is, the first equality in (2.36) holds. This follows directly from the fact that the set of functions φ of the form (2.38) is dense in $C(\Theta_0)$, by the Stone-Weierstrass Theorem.

Suppose further that $A_j \in \mathcal{L}(C(\Theta_j))$ for $1 \leq j \leq m$. For $1 \leq k \leq m$ define an operator $\widetilde{A}_k \in \mathcal{L}(C(\Theta_0))$ by (atil)

$$\widetilde{A}_k = I \otimes \cdots I \otimes A_k \otimes I \cdots \otimes I, \tag{2.42}$$

where the factor A_k occurs in the k^{th} position. Then if $\varphi_j \in C(\Theta_j)$ for $1 \leq j \leq m$ and with φ given by (2.37), we have that

$$(\widetilde{A}_k\varphi)(\theta_1, \ldots, \theta_m) = \varphi_1(\theta_1) \ldots \varphi_{k-1}(\theta_{k-1})[(A_k\varphi_k)(\theta_k)]\varphi_{k+1}(\theta_{k+1}) \cdots \varphi_m(\theta_m)$$

for every $(\theta_1, \ldots, \theta_m) \in \Theta_0$, that is, A_k acts upon the function φ_k with the other functions φ_j for $j \neq k$ untouched. More generally, for any $\varphi \in C(\Theta_0)$ not necessarily of the product form (2.37), one has that (cd)

$$(\widetilde{A}_k\varphi)(\theta_1, \ldots, \theta_m) = [A_k\varphi(\theta_1, \ldots, \theta_{k-1}, \cdot, \theta_{k+1}, \ldots, \theta_m)](\theta_k) \tag{2.43}$$

which is interpreted as follows. Let the points $\theta_j \in \Theta_j$ for $j \neq k$ be held fixed and regard

$$\varphi(\theta_1, \ldots, \theta_{k-1}, \cdot, \theta_{k+1}, \ldots, \theta_m) \in C(\Theta_k)$$

as a function of one variable represented by the centered dot "$\cdot$". Apply the operator A_k to this function, and then evaluate the resulting function at the point $\theta_k \in \Theta_k$ to get the right-hand side of (2.43). It follows that to calculate $(A_0\varphi)(\theta_1, \theta_2, \ldots, \theta_m)$ where $A_0 \in \mathcal{L}(C(\Theta_0))$ is the operator (a0)

$$A_0 = A_1 \otimes A_2 \otimes \cdots \otimes A_m = \widetilde{A}_1\widetilde{A}_2 \cdots \widetilde{A}_m, \tag{2.44}$$

one successively applies the operators A_k for $1 \leq k \leq m$ with the variable in the k^{th} position free, while holding the remaining $m - 1$ variables fixed. Note that one may apply these operators in any order, as the operators $\widetilde{A}_k$ commute with one another.

In the special case that all the spaces $X_j = X = C(\Theta)$ are the same, then $\Theta_0 = \Theta^m$, the m-fold cartesian product, and so we have the identification $C(\Theta)^{\otimes m} = C(\Theta^m)$. Further, it is clear that $C(\Theta)^{\wedge m}$ is identified with the subspace of $C(\Theta^m)$ consisting of all anti-symmetric functions, that is, (as)

$$\begin{gathered}C(\Theta)^{\wedge m} = \{\varphi \in C(\Theta^m) \mid \varphi(\theta_{\sigma(1)}, \theta_{\sigma(2)}, \dots, \theta_{\sigma(m)}) = \operatorname{sgn}(\sigma)\varphi(\theta_1, \theta_2, \dots, \theta_m) \\ \text{for every } (\theta_1, \theta_2, \dots, \theta_m) \in \Theta^m, \text{ and every } \sigma \in \mathcal{S}_m\}.\end{gathered} \tag{2.45}$$

As a practical matter, the above observations will be useful in evaluating tensor products of solution operators of linear delay-differential equations. In such applications we shall typically work with the exterior product space $C([-1, 0])^{\wedge m}$.

The following basic result from will be needed later. Although it is proved in [3], we provide a proof for completeness.

Proposition 2.5 (Ichinose [3, Lemma 3.6]). *Let X_j and Y_j be Banach spaces and $A_j \in \mathcal{L}(X_j, Y_j)$ for $1 \leq j \leq m$. Assume that each operator A_j is one-to-one. Then the operator $A_0 = A_1 \otimes A_2 \otimes \cdots \otimes A_m$ is one-to-one from $X_1 \otimes X_2 \otimes \cdots \otimes X_m$ to $Y_1 \otimes Y_2 \otimes \cdots \otimes Y_m$.*

Proof. Without loss it is enough to consider the case $m = 2$, as the case of general m can be proved inductively by writing $A_0 = A_* \otimes A_m$ where $A_* = A_1 \otimes A_2 \otimes \cdots \otimes A_{m-1}$. Further, if $m = 2$, then by writing $A_1 \otimes A_2 = (A_1 \otimes I_{Y_2})(I_{X_1} \otimes A_2)$ where I_{X_1} and I_{Y_2} denote the identity operators on X_1 and Y_2 respectively, we see that it is enough to prove that both $A_1 \otimes I_{Y_2}$ and $I_{X_1} \otimes A_2$ are one-to-one. In fact, it is enough to prove that the operator $A_1 \otimes Y_2$ is one-to-one.

Therefore, denoting $A = A_1$, $X = X_1$, $Y = Y_1$, and $Z = Y_2$, let us consider an operator $A \in \mathcal{L}(X, Y)$ which is one-to-one. We must prove that $A \otimes I \in \mathcal{L}(X \otimes Z, Y \otimes Z)$ is also one-to-one, where I denotes the identity operator on Z. Letting Z^* denote the dual space of Z, for any $\zeta \in Z^*$ define an operator $L_X(\zeta) \in \mathcal{L}(X \otimes Z, X)$ by setting

$$L_X(\zeta)(x \otimes z) = \zeta(z)x$$

for any $x \in X$ and $z \in Z$, and then extending $L_X(\zeta)$ to all of $X \otimes Z$ first by linearity and then by continuity. One easily sees that $L_X(\zeta)$ is well-defined, with operator norm

$$\|L_X(\zeta)\| = \|\zeta\|.$$

One also easily checks that (yy)

$$\|u\| = \sup_{\substack{\zeta \in Z^* \\ \|\zeta\|=1}} \|L_X(\zeta)u\| \tag{2.46}$$

for every $u \in X \otimes Z$, which in fact follows directly from the definition (2.2) of the injective norm. We also define the operator $L_Y(\zeta) \in \mathcal{L}(Y \otimes Z, Y)$ in an analogous fashion. Finally, let us note that (x)

$$L_Y(\zeta)(A \otimes I) = AL_X(\zeta) \tag{2.47}$$

for every $\zeta \in Z$, which one easily sees by showing that the operators in (2.47) agree on all elements $x \otimes z \in X \otimes Z$.

Now assume that $(A \times I)u = 0$ for some $u \in X \otimes Z$. Then from (2.47) we have for every $\zeta \in Z^*$ that $AL_X(\zeta)u = 0$, and hence that $L_X(\zeta)u = 0$ as A is one-to-one. But then (2.46) implies that $\|u\| = 0$, thus $u = 0$. We conclude that $A \otimes I$ is one-to-one, as desired. ■

3 Tensor Products of Linear Processes

Before specializing to the delay-differential equation (1.1), we begin with some general observations about abstract linear processes. These observations not only apply to (1.1), but also to a large class of linear nonautonomous delay-differential equations as well as to many other systems.

By a **linear process** (sometimes called a **linear evolutionary process**) $U(t,\tau)$ on a Banach space X, we mean a family of bounded linear operators $U(t,\tau) \in \mathcal{L}(X)$, for every $t, \tau \in \mathbf{R}$ with $t \geq \tau$, for which

(1) $U(\tau,\tau) = I$ for every $\tau \in \mathbf{R}$;

(2) $U(t,\sigma)U(\sigma,\tau) = U(t,\tau)$ for every $t, \sigma, \tau \in \mathbf{R}$ with $t \geq \sigma \geq \tau$; and

(3) $U(t,\tau)$ is strongly continuous in t and τ, that is, for every $x \in X$ it is the case that $U(t,\tau)x$ varies continuously in X as a function of t and τ, for $t \geq \tau$.

It is easy to check, using the uniform boundedness principle, that there is a bound $\|U(t,\tau)\| \leq K$ in the neighborhood of any point (t_0, τ_0) in the domain of $U(\cdot,\cdot)$, where such K depends on (t_0, τ_0).

Now fix an integer $m \geq 1$ and consider the m-fold tensor product $X^{\otimes m}$. For every k satisfying $1 \leq k \leq m$, and with t and τ as before, we may define an operator $U_k(t,\tau) \in \mathcal{L}(X^{\otimes m})$ by (uk)

$$U_k(t,\tau) = I \otimes \cdots \otimes I \otimes U(t,\tau) \otimes I \otimes \cdots \otimes I, \tag{3.1}$$

where the factor $U(t,\tau)$ occurs in the k^{th} place. It is easily checked that $U_k(t,\tau)$ is a linear process on $X^{\otimes n}$. Also, one has from (2.4) that (08)

$$U_k(t,\tau)U_j(t',\tau') = U_j(t',\tau')U_k(t,\tau) \tag{3.2}$$

for any real numbers $t \geq \tau$ and $t' \geq \tau'$, with $j \neq k$ in the range $1 \leq j, k \leq m$. Next define the operator $\mathbf{U}(t,\tau) \in \mathcal{L}(X^{\otimes m})$ for $t \geq \tau$ by (bigu)

$$\mathbf{U}(t,\tau) = U(t,\tau)^{\otimes m} = U_1(t,\tau)U_2(t,\tau)\cdots U_m(t,\tau), \tag{3.3}$$

where it does not matter in what order the above product is taken due to the commutativity (3.2). Again, $\mathbf{U}(t,\tau)$ is a linear process on $X^{\otimes m}$. It is also clear that the subspace $X^{\wedge m} \subseteq X^{\otimes m}$ is invariant under $\mathbf{U}(t,\tau)$, and we shall denote by (bigw)

$$\mathbf{W}(t,\tau) = U(t,\tau)^{\wedge m} = \mathbf{U}(t,\tau)|_{X^{\wedge m}} \tag{3.4}$$

the restriction of this linear process to $X^{\wedge m}$. Certainly, $\mathbf{W}(t,\tau) \in \mathcal{L}(X^{\wedge m})$ is itself a linear process on the space $X^{\wedge m}$.

It often happens that a linear process $U(t,\tau)$ is periodic, meaning that there exists some $\gamma > 0$ such that

$$U(t+\gamma, \tau+\gamma) = U(t,\tau)$$

for every t and τ with $t \geq \tau$. In this case, for each $\tau \in \mathbf{R}$ we define

$$M(\tau) = U(\tau+\gamma, \tau),$$

the so-called **monodromy operator** with initial time τ, and we note that $M(\tau+\gamma) = M(\tau)$. We refer to the nonzero spectrum of $M(\tau)$ as the **Floquet spectrum** of the linear process, and we call the set of nonzero elements in the point spectrum of $M(\tau)$ the set of **Floquet multipliers**. Let us observe that the Floquet spectrum does not depend on the initial time τ, that is, $\operatorname{spec}(M(\tau)) \setminus \{0\} = \operatorname{spec}(M(\tau')) \setminus \{0\}$ for every $\tau, \tau' \in \mathbf{R}$. To prove this, without loss we may take $\tau' = 0$ and let τ lie in the range $0 < \tau < \gamma$. With τ so fixed, we have that (decomp)

$$M(0) = AB, \qquad M(\tau) = BA, \qquad \text{where} \qquad A = U(\gamma,\tau), \qquad B = U(\tau,0), \tag{3.5}$$

and we must show that $\operatorname{spec}(AB) \setminus \{0\} = \operatorname{spec}(BA) \setminus \{0\}$. In fact, this is a well-known result for any pair of operators, whose proof we sketch. Taking any $\lambda \in \mathbf{C} \setminus \{0\}$ satisfying $\lambda \notin \operatorname{spec}(AB)$, one easily checks by multiplication that the operator $\lambda^{-1}I + \lambda^{-1}B(\lambda I - AB)^{-1}A$ is the inverse of $\lambda I - BA$, and so $\lambda \notin \operatorname{spec}(BA)$. Thus $\operatorname{spec}(AB) \setminus \{0\} \supseteq \operatorname{spec}(BA) \setminus \{0\}$, with the opposite inclusion proved similarly.

It is also the case that the nonzero point spectra of AB and BA are the same, and so the set of Floquet multipliers is independent of the initial time. Indeed, if $\lambda \neq 0$ is in the point spectrum of $M(0)$ then $ABx = \lambda x \neq 0$ for some $x \in X$. Letting $y = Bx \neq 0$, we thus have that $M(\tau)y = BAy = BABx = \lambda Bx = \lambda y \neq 0$, and so λ is in the point spectrum of $M(\tau)$.

It is easily seen from (3.5) that if for some $\tau' \in \mathbf{R}$ and some $n \geq 1$ the operator $M(\tau')^n$ is compact, then for every $\tau \in \mathbf{R}$ the operator $M(\tau)^{n+1}$ is compact. In this case the remarks above imply that the Floquet spectrum consists entirely of Floquet multipliers. This will indeed be the case in our studies of delay-differential equations below.

Remark. The above observations in connection with compound (exterior product) systems have great relevance for the stability of periodic solutions of nonlinear systems. For example, in the case of an autonomous ODE, say (nonl)

$$\dot{x} = f(x), \qquad x \in \mathbf{R}^n, \tag{3.6}$$

suppose that $x = \xi(t)$ is a nonconstant periodic solution of minimal period $\gamma > 0$. Consider the associated linearized system $\dot{y} = A(t)y$ where $A(t) = f'(\xi(t))$, with $U(t,\tau)$ the associated fundamental solution with $U(\tau,\tau) = I$. Let the Floquet multipliers (the characteristic multipliers) be ordered so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ with repetitions according to algebraic multiplicities and recall that $\lambda_k = 1$ for some k, the so-called trivial multiplier. Then the periodic solution $\xi(t)$ is exponentially asymptotically stable for the nonlinear system (3.6) if and only if (exst)

$$\lambda_1 = 1 > |\lambda_2|. \tag{3.7}$$

Further, consider the compound linear process $\mathbf{W}(t,\tau) = U(t,\tau)^{\wedge 2} = U(t,\tau) \wedge U(t,\tau)$, which acts on the space $\mathbf{R}^n \wedge \mathbf{R}^n$ of dimension $\binom{n}{2} = \frac{1}{2}n(n-1)$ and whose Floquet multipliers are precisely the quantities $\mu = \lambda_i\lambda_j$ for $1 \leq i < j \leq n$. Then $|\mu| < 1$ for every such μ if and only if (3.7) holds, that is, if and only if $\xi(t)$ is exponentially asymptotically stable. **REFERENCES TO MULDOWNEY/WANG?**

The appropriate generalizations of this conclusion hold for a wide variety of infinite dimensional systems, including a large class of retarded functional differential equations $\dot{x}(t) = f(x_t)$, not limited to a single delay. (Here we follow the notation as in [2].)

4 Tensor Products of Delay-Differential Equations

Consider the linear scalar delay-differential equation (1.1) which we write here (0)

$$\dot{y}(t) = -\alpha(t)y(t) - \beta(t)y(t-1) \tag{4.1}$$

using the variable y. The change of variables (cov)

$$x = \mu(t)y, \qquad \mu(t) = \exp\bigg(\int_0^t \alpha(s)\,ds\bigg), \tag{4.2}$$

transforms (4.1) into the equivalent equation (1)

$$\dot{x}(t) = -b(t)x(t-1), \tag{4.3}$$

where

$$b(t) = \frac{\mu(t)\beta(t)}{\mu(t-1)} = \beta(t)\exp\bigg(\int_{t-1}^t \alpha(s)\,ds\bigg).$$

Note that $\beta(t)$ and $b(t)$ have the same sign for every t.

As a standing hypothesis, throughout this section **AND THE NEXT (WHERE SOME OF THE RESULTS OF THE PRESENT SECTION ARE PROVED)** we shall always assume that $\alpha, \beta : \mathbf{R} \to \mathbf{R}$ and $b : \mathbf{R} \to \mathbf{R}$ are locally integrable functions. (Actually, it will be enough only to assume these properties on the interval $[\tau, \tau + \eta]$ considered in our results.)

Generally, we shall work with the simpler equation (4.3) and interpret our results back for equation (4.1). Both equations (4.1) and (4.3) generate linear processes, which we denote by $\widehat{U}(t, \tau)$ and $U(t, \tau)$, respectively, on the Banach space

$$X = C([-1, 0]),$$

where we keep this notation for the remainder of the paper. In particular, $U(t, \tau) \in \mathcal{L}(X)$ for any $t, \tau \in \mathbf{R}$ with $t \geq \tau$ denotes the associated solution operator on X to equation (4.3) defined as

$$U(t, \tau)\varphi = x_t.$$

Here $x(t)$ satisfies (4.3) for $t \geq \tau$, with $x_t \in X$ defined in the usual fashion [2] by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-1, 0]$, and where we take the initial condition $x_\tau = \varphi \in X$, that is, $x(\tau + \theta) = \varphi(\theta)$ for $\theta \in [-1, 0]$. Similarly, $\widehat{U}(t, \tau)$ denotes the analogous solution operator for equation (4.1), and one sees the relation (conj)

$$\widehat{U}(t,\tau) = \Sigma(t)^{-1} U(t,\tau) \Sigma(\tau) \tag{4.4}$$

between these two processes, where $\Sigma(t) \in \mathcal{L}(X)$ is defined to be the multiplication operator

$$[\Sigma(t)\varphi](\theta) = \mu(t+\theta)\varphi(\theta), \qquad \theta \in [-1,0],$$

for any $t \in \mathbf{R}$. Writing $t = \tau + \eta$, we see that if $0 \le \eta \le 1$ then we have the explicit formula (m1)

$$[U(\tau+\eta,\tau)\varphi](\theta) = \begin{cases} \varphi(\eta+\theta), & \text{for } -1 \le \theta \le -\eta, \\ \\ \varphi(0) - \displaystyle\int_0^{\eta+\theta} b(\tau+s)\varphi(s-1)\,ds, & \text{for } -\eta \le \theta \le 0. \end{cases} \tag{4.5}$$

Now fix an integer $m \ge 1$ and recall the identification

$$X^{\otimes m} = C([-1,0]^m)$$

of the m-fold tensor product as described above in Section 2. Generally, we shall denote the argument of a function in $X^{\otimes m}$ by $\theta = (\theta_1, \theta_2, \ldots, \theta_m) \in [-1,0]^m$. Also recall the operators $U_k(t,\tau)$ in (3.1), which for $1 \le k \le m$ are linear processes on $X^{\otimes m}$. From remarks at the end of Section 2, we have that $U_k(t,\tau)$ is simply the solution operator to equation (4.3) taken along the k^{th} coordinate in $[-1,0]^m$, with the remaining $m-1$ coordinates staying fixed. To see this more concretely, fix k and for any $\theta = (\theta_1, \theta_2, \ldots, \theta_m) \in [-1,0]^m$ let

$$\widetilde{\theta} = (\theta_1, \ldots, \theta_{k-1}, \theta_{k+1}, \ldots, \theta_m) \in [-1,0]^{m-1},$$

which is θ with the k^{th} coordinate removed. Then regarding $\widetilde{\theta}$ as a fixed parameter, consider $\varphi(\theta)$ as a function of θ_k alone and take this function as the initial condition for equation (4.3) at initial time τ. Denoting the resulting solution by $x(t,\widetilde{\theta})$ for $t \ge \tau$, we have that

$$[U_k(t,\tau)\varphi](\theta) = x(t+\theta_k, \widetilde{\theta}).$$

If $0 \le \eta \le 1$ then we also have the explicit formula (11)

$$[U_k(\tau+\eta,\tau)\varphi](\theta) = \begin{cases} \varphi(\eta+\theta_k, \widetilde{\theta}), & \text{for } -1 \le \theta_k \le -\eta, \\ \\ \varphi(0,\widetilde{\theta}) - \displaystyle\int_0^{\eta+\theta_k} b(\tau+s)\varphi(s-1,\widetilde{\theta})\,ds, & \text{for } -\eta \le \theta_k \le 0, \end{cases} \tag{4.6}$$

following (4.5), where we slightly abuse notation by writing $\varphi(\theta_k, \widetilde{\theta})$ for $\varphi(\theta)$.

Recall also the operator $\mathbf{U}(t,\tau) \in \mathcal{L}(X^{\otimes m})$ as in (3.3), and its restriction $\mathbf{W}(t,\tau) \in \mathcal{L}(X^{\wedge m})$ to the invariant subspace $X^{\wedge m} \subseteq X^{\otimes m}$ as in (3.4), which give linear processes in their respective spaces. Concerning the space $X^{\wedge m}$, note that if $\varphi \in X^{\wedge m}$ then the anti-symmetry property (2.45) implies that the values of $\varphi(\theta)$ for $\theta \in [-1,0]^m$ are completely determined by the values for which $\theta \in T_m$, where (tri)

$$T_m = \{\theta = (\theta_1, \theta_2, \ldots, \theta_m) \in [-1,0]^m \mid \theta_1 \le \theta_2 \le \cdots \le \theta_m\}. \tag{4.7}$$

Our main interest will be positivity properties of the operator $\mathbf{W}(t,\tau)$ on $X^{\wedge m}$, and to this end we define the set (km)

$$K_m = \{\varphi \in X^{\wedge m} \mid \varphi(\theta) \ge 0 \text{ for every } \theta \in T_m\}, \tag{4.8}$$

which is a closed, convex cone in the space $X^{\wedge m}$. Indeed, K_m is a so-called **reproducing cone** for $X^{\wedge m}$, meaning that every element of $X^{\wedge m}$ can be written as the difference of two element of K_m.

With respect to the cone K_m and to equation (4.3), we shall prove the following theorem, which is one of our main results.

Theorem 4.1 (Positivity Theorem). *Fix* $m \ge 1$, *and let* $\tau \in \mathbf{R}$ *and* $\eta \ge 0$. *Assume that* $(-1)^m b(t) \ge 0$ *for almost every* $t \in [\tau, \tau+\eta]$ *for the coefficient function in equation* (4.3). *Then the operator* $\mathbf{W}(\tau+\eta,\tau) \in \mathcal{L}(X^{\wedge m})$ *defined in* (3.4) *is a positive operator with respect to the cone* K_m *in* (4.8), *that is,* $\mathbf{W}(\tau+\eta,\tau)$ *maps* K_m *into itself.*

The corresponding result for equation (4.1) *holds, where one assumes that* $(-1)^m\beta(t) \ge 0$ *for almost every* $t \in [\tau, \tau+\eta]$.

The above Positivity Theorem for equation (4.3) is a straightforward consequence of the following result, Proposition 4.2, which provides more detailed information. The Positivity Theorem for equation (4.1) then follows using the conjugacy (4.4) and the fact that $\Sigma(t)^{\wedge m}$ and its inverse $[\Sigma(t)^{\wedge m}]^{-1}$ are positive operators on $X^{\wedge m}$ with respect to the cone K_m. **MENTION RESULTS ARE NEW EVEN FOR CONSTANT COEFFICIENTS.**

In the following result and below, we denote (brho)

$$b^{\rho}(s) = b(\rho+s) \tag{4.9}$$

for ease of notation. The superscript notation here is formally distinguished from the subscript notation $x_t(\theta) = x(t+\theta)$ used earlier wherein the argument θ was restricted to the interval $\theta \in [-1,0]$ and x_t

was regarded as an element of $C([-1,0])$. No restriction is imposed upon the argument s of $b^{\rho}(s)$, and $b^{\rho}(\cdot)$ is not viewed as an element of any particular space, but merely as a shorthand notation.

Proposition 4.2. *Fix $m \geq 1$, and let $\tau \in \mathbf{R}$ and $0 < \eta \leq 1$. Fix any $\theta = (\theta_1, \theta_2, \ldots, \theta_m) \in T_m$ where T_m is as in (4.7), and let a be any integer satisfying* (thet)

$$\theta_a \leq -\eta \leq \theta_{a+1}, \tag{4.10}$$

where $a = 0$ is allowed in case $-\eta \leq \theta_1$, and $a = m$ is allowed in case $\theta_m \leq -\eta$. Then for every $\varphi \in X^{\wedge m}$ we have that (uform)

$$\begin{aligned}[\mathbf{W}(\tau+\eta,\tau)\varphi](\theta) = {} & (-1)^{am} \int_{\eta+\theta_{a+1}-1}^{\eta+\theta_{a+2}-1} \cdots \int_{\eta+\theta_{m-1}-1}^{\eta+\theta_m-1} b^{\tau+1}(t_1) \cdots b^{\tau+1}(t_{m-a-1}) \\ & \quad \times \ \varphi(t_1, \ldots, t_{m-a-1}, \eta+\theta_1, \ldots, \eta+\theta_a, 0)\, dt_{m-a-1} \cdots dt_1 \\ & + (-1)^{(a+1)m} \int_{-1}^{\eta+\theta_{a+1}-1} \int_{\eta+\theta_{a+1}-1}^{\eta+\theta_{a+2}-1} \cdots \int_{\eta+\theta_{m-1}-1}^{\eta+\theta_m-1} b^{\tau+1}(t_1) \cdots b^{\tau+1}(t_{m-a}) \\ & \quad \times \ \varphi(t_1, \ldots, t_{m-a}, \eta+\theta_1, \ldots, \eta+\theta_a)\, dt_{m-a} \cdots dt_1,\end{aligned} \tag{4.11}$$

if $0 \leq a \leq m-2$, where the terms $\eta + \theta_j$ in the arguments of φ are absent if $a = 0$. If $a = m-1$ then (uformx)

$$\begin{aligned}[\mathbf{W}(\tau+\eta,\tau)\varphi](\theta) = {} & \varphi(\eta+\theta_1, \ldots, \eta+\theta_{m-1}, 0) \\ & + (-1)^m \int_{-1}^{\eta+\theta_m-1} b^{\tau+1}(t_1)\varphi(t_1, \eta+\theta_1, \ldots, \eta+\theta_{m-1})\, dt_1,\end{aligned} \tag{4.12}$$

while (uformz)

$$[\mathbf{W}(\tau+\eta,\tau)\varphi](\theta) = \varphi(\eta+\theta_1, \ldots, \eta+\theta_m) \tag{4.13}$$

if $a = m$.

Remark. The integer a in the statement of Proposition 4.2 need not be unique; indeed, this is the case if $\theta_k = -\eta$ for some k, where either $a = k-1$ or $a = k$ could be taken. Indeed, if a is not unique then any value permitted by the statement of the proposition may be taken. In any case, $0 \leq a \leq m$ must hold.

Before proving Theorem 4.1 and Proposition 4.2, we believe it is instructive to verify them in the simplest nontrivial case of $m = 2$ and $\eta = 1$. First, by equation (4.6) we have that

$$[U_1(\tau+1,\tau)\varphi](\theta) \,=\, \varphi(0,\theta_2) - \int_0^{1+\theta_1} b(\tau+s)\varphi(s-1,\theta_2)\,ds,$$

$$[U_2(\tau+1,\tau)\varphi](\theta) \,=\, \varphi(\theta_1,0) - \int_0^{1+\theta_2} b(\tau+s)\varphi(\theta_1,s-1)\,ds,$$

for every $\varphi \in X^{\otimes 2} = C([-1,0]^2)$, where $\theta = (\theta_1,\theta_2) \in [-1,0]^2$. We next compose these two formulas as in (3.3), substituting the second into the first. Denoting $\psi(\theta) = [U_2(\tau+1,\tau)\varphi](\theta)$, we have that (u2)

$$\begin{aligned}
[\mathbf{U}(\tau+1,\tau)\varphi](\theta) \,&=\, \psi(0,\theta_2) - \int_0^{1+\theta_1} b(\tau+s)\psi(s-1,\theta_2)\,ds \\
&= \varphi(0,0) - \int_0^{1+\theta_2} b(\tau+s)\varphi(0,s-1)\,ds - \int_0^{1+\theta_1} b(\tau+s)\psi(s-1,\theta_2)\,ds \\
&= \varphi(0,0) - \int_0^{1+\theta_2} b(\tau+s)\varphi(0,s-1)\,ds \\
&\quad - \int_0^{1+\theta_1} b(\tau+s)\Big(\varphi(s-1,0) - \int_0^{1+\theta_2} b(\tau+r)\varphi(s-1,r-1)\,dr\Big)\,ds \\
&= \varphi(0,0) - \int_0^{1+\theta_2} b(\tau+s)\varphi(0,s-1)\,ds - \int_0^{1+\theta_1} b(\tau+s)\varphi(s-1,0)\,ds \\
&\quad + \int_0^{1+\theta_1}\int_0^{1+\theta_2} b(\tau+s)b(\tau+r)\varphi(s-1,r-1)\,dr\,ds.
\end{aligned} \tag{4.14}$$

The formula occurring after the final equal sign in (4.14) simplifies in the anti-symmetric case $\varphi \in X^{\wedge 2}$, that is, where $\varphi(\theta_1,\theta_2) \equiv -\varphi(\theta_2,\theta_1)$ holds identically. For such φ we have that $\varphi(0,0) = 0$. Moreover, we have that

$$-\int_0^{1+\theta_2} b(\tau+s)\varphi(0,s-1)\,ds - \int_0^{1+\theta_1} b(\tau+s)\varphi(s-1,0)\,ds = \int_{1+\theta_1}^{1+\theta_2} b(\tau+s)\varphi(s-1,0)\,ds,$$

and also that

$$\begin{aligned}\int_0^{1+\theta_1}\int_0^{1+\theta_2} & b(\tau+s)b(\tau+r)\varphi(s-1,r-1)\,dr\,ds\\ &= \int_0^{1+\theta_1}\int_0^{1+\theta_1} b(\tau+s)b(\tau+r)\varphi(s-1,r-1)\,dr\,ds\\ &\quad+\int_0^{1+\theta_1}\int_{1+\theta_1}^{1+\theta_2} b(\tau+s)b(\tau+r)\varphi(s-1,r-1)\,dr\,ds\\ &= \int_0^{1+\theta_1}\int_{1+\theta_1}^{1+\theta_2} b(\tau+s)b(\tau+r)\varphi(s-1,r-1)\,dr\,ds,\end{aligned}$$

where the integral taken over the square $[0,1+\theta_1]^2$ vanishes on account of the anti-symmetry. With this we obtain from (4.14) that (uu2)

$$\begin{aligned}[\mathbf{W}(\tau+1,\tau)\varphi](\theta) = &\int_{1+\theta_1}^{1+\theta_2} b(\tau+s)\varphi(s-1,0)\,ds\\ &+\int_0^{1+\theta_1}\int_{1+\theta_1}^{1+\theta_2} b(\tau+s)b(\tau+r)\varphi(s-1,r-1)\,dr\,ds,\end{aligned} \tag{4.15}$$

and one sees this formula coincides with (4.11), where $a = 0$ is taken. It follows directly from equation (4.15) that $\mathbf{W}(\tau+1,\tau)$ is a positive operator with respect to the cone K_2 provided that $(-1)^m b(t) = b(t) \geq 0$ in $[\tau,\tau+1]$. To see this, first let $\theta \in T_2$, that is, $-1 \leq \theta_1 \leq \theta_2 \leq 0$. Then taking $\varphi \in K_2$, one notes for the first term in (4.15) that $s-1 \leq \theta_2 \leq 0$, thus $(s-1,0) \in T_2$ and so $\varphi(s-1,0) \geq 0$. Similarly, for the second term in (4.15) we have that $s-1 \leq \theta_1 \leq r-1$, thus $(s-1,r-1) \in T_2$ and $\varphi(s-1,r-1) \geq 0$. It follows that the expression in (4.15) is nonnegative, so $\mathbf{W}(\tau+1,\tau)\varphi \in K_2$, as desired.

We end this section by describing several properties of the above linear processes. The first is a well-known compactness property of $U(\tau+\eta,\tau)$ and $\widehat{U}(\tau+\eta,\tau)$ for $\eta > 0$. Its significance is that in the case of a periodic process, some power of the monodromy operator is compact, and so the Floquet spectrum consists entirely of Floquet multipliers, and these are isolated values $\lambda \in \mathbf{C}\setminus\{0\}$ of finite multiplicity which can only cluster at $\lambda = 0$.

Proposition 4.3. *If $\eta \geq 1$ then the solution operator $U(\tau+\eta,\tau)$ for equation* (4.3) *is compact. More generally, if $\eta \geq \frac{1}{n}$ for some $n \geq 1$ then the n^{th} power $U(\tau+\eta,\tau)^n$ is compact. The same conclusions hold for the solution operator $\widehat{U}(t+\eta,\tau)$ for equation* (4.1).

Proof. For simplicity we consider only the operator $U(\tau+\eta,\tau)$. The operator $U(\tau+1,\tau)$ is easily seen from (4.5) to be compact, being the sum of the rank-one operator $\varphi(0)$ and an integral operator, and thus if $\eta \geq 1$ then the operator $U(\tau+\eta,\tau) = U(\tau+\eta-1,\tau+1)U(\tau+1,\tau)$ is also compact. Now suppose that $\eta \geq \frac{1}{n}$ for some $n \geq 1$. Define a new function $\widetilde{b}(t)$ by setting $\widetilde{b}(t) = b(t)$ for $\tau \leq t < \tau+\eta$, and extending it periodically so that $\widetilde{b}(t+\eta) = \widetilde{b}(t)$ for every $t \in \mathbf{R}$. Let $\widetilde{U}(\widetilde{\tau}+\widetilde{\eta},\widetilde{\tau})$ denote the linear process associated to equation (4.3) but with $\widetilde{b}(t)$ replacing $b(t)$, where $\widetilde{\tau} \in \mathbf{R}$ and $\widetilde{\eta} \geq 0$ are general arguments. Note that $\widetilde{U}(\tau+\eta,\tau) = U(\tau+\eta,\tau)$ for our specific τ and η, as these operators only involve the range where $\widetilde{b}(t)$ and $b(t)$ agree. Also note that $\widetilde{U}(\widetilde{\tau}+\widetilde{\eta}+\eta,\widetilde{\tau}+\eta) = \widetilde{U}(\widetilde{\tau}+\widetilde{\eta},\widetilde{\tau})$ for every $\widetilde{\tau}$ and $\widetilde{\eta}$, due to the η-periodicity of $\widetilde{b}(t)$. From this it follows that $U(\tau+\eta,\tau)^n = \widetilde{U}(\tau+\eta,\tau)^n = \widetilde{U}(\tau+n\eta,\tau)$, which is a compact operator by our earlier remarks as $n\eta \geq 1$. ■

The next result concerns one-to-oneness of the above linear processes.

Proposition 4.4. *Assume, for some $\tau \in \mathbf{R}$ and $\eta > 0$, that $b(t) \neq 0$ for almost every $t \in [\tau,\tau+\eta]$ for the coefficient in equation* (4.3)*. Let $m \geq 1$. Then the operators $\mathbf{U}(\tau+\eta,\tau) \in \mathcal{L}(X^{\otimes m})$ and thus $\mathbf{W}(\tau+\eta,\tau) \in \mathcal{L}(X^{\wedge m})$ are one-to-one. The corresponding results also hold for equation* (4.1)*, where we assume that $\beta(t) \neq 0$ for almost every $t \in [\tau,\tau+\eta]$.*

Proof. By Proposition 2.5 it is enough to show that $U(\tau+\eta,\tau) \in \mathcal{L}(X)$ is one-to-one. Also, we may assume without loss that $0 < \eta \leq 1$ due to the fact that $U(t,\tau)$ is a linear process. Assume for such η that $U(\tau+\eta,\tau)\varphi = 0$ for some $\varphi \in X$. Then from (4.5) we have that $\varphi(\theta) = 0$ for every $\theta \in [\eta-1,0]$ and that

$$\int_0^\theta b(\tau+s)\varphi(s-1)\,ds = 0 \qquad \text{for every } \theta \in [0,\eta].$$

Differentiating the above integral shows that $b(\tau+s)\varphi(s-1) = 0$ for almost every $s \in [0,\eta]$, and as $b(\tau+s) \neq 0$ for almost every such s, we conclude that $\varphi(s-1) = 0$ for $s \in [0,\eta]$. Thus $\varphi(\theta) = 0$ for every $\theta \in [-1,0]$, and this gives the result. ■

5 Positivity and Floquet Theory

In this section we describe some basic consequences of the Positivity Theorem in the context of Floquet theory. If the coefficients in equation (4.1) are γ-periodic for some $\gamma > 0$, that is (gper)

$$\alpha(t+\gamma) = \alpha(t), \qquad \beta(t+\gamma) = \beta(t), \tag{5.1}$$

for almost every t, recall the monodromy operator $\widehat{M}(0) = \widehat{U}(\gamma, 0)$ where $\widehat{U}(t, \tau)$ denotes the linear process on $X = C([-1, 0])$ associated to equation (4.1). As some power of $\widehat{M}(0)$ is compact, the Floquet spectrum $\operatorname{spec}(\widehat{M}(0)) \setminus \{0\}$ consists entirely of point spectrum (Floquet multipliers), of which there are at most countably many, and each of finite multiplicity. Our main result of this section, Theorem 5.1 below, provides additional structure to the the Floquet multipliers and their associated eigenfunctions in the case that the feedback coefficient $\beta(t)$ is of constant sign. These results supplement results of Sell and one of the authors [5] which provided partial information on the multipliers.

Theorem 5.1. *Consider equation* (4.1) *where* $\alpha : \mathbf{R} \to \mathbf{R}$ *and* $\beta : \mathbf{R} \to \mathbf{R}$ *are locally integrable and* γ*-periodic for some* $\gamma > 0$, *and so satisfy* (5.1) *for almost every* t. *Also assume that*

$$(-1)^m \beta(t) \geq (-1)^m \beta_0 > 0$$

for almost every t, *for some integer* m *and some* $\beta_0 \neq 0$. *Then there are countably infinitely many Floquet multipliers* $\{\lambda_k\}_{k=1}^{\infty}$, *that is, spectra of the monodromy operator. Further, if the multipliers are labelled so that* (mults)

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots, \tag{5.2}$$

with repetitions according to algebraic multiplicity, then it is the case that the strict inequality

$$|\lambda_k| > |\lambda_{k+1}|$$

holds whenever $k - m$ *is even.*

We shall prove Theorem 5.1 below, although we defer the proof of the Positivity Theorem (Theorem 4.1) and also Proposition 4.2, on which the proof of Theorem 5.1 relies, to the next section. Before presenting the proof of Theorem 5.1, we need to develop several concepts.

Suppose Y is a Banach space and $K \subseteq Y$ is a closed, convex cone, that is, K is closed, convex set such that if $u \in K$ then $\sigma u \in K$ for every $\sigma \geq 0$, and such that if both $u, -u \in K$ then $u = 0$. We say the cone K is **total** if the set $\{u - v \mid u, v \in K\}$ is dense in Y. Also, if $A\mathcal{L}(Y)$ then we say the operator A is **positive** if it maps K into K, that is, $Au \in K$ whenever $u \in K$, and we write $A \geq 0$. The following basic result will be needed. **PUT DISCUSSION OF CONES EARLIER IN PAPER?**

Proposition 5.2. *Let Y be a Banach space and $K \subseteq Y$ a total cone. Suppose $A, B \in \mathcal{L}(Y)$ satisfy $B \geq A \geq 0$, that is, $A \geq 0$ and $B - A \geq 0$. Then*

$$r(A) \leq r(B)$$

for the spectral radii of these operators.

Proof of Theorem 5.1. Without loss we prove the result only for the simpler equation (4.3), where we assume that $b : \mathbf{R} \to \mathbf{R}$ is locally integrable, γ-periodic, and enjoys the bound

$$(-1)^m b(t) \geq (-1)^m b_0 > 0$$

for some $b_0 \neq 0$. The proof for the full equation (4.1) follows straightforwardly via the change of variables (4.2) as described in the previous section.

Let $U(t, \tau)$ denote the linear process on $X = C([-1, 0])$ associated to equation (4.3), and let $U_0(t)$ denote the semiflow on X associated to the autonomous linear equation (cc)

$$\dot{x}(t) = -b_0 x(t-1). \tag{5.3}$$

We let $M = M(0) = U(\gamma, 0)$ denote the monodromy operator associated to equation (4.3). We denote the characteristic multipliers of this equation, that is, the nonzero spectra of M, by $\{\lambda_k\}_{k=1}^{\infty}$ ordered so that (5.2) holds. In case there are only finitely many such multipliers, say k_0 of them, we write $\lambda_k = 0$ for $k > k_0$.

We observe directly from Proposition 4.2, and in particular the formulas (4.11), (4.12), and (4.13), that $U(t, \tau)^{\wedge m} \geq U_0(t - \tau)^{\wedge m}$ whenever $t \geq \tau$, and thus

$$M^{\wedge m} \geq U_0(\gamma)^{\wedge m},$$

where the relation $\geq$ is taken with respect to the cone K_m given by (4.8). **NEED TO EXPLAIN THIS. MAY OVERLAP WITH PROOF OF POSITIVITY THEOREM 4.1 IN NEXT SECTION.** As the cone K_m is total (in fact, reproducing), it follows from Proposition 5.2 that

$$r(M^{\wedge m}) \geq r(U_0(\gamma)^{\wedge m}).$$

The nonzero spectrum of $U_0(t)$ consists of the points $e^{\zeta t}$ where ζ satisfies the characteristic equation $\zeta = -b_0 e^{-\zeta}$ of equation (5.3). Let the roots of the characteristic equation (of which there are countably many, with real parts bounded above) be ordered so that

$$\operatorname{Re} \zeta_1 \geq \operatorname{Re} \zeta_2 \geq \operatorname{Re} \zeta_3 \geq \cdots,$$

with repetitions according to multiplicity. Denote $\lambda_{0,k} = e^{\zeta_k \gamma}$, and so $|\lambda_{0,k}| = e^{\operatorname{Re}\zeta_k \gamma}$. Then

$$r(M^{\wedge m}) = |\lambda_1 \lambda_2 \cdots \lambda_m|, \qquad r(U_0(\gamma)) = |\lambda_{0,1}\lambda_{0,2} \cdots \lambda_{0,m}|,$$

and so (linq)

$$|\lambda_1 \lambda_2 \cdots \lambda_m| \geq |\lambda_{0,1}\lambda_{0,2} \cdots \lambda_{0,m}| > 0. \tag{5.4}$$

In particular, $\lambda_m \neq 0$ and so equation (4.3) possesses at least m characteristic multipliers. But m can be replaced with any integer of the same parity, and thus can be taken arbitrarily large. It follows that equation (4.3) possesses countably infintely many characteristic multipliers, as claimed. ■

Remark. An examination of the above proof shows that a computable lower bound (albeit probably not a sharp bound) can be found for the characteristic multipliers. We have $|\lambda_k| \leq r(M)$ for each k, and with (5.4) this gives

$$|\lambda_m| \geq r(M)^{-(m-1)} |\lambda_{0,1}\lambda_{0,2} \cdots \lambda_{0,m}|,$$

at least if m has the parity for which $(-1)^m b(t) > 0$. For indices of the opposite parity, say for $m-1$, the inequality $|\lambda_{m-1}| \geq |\lambda_m|$ provides the requisite bound.

STILL NEED TO PROVE GAP BETWEEN MULTIPLIERS.

6 The Proof of the Positivity Theorem

Here we prove Proposition 4.2 for general m, and from it we will obtain Theorem 4.1, the Positivity Theorem, for general m. Our approach follows that of the special case with $m = 2$ and $\eta = 1$ above. Namely, with $\eta \leq 1$ in Proposition 4.2, we have an explicit expression (4.6) for each $U_k(\tau + \eta, \tau)$, and composing these expressions will provide an explicit formula for $\mathbf{U}(\tau + \eta, \tau)\varphi$, for any $\varphi \in X^{\otimes m}$. Then, assuming that $\varphi \in X^{\wedge m}$, namely that φ is anti-symmetric, we will observe significant cancellations in this formula, and this will yield a much simpler formula for $\mathbf{W}(\tau + \eta, \tau)\varphi$.

Proof of Proposition 4.2. Fix m, τ, η, and $b(t)$, as in the statement of the proposition. Define operators $Z_k, B_k \in \mathcal{L}(X^{\otimes m})$ for $1 \leq k \leq m$ by

$$\begin{aligned} (Z_k\psi)(\theta) &= \psi(\theta_1, \ldots, \theta_{k-1}, \omega(\theta_k), \theta_{k+1}, \ldots, \theta_m), \\ (B_k\psi)(\theta) &= \int_{-1}^{\Omega(\theta_k)} b^{\tau+1}(s)\psi(\theta_1, \ldots, \theta_{k-1}, s, \theta_{k+1}, \ldots, \theta_m)\, ds, \end{aligned}$$

where we denote

$$\omega(s) = \min\{\eta + s, 0\}, \qquad \Omega(s) = \max\{\eta + s, 0\} - 1,$$

and we recall the notation (4.9). Then from (4.6) one sees that

$$U_k(\tau + \eta, \tau) = Z_k - B_k,$$

and so (bigu2)

$$\mathbf{U}(\tau + \eta, \tau) = (Z_1 - B_1)(Z_2 - B_2) \cdots (Z_m - B_m) \tag{6.1}$$

by equation (3.3). Next observe the commutativity properties (com)

$$Z_j Z_k = Z_k Z_j, \qquad B_j B_k = B_k B_j, \qquad Z_j B_k = B_k Z_j, \qquad \text{provided } j \neq k. \tag{6.2}$$

Concerning symmetries, let us define the swap operators $S_{j,k} \in \mathcal{L}(X^{\otimes m})$ for $1 \leq j, k \leq m$ by (swap)

$$(S_{j,k}\psi)(\theta) = \psi(\overline{\theta}), \qquad \overline{\theta}_i = \begin{cases} \theta_k, & \text{if } i = j, \\ \theta_j, & \text{if } i = k, \\ \theta_i, & \text{if } i \neq j \text{ and } i \neq k. \end{cases} \tag{6.3}$$

Note that $S_{j,k} = S_{\sigma_{j,k}}$ in the notation (2.16), where $\sigma_{j,k} \in \mathcal{S}_m$ is the permutation satisfying $\sigma_{j,k}(j) = k$ and $\sigma_{j,k}(k) = j$, with $\sigma_{j,k}(i) = i$ if $i \neq j$ and $i \neq k$. We write $S_{j,k}$ rather than $S_{\sigma_{j,k}}$ for simplicity of notation. Each $S_{j,k}$ is an isometry on the space $X^{\otimes m}$, and of course $S_{j,k} = S_{k,j} = S_{j,k}^{-1}$. One easily checks that (inter)

$$\begin{aligned} S_{j,k} Z_k &= Z_j S_{j,k}, \qquad & S_{j,k} B_k &= B_j S_{j,k}, \\ S_{j,k} Z_i &= Z_i S_{j,k}, \qquad & S_{j,k} B_i &= B_i S_{j,k}, \end{aligned} \tag{6.4}$$

in the last two cases provided $i \neq j$ and $i \neq k$.

Let us note that $S_{j,k}\mathbf{U}(\tau + \eta, \tau) = \mathbf{U}(\tau + \eta, \tau) S_{j,k}$ by (6.2) and (6.4), and so the space $X^{\wedge m}$ of anti-symmetric functions is invariant under the operator $\mathbf{U}(\tau + \eta, \tau)$. However, also note that $X^{\wedge m}$ is not in general invariant under either of the operators Z_k or B_k.

Fix $\varphi \in X^{\wedge m}$ and let the right-hand side of (6.1) be expanded and act on φ. We obtain a sum of all terms of the form (cprod)

$$\pm C_1 C_2 \cdots C_m \varphi, \qquad C_k = Z_k \text{ or } B_k, \tag{6.5}$$

where the sign $\pm$ is $(-1)^i$, where i is the number of B_k appearing in the product (6.5). Now fix $\theta \in T_m$, along with the integer a satisfying (4.10) as in the statement of the proposition. Both θ and a will stay fixed for the remainder of this proof. Here we make two crucial observations. First, suppose that $C_k = B_k$ for some k satisfying $1 \leq k \leq a$. Note that $\Omega(\theta_k) = -1$ and thus $(B_k\psi)(\theta) = 0$ for every $\psi \in X^{\otimes m}$, for our chosen θ. (We are not claiming that $B_k\psi$ is the zero function, however, but simply that it vanishes at this particular θ.) In particular, taking $\psi = C_1 \cdots C_{k-1}\widehat{C}_k C_{k+1} \cdots C_m\varphi$ where the hat $\widehat{\ }$ indicates the term is omitted, and using the commutativity properties (6.2), we have that $C_1C_2 \cdots C_m\varphi = B_k\psi$ and thus (czer)

$$(C_1C_2 \cdots C_m\varphi)(\theta) = 0. \tag{6.6}$$

Secondly, suppose that $C_{k_1} = Z_{k_1}$ and $C_{k_2} = Z_{k_2}$ for two values k_1, k_2 satisfying $a < k_1 < k_2 \leq m$. Then $\omega(\theta_{k_1}) = \omega(\theta_{k_2}) = 0$ and so

$$(Z_{k_1}Z_{k_2}\psi)(\theta) = \psi(\theta_1, \ldots, \theta_{k_1-1}, 0, \theta_{k_1+1}, \ldots, \theta_{k_2-1}, 0, \theta_{k_2+1}, \ldots, \theta_m)$$

for every $\psi \in X^{\otimes m}$. If it is further the case that ψ satisfies (k1k2)

$$S_{k_1,k_2}\psi = -\psi \tag{6.7}$$

identically as functions, then in fact $(Z_{k_1}Z_{k_2}\psi)(\theta) = 0$. (Again, this is for our chosen θ, and there is no claim that $Z_{k_1}Z_{k_2}\psi$ is the zero function.) Taking

$$\psi = C_1 \cdots C_{k_1-1}\widehat{C}_{k_1}C_{k_1+1} \cdots C_{k_2-1}\widehat{C}_{k_2}C_{k_2+1} \cdots C_m\varphi$$

and recalling that $\varphi \in X^{\wedge m}$, we see from the anti-symmetry of φ and the properties (6.4) that (6.7) holds. Thus $C_1C_2 \cdots C_m\varphi = Z_{k_1}Z_{k_2}\psi$ and again (6.6) holds.

Following the above two crucial observation, we see that only a few select terms survive in the expansion of (6.1) when applied to $\varphi \in X^{\wedge m}$ and evaluated at our chosen θ. These are the terms as in (6.5) for which $C_k = Z_k$ for every k satisfying $1 \leq k \leq a$, and for which also $C_k = Z_k$ for at most one k in the range $a < k \leq m$. We conclude that for every $\varphi \in X^{\wedge m}$, we have that (gam)

$$[\mathbf{W}(\tau + \eta, \tau)\varphi](\theta) = (-1)^{m-a}[(\Gamma_0\varphi)(\theta)] + (-1)^{m-a+1}\sum_{k=1}^{m-a}(\Gamma_k\varphi)(\theta), \tag{6.8}$$

where (gammas)

$$\begin{aligned} \Gamma_0 &= Z_1 \cdots Z_a B_{a+1} \cdots B_m, \\ \Gamma_k &= Z_1 \cdots Z_a B_{a+1} \cdots B_{a+k-1} Z_{a+k} B_{a+k+1} \cdots B_m, \end{aligned} \tag{6.9}$$

for $1 \leq k \leq m - a$. The signs $(-1)^{m-a}$ and $(-1)^{m-a+1}$ occurring in the above formulas count the number of B_j terms.

(We remark on these formulas in the extreme cases $a = 0$ and $a = m$. If $a = 0$ then the factors $Z_1 \cdots Z_a$ in (6.9) are simply absent. If $a = m$ then the summation in (6.8) is empty, so no Γ_k are defined. Also, if $a = m$ then $\Gamma_0 = Z_1 \cdots Z_m$. We leave the verification of these facts to the reader.)

We note a peculiarity of the formula (6.8), namely that the operators Γ_0 and Γ_k depend on a, which in turn depends on θ. Thus it is not the case, in general, that the operator $\mathbf{W}(\tau + \eta, \tau)$ is the sum of the operators Γ_0 and Γ_k with the indicated signs as in (6.8). Rather, equation (6.8) is only valid pointwise for those θ which satisfy (4.10). We emphasize that for this reason, we work with a fixed $\theta \in T_m$.

Let us now evaluate the terms in (6.8). We first consider the term involving Γ_k with $1 \le k \le m-a$. As noted, the case $a = m$ is vacuous. If $a = m - 1$, then $k = 1$ and $\Gamma_1 = Z_1 \cdots Z_{m-1} Z_m$, which immediately gives (ma1)

$$(\Gamma_1 \varphi)(\theta) = \varphi(\eta + \theta_1, \ldots, \eta + \theta_{m-1}, 0), \tag{6.10}$$

as we note that $\omega(\theta_j) = \eta + \theta_j$ for $1 \le j \le m-1$ but $\omega(\theta_m) = 0$. Now let us assume that $0 \le a \le m-2$. We have, from the above formula (6.9) for Γ_k, and also from the formulas for Z_k and B_k, that (gk)

$$\begin{aligned}(\Gamma_k \varphi)(\theta) = &\int_{-1}^{\Omega(\theta_{a+1})} \cdots \int_{-1}^{\widehat{\Omega(\theta_{a+k})}} \cdots \int_{-1}^{\Omega(\theta_m)} b^{\tau+1}(s_1) \cdots \widehat{b^{\tau+1}(s_k)} \cdots b^{\tau+1}(s_{m-a}) \\ &\times \varphi(\eta + \theta_1, \ldots, \eta + \theta_a, s_1, \ldots, s_{k-1}, 0, s_{k+1}, \ldots, s_{m-a}) \, ds_{m-a} \cdots \widehat{ds_k} \cdots ds_1,\end{aligned} \tag{6.11}$$

where here again (and below) the hat $\widehat{\ }$ denotes that the indicated expression is omitted, and where we note that $\omega(\theta_j) = \eta + \theta_j$ for $1 \le j \le a$ but $\omega(\theta_{a+k}) = 0$. (If $a = 0$, then the terms $\eta + \theta_j$ in the integral (6.11) are simply absent.) Next, by permuting the arguments of φ in (6.11) and using the anti-symmetry of φ, we see that (perm)

$$\begin{aligned}&\varphi(\eta + \theta_1, \ldots, \eta + \theta_a, s_1, \ldots, s_{k-1}, 0, s_{k+1}, \ldots, s_{m-a}) \\ &\qquad = (-1)^{a+k+(a+1)m} \varphi(t_1, \ldots, t_{k-1}, t_k, \ldots, t_{m-a-1}, \eta + \theta_1, \ldots, \eta + \theta_a, 0), \\ &\text{where } t_j = \begin{cases} s_j, & \text{for } 1 \le j \le k-1, \\ s_{j+1}, & \text{for } k \le j \le m-a-1. \end{cases}\end{aligned} \tag{6.12}$$

The explanation for the term $(-1)^{a+k+(a+1)m}$, arising from the anti-symmetry of φ, is as follows. Each term s_j, for $1 \le j \le k-1$, is moved leftward a places by means of swaps with adjacent terms $\eta + \theta_i$ for $1 \le i \le a$. This is a total of $a(k-1)$ swaps of such terms. Each term s_j, for $k+1 \le j \le m-a$, is moved leftward $a+1$ places by means of swaps with adjacent terms 0 and then $\eta + \theta_i$. This is an additional $(a+1)(m-a-k)$ swaps. The total number of swaps is thus $a(k-1)+(a+1)(m-a-k)$, and one sees easily that this number has the same parity as $a+k+(a+1)m$.

Let us now introduce some notation which will simplify our calculations. We define a bounded linear operator $E \in \mathcal{L}(X^{\wedge m}, X^{\wedge(m-a-1)})$ by (q)

$$(E\psi)(t_1, \ldots, t_{m-a-1}) = \psi(t_1, \ldots, t_{m-a-1}, \eta+\theta_1, \ldots, \eta+\theta_a, 0). \tag{6.13}$$

We also define operators $I^i_j, J^i_j \in \mathcal{L}(X^{\otimes(m-a-1)})$ by (iop)

$$\begin{aligned} (I^i_j\psi)(t_1, \ldots, t_{m-a-1}) &= \int_{\Omega(\theta_{a+i-1})}^{\Omega(\theta_{a+i})} b^{\tau+1}(s)\psi(t_1, \ldots, t_{j-1}, s, t_{j+1}, \ldots, t_{m-a-1})\,ds, \\ (J^i_j\psi)(t_1, \ldots, t_{m-a-1}) &= \int_{-1}^{\Omega(\theta_{a+i})} b^{\tau+1}(s)\psi(t_1, \ldots, t_{j-1}, s, t_{j+1}, \ldots, t_{m-a-1})\,ds, \end{aligned} \tag{6.14}$$

where $1 \le i \le m-a$ and $1 \le j \le m-a-1$, and where we note that $\Omega(\theta_a) = -1$. (If $a=0$, then by convention we set $\Omega(\theta_0) = -1$.) Observe that (jdef)

$$J^i_j = I^1_j + I^2_j + \cdots + I^i_j \tag{6.15}$$

holds. With this notation, and upon inserting the formula (6.12) into (6.11), one sees that (6.11) takes the form (jform)

$$(\Gamma_k\varphi)(\theta) = (-1)^{a+k+(a+1)m} J^1_1 J^2_2 \cdots J^{k-1}_{k-1} J^{k+1}_k \cdots J^{m-a}_{m-a-1} E\varphi. \tag{6.16}$$

We remind the reader again, that θ has been fixed and does not serve as the argument of the functions $E\psi$, $I^i_j\psi$, and $J^i_j\psi$ above. Rather, these are functions of the variables $(t_1, \ldots, t_{m-a-1})$. The functions $I^i_j\psi$ and $J^i_j\psi$, in particular, are constant in the variable t_j, as the right-hand sides in (6.14) are independent of t_j. Thus the right-hand side of (6.16) is formally a function of $(t_1, \ldots, t_{m-a-1})$, and in fact is a constant function of those variables. That constant value is the value of the function $\Gamma_k\varphi$ evaluated at the point θ.

The operators I^i_j and J^i_j act on the full tensor product, and not just the wedge product. That is, no symmetry assumption is made on the argument function $\psi \in X^{\otimes(m-a-1)}$ in (6.14). Observe that (asym)

$$\begin{gathered} S_{j_1,j_2} I^i_j = I^i_j S_{j_1,j_2}, \qquad S_{j_1,j_2} J^i_j = J^i_j S_{j_1,j_2}, \\ \text{in both cases provided } j \ne j_1 \text{ and } j \ne j_2. \end{gathered} \tag{6.17}$$

Thus if it is the case that ψ is anti-symmetric in t_{j_1} and t_{j_2}, meaning that $S_{j_1,j_2}\psi = -\psi$, then $I^i_j\psi$ and $J^i_j\psi$ are also anti-symmetric in these variables as long as $j \neq j_1, j_2$.

It is easily seen that (zer2)

$$J^i_{j_1}J^i_{j_2}\psi = 0, \qquad J^i_{j_1}J^{i+1}_{j_2}\psi = J^i_{j_1}I^{i+1}_{j_2}\psi, \qquad J^i_{j_1}J^{i+2}_{j_2}\psi = J^i_{j_1}(I^{i+1}_{j_2} + I^{i+2}_{j_2})\psi, \tag{6.18}$$
$$\text{in every case provided } j_1 \neq j_2 \text{ and } S_{j_1,j_2}\psi = -\psi.$$

Indeed, the first equation in (6.18) holds as it is simply the integral over the square $[-1, \Omega(\theta_{a+i})]^2$ of an anti-symmetric function of (t_{j_1}, t_{j_2}). The second and third equations in (6.18) follow from the first equation because $J^{i+1}_j = J^i_j + I^{i+1}_j$ and $J^{i+2}_j = J^i_j + I^{i+1}_j + I^{i+2}_j$.

We now use the identities (6.18) to obtain a simplification of equation (6.16) when the function φ is anti-symmetric. We begin with the rightmost pair of J-operators in (6.16), namely $J^{m-a-1}_{m-a-2}J^{m-a}_{m-a-1}$ and move to the left. The result is that each factor J^{i+1}_i is replaced with I^{i+1}_i, and each J^i_i is replaced with I^i_i, except for the factor J^{k+1}_k which is replaced with $I^k_k + I^{k+1}_k$. At each stage we observe, using (6.17), that the relevant function is anti-symmetric in the appropriate variables, as required by (6.18). Finally noting that $J^1_1 = I^1_1$, we conclude directly that (iform)

$$(\Gamma_k\varphi)(\theta) = (-1)^{a+k+(a+1)m} I^1_1 I^2_2 \cdots I^{k-1}_{k-1}(I^k_k + I^{k+1}_k) I^{k+2}_{k+1} \cdots I^{m-a}_{m-a-1} E\varphi. \tag{6.19}$$

The reader can verify that the formula (6.19) degenerates in the extreme cases of the indices, as follows. If $m - a \geq 4$ then

$$\begin{aligned}
(\Gamma_1\varphi)(\theta) &= (-1)^{a+1+(a+1)m}(I^1_1 + I^2_1)I^3_2 \cdots I^{m-a}_{m-a-1}E\varphi,\\
(\Gamma_2\varphi)(\theta) &= (-1)^{a+2+(a+1)m}I^1_1(I^2_2 + I^3_2)I^4_3 \cdots I^{m-a}_{m-a-1}E\varphi,\\
(\Gamma_{m-a-1}\varphi)(\theta) &= (-1)^{m-1+(a+1)m}I^1_1 I^2_2 \cdots I^{m-a-2}_{m-a-2}(I^{m-a-1}_{m-a-1} + I^{m-a}_{m-a-1})E\varphi,\\
(\Gamma_{m-a}\varphi)(\theta) &= (-1)^{m+(a+1)m}I^1_1 I^2_2 \cdots I^{m-a-1}_{m-a-1}E\varphi.
\end{aligned}$$

If $m - a = 3$ then we have

$$\begin{aligned}
(\Gamma_1\varphi)(\theta) &= (-1)^{a+1+(a+1)m}(I^1_1 + I^2_1)I^3_2 E\varphi,\\
(\Gamma_2\varphi)(\theta) &= (-1)^{a+2+(a+1)m}I^1_1(I^2_2 + I^3_2)E\varphi,\\
(\Gamma_3\varphi)(\theta) &= (-1)^{a+3+(a+1)m}I^1_1 I^2_2 E\varphi,
\end{aligned}$$

while if $m-a=2$ then we have

$$(\Gamma_1\varphi)(\theta) \;=\; (-1)^{a+1+(a+1)m}(I_1^1+I_1^2)E\varphi,$$

$$(\Gamma_2\varphi)(\theta) \;=\; (-1)^{a+2+(a+1)m}I_1^1E\varphi.$$

These formulas follow easily from (6.16) using the identities (6.18). Now define the operators

$$R_1 = I_1^2I_2^3\cdots I_{m-a-1}^{m-a},$$

$$R_k = I_1^1\cdots I_{k-1}^{k-1}I_k^{k+1}\cdots I_{m-a-1}^{m-a}, \qquad 2\le k\le m-a-1,$$

$$R_{m-a} = I_1^1I_2^2\cdots I_{m-a-1}^{m-a-1},$$

$$R_{m-a+1}=0.$$

Then (6.19) can be rewritten as (rform)

$$(\Gamma_k\varphi)(\theta) = (-1)^{a+k+(a+1)m}(R_{k+1}+R_k)E\varphi, \tag{6.20}$$

and we see that the formula is valid for all values $1\le k\le m-a$ with $m-a\ge 2$, including the extreme cases above. It follows immediately that the summation in (6.8) telescopes to give (gammak)

$$\begin{aligned}(-1)^{a+1+(a+1)m}\sum_{k=1}^{m-a}(\Gamma_k\varphi)(\theta) &= R_1E\varphi\\ &= \int_{\Omega(\theta_{a+1})}^{\Omega(\theta_{a+2})}\cdots\int_{\Omega(\theta_{m-1})}^{\Omega(\theta_m)} b^{\tau+1}(t_1)\cdots b^{\tau+1}(t_{m-a-1})[(E\varphi)(t_1,\ldots,t_{m-a-1})]\,dt_{m-a-1}\cdots dt_1.\end{aligned} \tag{6.21}$$

Upon multiplying the above formula by $(-1)^{am}$ and noting that $(-1)^{am}(-1)^{a+1+(a+1)m}=(-1)^{m-a+1}$, we obtain (x1)

$$\begin{aligned}(-1)^{m-a+1}\sum_{k=1}^{m-a}(\Gamma_k\varphi)(\theta) \;&=\; (-1)^{am}\int_{\eta+\theta_{a+1}-1}^{\eta+\theta_{a+2}-1}\cdots\int_{\eta+\theta_{m-1}-1}^{\eta+\theta_m-1} b^{\tau+1}(t_1)\cdots b^{\tau+1}(t_{m-a-1})\\ &\qquad\times\;\varphi(t_1,\ldots,t_{m-a-1},\eta+\theta_1,\ldots,\eta+\theta_a,0)\,dt_{m-a-1}\cdots dt_1,\end{aligned} \tag{6.22}$$

where the formula (6.13) for E is used along with the fact that $\Omega(\theta_j)=\eta+\theta_j-1$ for $a+1\le j\le m$.

The calculation of $(\Gamma_0\varphi)(\theta)$ is handled in a similar fashion, and in fact is slightly simpler. If $a=m$ then $\Gamma_0=Z_1Z_2\cdots Z_m$, and so (gzz)

$$(\Gamma_0\varphi)(\theta) = \varphi(\eta+\theta_1,\ldots,\eta+\theta_m). \tag{6.23}$$

Let us therefore assume that $0 \le a \le m-1$. We have first that (gz)

$$(\Gamma_0\varphi)(\theta) = \int_{-1}^{\Omega(\theta_{a+1})} \cdots \int_{-1}^{\Omega(\theta_m)} b^{\tau+1}(s_1)\cdots b^{\tau+1}(s_{m-a}) \\ \times\ \varphi(\eta+\theta_1,\ldots,\eta+\theta_a,s_1,\ldots,s_{m-a})\,ds_{m-a}\cdots ds_1, \tag{6.24}$$

and also, using the anti-symmetry of φ, that

$$\varphi(\eta+\theta_1,\ldots,\eta+\theta_a,s_1,\ldots,s_{m-a}) = (-1)^{a(m-a)}\varphi(s_1,\ldots,s_{m-a},\eta+\theta_1,\ldots,\eta+\theta_a).$$

Introducing the operator $E_0 \in \mathcal{L}(X^{\wedge m}, X^{\wedge(m-a)})$ given by (q0)

$$(E_0\psi)(t_1,t_2,\ldots,t_{m-a}) = \psi(t_1,\ldots,t_{m-a},\eta+\theta_1,\ldots,\eta+\theta_a), \tag{6.25}$$

we have from (6.24) that

$$(\Gamma_0\varphi)(\theta) = (-1)^{a(m-a)}J_1^1J_2^2\cdots J_{m-a}^{m-a}E_0\varphi.$$

Here the operators J_j^i, and I_j^i are as before, except they now operate on functions of $m-a$ variables rather than $m-a-1$ variables. Using (6.17) and (6.18) as before, we obtain (gammaz)

$$\begin{aligned}(-1)^{a(m-a)}[(\Gamma_0\varphi)(\theta)] &= I_1^1I_2^2\cdots I_{m-a}^{m-a}E_0\varphi \\ &= \int_{\Omega(\theta_a)}^{\Omega(\theta_{a+1})} \cdots \int_{\Omega(\theta_{m-1})}^{\Omega(\theta_m)} b^{\tau+1}(t_1)\cdots b^{\tau+1}(t_{m-a})[(E_0\varphi)(t_1,\ldots,t_{m-a})]\,dt_{m-a}\cdots dt_1.\end{aligned} \tag{6.26}$$

Upon multiplying the above formula by $(-1)^{(a+1)m}$ and noting that $(-1)^{(a+1)m}(-1)^{a(m-a)} = (-1)^{m-a}$, we obtain (x2)

$$(-1)^{m-a}[(\Gamma_0\varphi)(\theta)] = (-1)^{(a+1)m}\int_{-1}^{\eta+\theta_{a+1}-1}\int_{\eta+\theta_{a+1}-1}^{\eta+\theta_{a+2}-1} \cdots \int_{\eta+\theta_{m-1}-1}^{\eta+\theta_m-1} b^{\tau+1}(t_1)\cdots b^{\tau+1}(t_{m-a}) \\ \times\ \varphi(t_1,\ldots,t_{m-a},\eta+\theta_1,\ldots,\eta+\theta_a)\,dt_{m-a}\cdots dt_1, \tag{6.27}$$

where the formula (6.25) for E_0 is used, and where we have that $\Omega(\theta_a) = -1$. Adding the two equations (6.22) and (6.27) and using (6.8) gives (4.11), as desired, at least in the case that $0 \le a \le m-2$. If $a = m-1$ then (6.22) must be replaced by (6.10) to give the desired formula (4.12). Finally, if $a = m$ then the term corresponding to (6.22) is absent, while (6.27) is given by (6.23) to give (4.13), again as desired. With this, the proposition is proved. ■

We now prove the Positivity Theorem.

Proof of Theorem 4.1. We prove the result only for equation (4.3). The corresponding result for the more general equation (4.1) follows directly from the conjugacy (4.4) and the positivity of the operators $\Sigma(t)^{\wedge m}$ and $[\Sigma(t)^{\wedge m}]^{-1}$.

Due to the fact that $\mathbf{W}(\tau+\eta,\tau)$ is a linear process, it is enough to prove the theorem in the case that $0<\eta\leq 1$. Taking such η, we assume that $(-1)^m b(t)\geq 0$ almost everywhere in $[\tau,\tau+\eta]$. With $\varphi\in K_m\subseteq X^{\wedge m}$ and $\theta\in T_m$ fixed, and with a as in the statement of Proposition 4.2, consider the formulas (4.11), (4.12), and (4.13) in that result for $[\mathbf{W}(\tau+\eta,\tau)\varphi](\theta)$. Note that

$$(t_1,\ldots,t_{m-a-1},\eta+\theta_1,\ldots,\eta+\theta_a,0)\in T_m,\qquad (t_1,\ldots,t_{m-a},\eta+\theta_1,\ldots,\eta+\theta_a)\in T_m,$$

both hold for the arguments of φ in these formulas, in particular because $\eta+\theta_m-1\leq\eta+\theta_1$ and $\eta+\theta_a\leq 0$. Therefore φ evaluated at these points is nonnegative. Thus if m is even, so $b(t)\geq 0$, it is immediate from these formulas that $[\mathbf{W}(\tau+\eta,\tau)\varphi](\theta)\geq 0$. If m is odd, so $b(t)\leq 0$, the same conclusion holds after noting that $(-1)^{am}=(-1)^{m-a-1}$ and $(-1)^{(a+1)m}=(-1)^{m-a}$. In either case one concludes that $\mathbf{W}(\tau+\eta,\tau)\varphi\in K_m$, as desired. ■

7 u_0-Positivity

Here we consider the question of u_0-positivity of the linear process $\mathbf{W}(t,\tau)\in\mathcal{L}(X^{\wedge m})$ under the assumption that $(-1)^m b(t)\geq 0$ as in Theorem 4.1. We maintain the same notation as in the **PREVIOUS TWO SECTIONS,** with $X=C([-1,0])$ and the cone $K_m\subseteq X^{\wedge m}$ given by (4.8), with the set $T_m\subseteq[-1,0]$ given by (4.7), and where $\mathbf{W}(t,\tau)\in\mathcal{L}(X^{\wedge m})$ is the m-fold wedge product of the linear process associated to the delay-differential equation (4.3). Corresponding questions for the more general equation (4.1) are also addressed via the conjugacy (4.4), as before.

Generally, if Y is a Banach space and $K\subseteq Y$ is a closed, convex cone, then we say two elements $u,v\in K\setminus\{0\}$ are **comparable** in case there exist quantities $M_2\geq M_1>0$ such that

$$M_1 v\leq u\leq M_2 v,$$

where $\leq$ denotes the ordering with respect to K. We denote this relation by

$$u\sim v.$$

Clearly, $\sim$ is an equivalence relation on K. The following definition is classical.

Definition. Suppose that $A \in \mathcal{L}(Y)$ is a positive operator with respect to a closed, convex cone $K \subseteq Y$ in a Banach space Y, that is, $Au \in K$ whenever $u \in K$. Let $u_0 \in K \setminus \{0\}$. Then we say that the operator A is $\boldsymbol{u_0}$**-positive** in case there exists an integer $k_0 \geq 1$ such that $A^k u \sim u_0$ for every $u \in K \setminus \{0\}$ and every $k \geq k_0$.

In the case of a linear process, we make a related definition which accounts for continuous rather than discrete time, with constants M_1 and M_2 which are uniform with respect to compact time-intervals.

Definition. Suppose that $U(t,\tau) \in \mathcal{L}(Y)$, for $t \geq \tau$, is a linear process on a Banach space Y. Suppose also that $U(t,\tau)$ is a positive operator with respect to a closed, convex cone $K \subseteq Y$, for every $t, \tau \in \mathbf{R}$ with $t \geq \tau$. Let $u_0 \in K \setminus \{0\}$. Then we say that the process $U(t,\tau)$ is $\boldsymbol{u_0}$**-positive** in case there exists $\eta_0 > 0$ such that the following holds. Given any $u \in K \setminus \{0\}$, and given $\tau \in \mathbf{R}$ and $\eta_* \geq \eta_0$, then there exist $M_1 > 0$ and $M_2 > 0$ such that

$$M_1 u_0 \leq U(\tau + \eta, \tau) u \leq M_2 u_0$$

for every $\eta \in [\eta_0, \eta_*]$.

For each $m \geq 2$ let us define a function (u0)

$$u_m(\theta) = \Bigg(\prod_{1 \leq i < j \leq m} (\theta_j - \theta_i) \Bigg) \Bigg(\prod_{1 \leq i < j \leq m-1} (1 + \theta_i - \theta_j) \Bigg) \qquad \text{for } \theta \in T_m, \tag{7.1}$$

and extend u_m to all of $[-1,0]^m$ as an anti-symmetric function, so that $(S_\sigma u_m)(\theta) = \operatorname{sgn}(\sigma) u_m(\theta)$ for every $\sigma \in \mathcal{S}_m$ and every $\theta \in [-1,0]^m$. Note that for $m = 2$ the range $1 \leq i < j \leq m-1$ of the indices in the second factor of (7.1) is empty. In this and other such cases, here and below, we interpret such an empty product to be equal to $+1$ identically. Also note that the extended function u_m is continuous throughout $[-1,0]^m$, that is $u_m \in X^{\wedge m}$. This holds because $u_m(\theta) = 0$ whenever $\theta \in T_m$ is such that $\theta_j = \theta_{j+1}$ for some j with $1 \leq j \leq m-1$. Of course the polynomial formula (7.1) is not generally valid for $\theta \in [-1,0]^m \setminus T_m$.

The following theorem is the main result of this section. It is followed by a conjecture concerning the natural generalization of this result.

Theorem 7.1. *Let $m = 2$ or $m = 3$ be fixed. Assume that the coefficient function $b : \mathbf{R} \to \mathbf{R}$ in* *equation* (4.3) *is measurable and that for every compact interval* $[t_1, t_2] \subseteq \mathbf{R}$ *there exist* $M_2 \geq M_1 > 0$ *such that*

$$M_1 \leq (-1)^m b(t) \leq M_2 \qquad \textit{for almost every } t \in [t_1, t_2].$$

Then the linear process $\mathbf{W}(t, \tau)$ *on* $X^{\wedge m}$ *is* u_m*-positive with respect to the cone* K_m*, with the above function* u_m*. Moreover, we have that* $\eta_0 = 3$ *if* $m = 2$ *and* $\eta_0 = 5$ *if* $m = 3$ *for the quantity* η_0 *in the above definition of a* u_0*-positive linear process.*

The corresponding result for equation (4.1) *holds, where one assumes the same condition on the coefficient* $\beta : \mathbf{R} \to \mathbf{R}$ *as for* b*, and where the coefficient* $\alpha : \mathbf{R} \to \mathbf{R}$ *is locally integrable.*

Conjecture A. Let $m \geq 4$. Then the conclusions of Theorem 7.1 hold for this m, with $\eta_0 = 2m-1$.

For our purpose here it will be sufficient to take $\eta = 1$ in the formula (4.11), and so we may take $a = 0$ in that formula, as per the statement of Proposition 4.2. Assume that $(-1)^m b(t) \geq 0$ for almost every t satisfying $\tau \leq t \leq \tau + 1$. Then (4.11) gives (eta1x)

$$\begin{aligned} [\mathbf{W}(\tau+1, \tau)\varphi](\theta) = & \int_{\theta_1}^{\theta_2} \cdots \int_{\theta_{m-1}}^{\theta_m} |b^{\tau+1}(t_1) \cdots b^{\tau+1}(t_{m-1})| \varphi(t_1, \ldots, t_{m-1}, 0)\, dt_{m-1} \cdots dt_1 \\ & + \int_{-1}^{\theta_1} \int_{\theta_1}^{\theta_2} \cdots \int_{\theta_{m-1}}^{\theta_m} |b^{\tau+1}(t_0) \cdots b^{\tau+1}(t_{m-1})| \varphi(t_0, \ldots, t_{m-1})\, dt_{m-1} \cdots dt_0 \end{aligned} \tag{7.2}$$

for every $\varphi \in X^{\wedge m}$ provided that $\theta \in T_m$. (For convenience later, we have reindexed the variables t_j in the final term of (7.2).) Note that in the case of odd m, where $b(t) \leq 0$, the identities $(-1)^{am} = (-1)^{m-a-1}$ and $(-1)^{(a+1)m} = (-1)^{m-a}$ are used in taking the absolute values of $b(t)$.

We shall need both positive upper and lower bounds for the operator $\mathbf{W}(\tau+1, \tau)$, which is why we assume in Theorem 7.1 that there are uniform positive upper and lower bounds for $|b(t)|$ on compact intervals. The bounds for $\mathbf{W}(\tau+1, \tau)$ will then be given by appropriate multiples of the operator $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1$, where (aa)

$$\begin{aligned} (\mathbf{A}_0\varphi)(\theta) &= \int_{\theta_1}^{\theta_2} \cdots \int_{\theta_{m-1}}^{\theta_m} \varphi(t_1, \ldots, t_{m-1}, 0)\, dt_{m-1} \cdots dt_1, \\ (\mathbf{A}_1\varphi)(\theta) &= \int_{-1}^{\theta_1} \int_{\theta_1}^{\theta_2} \cdots \int_{\theta_{m-1}}^{\theta_m} \varphi(t_0, \ldots, t_{m-1})\, dt_{m-1} \cdots dt_0. \end{aligned} \tag{7.3}$$

The operator $\mathbf{A}$ is a central object of study below. Although we prove some general results (Propositions 7.5 and 7.6) valid for every $m \geq 1$, our focus is ultimately on the cases $m = 2$ and $m = 3$, as in

Theorem 7.1.

We shall consider $\mathbf{A}$ as acting on the space $C(T_m)$ of all continuous functions $\varphi : T_m \to \mathbf{R}$, which is in contrast to earlier sections where we worked with the space $X^{\wedge m}$. However, note that $X^{\wedge m}$ is isometrically isomorphic to the subspace (iso)

$$X_{\mathrm{r}}^{\wedge m} = \{\varphi \in C(T_m) \mid \varphi(\theta) = 0 \text{ whenever } \theta_j = \theta_{j+1} \text{ for some } j \text{ satisfying } 1 \leq j \leq m-1\} \tag{7.4}$$

of $C(T_m)$ consisting of all restrictions $\varphi|_{T_m}$ of functions $\varphi \in X^{\wedge m}$ to $T_m \subseteq [-1,0]^m$. As such, we shall freely regard the function u_m in (7.1) to be an element of $C(T_m)$, in fact, $u_m \in X_{\mathrm{r}}^{\wedge m} \subseteq C(T_m)$. Also, without loss, we may regard $\mathbf{W}(t,\tau)$ to be an operator on $X_{\mathrm{r}}^{\wedge m}$ rather than on $X^{\wedge m}$, as we wish to compare $\mathbf{W}(t,\tau)$ with powers of the operator $\mathbf{A}$. Note that the ranges of $\mathbf{A}_0$ and $\mathbf{A}_1$ on $C(T_m)$ lie in the subspace $X_{\mathrm{r}}^{\wedge m}$, and so the subspace $X_{\mathrm{r}}^{\wedge m} \subseteq C(T_m)$ is invariant under these operators.

Let us also denote the positive cone in $C(T_m)$ by (pm)

$$C(T_m)^+ = \{\varphi \in C(T_m) \mid \varphi(\theta) \geq 0 \text{ for every } \theta \in T_m\}. \tag{7.5}$$

The crucial part in proving Theorem 7.1 is to show that the operator $\mathbf{A}$ is u_m-positive with respect to $C(T_m)^+$. Indeed, we have the following result.

Proposition 7.2. *Let $m \geq 2$ and suppose it is the case that the operator $\mathbf{A} \in \mathcal{L}(C(T_m))$ given above is a u_m-positive operator with respect to the cone $C(T_m)^+$, with u_m as in* (7.1). *Then the conclusion of Theorem 7.1 holds, but with the value of m chosen here. Further, we have that $\eta_0 = k_0$ for the quantities in the above definitions of u_0-positive operator and u_0-positive linear process, corresponding to the operator $\mathbf{A}$ and to the linear process $\mathbf{W}(t,\tau)$.*

Proposition 7.2 implies that in order to prove Theorem 7.1, it is sufficient to prove the following result.

Theorem 7.3. *Let $m = 2$ or $m = 3$. Then the operator $\mathbf{A}$ acting on $C(T_m)$ is u_m-positive with respect to the cone $C(T_m)^+$, with u_m as in* (7.1). *Further, if $\varphi \in C(T_m)^+ \setminus \{0\}$ then $\mathbf{A}^k\varphi \sim u_m$ for every $k \geq 3$ if $m = 2$, and for every $k \geq 5$ if $m = 3$.*

Remark. The fact that $u_m \in X_{\mathrm{r}}^{\wedge m}$, along with the invariance of $X_{\mathrm{r}}^{\wedge m}$ under $\mathbf{A}$, implies that $\mathbf{A}$, as an operator on $X_{\mathrm{r}}^{\wedge m}$, is also u_m-positive for that space with respect to the cone $C(T_m)^+ \cap X_{\mathrm{r}}^{\wedge m}$.

Conjecture B. Let $m \geq 4$. Then the conclusions of Theorem 7.3 hold for this m, but where $\mathbf{A}^k\varphi \sim u_m$ for every $k \geq 2m-1$.

It is clear from Proposition 7.2 that if Conjecture B holds, then so does Conjecture A.

Proof of Proposition 7.2. For simplicity, we prove only the conclusions of Theorem 7.1 pertaining to equation (4.3), that is, for the linear process $\mathbf{W}(t,\tau)$. The corresponding conclusions for equation (4.1) can be obtained from these using the conjugacy (4.4).

By assumption, there exists $k_0 \geq 1$ such that $\mathbf{A}^k\varphi \sim u_m$ for every $k \geq k_0$ and every $\varphi \in C(T_m)^+ \setminus \{0\}$. Let $\tau \in \mathbf{R}$ and let $\eta_* \geq k_0$ be given. Then there exist $M_2 \geq M_1 > 0$ such that (bbnd)

$$M_1 \leq (-1)^m b(t) \leq M_2 \qquad \text{for almost every } t \in [\tau, \tau+\eta_*], \tag{7.6}$$

as in the statement of Theorem 7.1. We shall work in the space $X_{\mathrm{r}}^{\wedge m} \subseteq C(T_m)$, and we note that $C(T_m)^+ \cap X_{\mathrm{r}}^{\wedge m}$ is a closed, convex cone in that space. First note that if $\varphi \in C(T_m)^+ \cap X_{\mathrm{r}}^{\wedge m}$, then from the formulas (7.2) and (7.3), and the bounds (7.6), we have that if $[\sigma, \sigma+1] \subseteq [\tau, \tau+\eta_*]$ then

$$M_3\mathbf{A}\varphi \leq (M_1^{m-1}\mathbf{A}_0 + M_1^m\mathbf{A}_1)\varphi \leq \mathbf{W}(\sigma+1,\sigma)\varphi \leq (M_2^{m-1}\mathbf{A}_0 + M_2^m\mathbf{A}_1)\varphi \leq M_4\mathbf{A}\varphi$$

where

$$M_3 = \min\{M_1^{m-1},\, M_1^m\}, \qquad M_4 = \max\{M_2^{m-1},\, M_2^m\},$$

with $\leq$ denoting the ordering in the cone $C(T_m)^+ \cap X_{\mathrm{r}}^{\wedge m}$. It follows by iteration that if $[\sigma, \sigma+k] \subseteq [\tau, \tau+\eta_*]$ for some integer $k \geq 1$, then (kbnd)

$$M_3^k\mathbf{A}^k\varphi \leq \mathbf{W}(\sigma+k,\sigma)\varphi \leq M_4^k\mathbf{A}^k\varphi. \tag{7.7}$$

Now let $\varphi \in [C(T_m)^+ \cap X_{\mathrm{r}}^{\wedge m}] \setminus \{0\}$ be given; we shall keep φ fixed for the remainder of the proof. Given any $\eta \in [k_0, \eta_*]$, let $\psi = \mathbf{W}(\tau+\eta-k_0,\tau)\varphi$, and note that $\psi \in [C(T_m)^+ \cap X_{\mathrm{r}}^{\wedge m}] \setminus \{0\}$ where Proposition 4.4 is used. Thus there exists $\varepsilon = \varepsilon(\eta) > 0$ such that if we define

$$\psi_-(\theta) = \inf_{\eta' \in [k_0,\eta_*] \cap (\eta-\varepsilon,\eta+\varepsilon)} [\mathbf{W}(\tau+\eta'-k_0,\tau)\varphi](\theta),$$

for $\theta \in T_m$, then $\psi_-(\theta) > 0$ for some θ and so $\psi_- \in [C(T_m)^+ \cap X_{\mathrm{r}}^{\wedge m}] \setminus \{0\}$. Fix such ε and let

$$\psi_+(\theta) = \sup_{\eta' \in [k_0,\eta_*] \cap (\eta-\varepsilon,\eta+\varepsilon)} [\mathbf{W}(\tau+\eta'-k_0,\tau)\varphi](\theta),$$

and so also $\psi_+ \in [C(T_m)^+ \cap X_{\mathrm{r}}^{\wedge m}] \setminus \{0\}$. Then for any $\eta' \in [k_0, \eta_*] \cap (\eta - \varepsilon, \eta + \varepsilon)$, we have that (ord)

$$\psi_- \leq \mathbf{W}(\tau + \eta' - k_0, \tau)\varphi \leq \psi_+, \tag{7.8}$$

and upon applying the positive operator $\mathbf{W}(\tau + \eta', \tau + \eta' - k_0)$ to (7.8), one obtains

$$\mathbf{W}(\tau + \eta', \tau + \eta' - k_0)\psi_- \leq \mathbf{W}(\tau + \eta', \tau)\varphi \leq \mathbf{W}(\tau + \eta', \tau + \eta' - k_0)\psi_+.$$

It follows, by (7.7), that (ord4)

$$Q_1 u_m \leq M_3^{k_0} \mathbf{A}^{k_0} \psi_- \leq \mathbf{W}(\tau + \eta', \tau)\varphi \leq M_4^{k_0} \mathbf{A}^{k_0} \psi_+ \leq Q_2 u_m \tag{7.9}$$

for some $Q_2 \geq Q_1 > 0$, where the existence of Q_1 and Q_2 follows directly from the assumption that $\mathbf{A}$ is u_m-positive, specifically, that $\mathbf{A}^{k_0}\psi_\pm \sim u_m$.

To complete the proof of the theorem, let us denote the constants in (7.9) by $Q_{j,\eta}$, for $j = 1, 2$ and any $\eta \in [k_0, \eta_*]$. We observe that the open intervals $(\eta - \varepsilon(\eta), \eta + \varepsilon(\eta))$ for such η form an open cover of $[k_0, \eta_*]$, so we may extract a finite subcover, corresponding to points η_i for $1 \leq i \leq p$. Then upon setting $Q_{1,*} = \min\limits_{1 \leq i \leq p} \{Q_{1,\eta_i}\}$ and $Q_{2,*} = \max\limits_{1 \leq i \leq p} \{Q_{2,\eta_i}\}$, we see that

$$Q_{1,*} u_m \leq \mathbf{W}(\tau + \eta', \tau)\varphi \leq Q_{2,*} u_m$$

for every $\eta' \in [k_0, \eta_*]$. With this, the proof is complete. ■

Moving toward the proof of Theorem 7.3, we shall first obtain a pointwise upper bound for $|(\mathbf{A}^k\varphi)(\theta)|$ in Proposition 7.5 below, and in fact we shall obtain such for every $m \geq 2$. To this end we define functions $u_m^q \in C(T_m)$ by (umq)

$$u_m^q(\theta) = \Bigg(\prod_{\substack{1 \leq i,j \leq m \\ 1 \leq j-i \leq q}} (\theta_j - \theta_i) \Bigg) \Bigg(\prod_{\substack{1 \leq i,j \leq m-1 \\ j-i \geq m-q}} (1 + \theta_i - \theta_j) \Bigg), \tag{7.10}$$

for $0 \leq q \leq m - 1$. Here and below we shall always assume q is in this range, although we shall sometimes impose additional restrictions on q. We assume that $m \geq 2$ and that $\theta \in T_m$. Note that all factors in the products (7.10) are nonnegative and bounded above by $+1$. Now define (wdef)

$$w_m(\theta) = \prod_{j=1}^{m-1} (\theta_{j+1} - \theta_j), \qquad \widetilde{w}_m(\theta) = (1 + \theta_1 - \theta_{m-1}) w_m(\theta_1, \ldots, \theta_m), \tag{7.11}$$

and let polynomials v_m^q and $\widetilde{v}_m^q$ be defined by (vdef)

$$\begin{aligned} u_m^q(\theta) &= v_m^q(\theta) w_m(\theta), && \text{for } 1 \le q \le m-1, \\ u_m^q(\theta) &= \widetilde{v}_m^q(\theta) \widetilde{w}_m(\theta), && \text{for } 2 \le q \le m-1, \\ \widetilde{v}_m^1(\theta) &\equiv 1 \text{ identically.} \end{aligned} \tag{7.12}$$

It is easy to check that v_m^q and $\widetilde{v}_m^q$ are well-defined polynomials, as every factor of of w_m and $\widetilde{w}_m$ occurs as a factor of the polynomial u_m^q for the indicated ranges of q. Note that $u_m^0(\theta) \equiv 1$ identically, while $u_m^1(\theta) = w_m(\theta)$, where some of the products in (7.10) are empty hence take the value $+1$, as noted earlier. Also observe that $u_m^{m-1}(\theta) = u_m(\theta)$ as in (7.1).

Now let us take quantities t_j for $0 \le j \le m-1$ satisfying (tthet)

$$t_0 \in [-1, \theta_1], \qquad t_j \in [\theta_j, \theta_{j+1}] \qquad \text{for } 1 \le j \le m-1, \tag{7.13}$$

as in the integrands of (7.3). The following lemma provides a crucial estimate needed for the proof of Proposition 7.5.

Lemma 7.4. *With $m \ge 2$, let $\theta \in T_m$ and let t_j for $0 \le j \le m-1$ satisfy* (7.13). *Then* (uinq)

$$0 \le u_m^q(t_1, \ldots, t_{m-1}, 0) \le v_m^{q+1}(\theta), \qquad 0 \le u_m^q(t_0, \ldots, t_{m-1}) \le \widetilde{v}_m^{q+1}(\theta), \tag{7.14}$$

for $0 \le q \le m-2$. Further, (uinq2)

$$0 \le u_m^{m-1}(t_1, \ldots, t_{m-1}, 0) \le v_m^{m-1}(\theta), \qquad 0 \le u_m^{m-1}(t_0, \ldots, t_{m-1}) \le \widetilde{v}_m^{m-1}(\theta), \tag{7.15}$$

holds.

Proof. In this proof care must be taken to ensure the correct ranges of the indices i and j, and it will be helpful to note that $i < j$ in many places.

Assume that $\theta \in T_m$ and that (7.13) holds. We begin by observing that (tinq)

$$\begin{aligned} 0 \le t_j - t_i \ &\le \theta_{j+1} - \theta_i \le 1, && \text{for } 1 \le i < j \le m-1, \\ 0 \le t_j - t_0 \ &\le 1 + \theta_{j+1} - \theta_{m-1} \le 1, && \text{for } 1 \le j \le m-2, \\ 0 \le -t_i \ &\le 1 + \theta_1 - \theta_i \le 1, && \text{for } 1 \le i \le m-1, \\ 0 \le 1 + t_i - t_j \ &\le 1 + \theta_{i+1} - \theta_j \le 1, && \text{for } 0 \le i < j \le m-1. \end{aligned} \tag{7.16}$$

We first establish (7.14). Suppose that $0 \le q \le m-2$. Then from (7.10) we have that (prod0)

$$u_m^q(t_1,\dots,t_{m-1},0) = \Bigg(\prod_{\substack{1\le i,j\le m-1\\ 1\le j-i\le q}} (t_j - t_i)\Bigg)\Bigg(\prod_{m-q\le i\le m-1} (-t_i)\Bigg)\Bigg(\prod_{\substack{1\le i,j\le m-1\\ j-i\ge m-q}} (1+t_i-t_j)\Bigg). \tag{7.17}$$

Using (7.16) we see that (first)

$$\prod_{\substack{1\le i,j\le m-1\\ 1\le j-i\le q}} (t_j - t_i) \;\le\; \prod_{\substack{1\le i,j\le m-1\\ 1\le j-i\le q}} (\theta_{j+1}-\theta_i) \;=\; \prod_{\substack{1\le i,j\le m\\ 2\le j-i\le q+1}} (\theta_j-\theta_i) \tag{7.18}$$

for the first product in (7.17). We also have that (third)

$$\prod_{\substack{1\le i,j\le m-1\\ j-i\ge m-q}} (1+t_i-t_j) \;\le\; \prod_{\substack{1\le i,j\le m-1\\ j-i\ge m-q}} (1+\theta_{i+1}-\theta_j) \;=\; \prod_{\substack{2\le i,j\le m-1\\ j-i\ge m-q-1}} (1+\theta_i-\theta_j) \tag{7.19}$$

for the third product in (7.17). Using the inequality $-t_i \le 1+\theta_1-\theta_i$ from (7.16) in the second product in (7.17), and reindexing using j instead of i, we may combine this with (7.19) to obtain

$$\Bigg(\prod_{m-q\le i\le m-1} (-t_i)\Bigg)\Bigg(\prod_{\substack{1\le i,j\le m-1\\ j-i\ge m-q}} (1+t_i-t_j)\Bigg) \;\le\; \prod_{\substack{1\le i,j\le m-1\\ j-i\ge m-q-1}} (1+\theta_i-\theta_j).$$

Combining this further with (7.18), we see that with (7.17) this gives

$$u_m^q(t_1,\dots,t_{m-1},0) \le \Bigg(\prod_{\substack{1\le i,j\le m\\ 2\le j-i\le q+1}} (\theta_j-\theta_i)\Bigg)\Bigg(\prod_{\substack{1\le i,j\le m-1\\ j-i\ge m-q-1}} (1+\theta_i-\theta_j)\Bigg) = v_m^{q+1}(\theta),$$

to give the first half of (7.14).

Next observe that (prod1)

$$u_m^q(t_0,\dots,t_{m-1}) = \Bigg(\prod_{\substack{0\le i,j\le m-1\\ 1\le j-i\le q}} (t_j - t_i)\Bigg)\Bigg(\prod_{\substack{0\le i,j\le m-2\\ j-i\ge m-q}} (1+t_i-t_j)\Bigg). \tag{7.20}$$

For the first product in (7.20) we have, again using (7.16), that (first2)

$$\begin{aligned}\prod_{\substack{0\le i,j\le m-1\\ 1\le j-i\le q}} (t_j - t_i) &\le \Bigg(\prod_{\substack{1\le i,j\le m-1\\ 1\le j-i\le q}} (\theta_{j+1}-\theta_i)\Bigg)\Bigg(\prod_{1\le j\le q} (1+\theta_{j+1}-\theta_{m-1})\Bigg)\\ &\le \Bigg(\prod_{\substack{1\le i,j\le m\\ 2\le j-i\le q+1}} (\theta_j-\theta_i)\Bigg)\Bigg(\prod_{2\le i\le q} (1+\theta_i-\theta_{m-1})\Bigg).\end{aligned} \tag{7.21}$$

Note that in the second inequality of (7.21), we have used the estimate $1+\theta_{q+1}-\theta_{m-1}\le 1$, which holds because $q\le m-2$, and which allows us to drop the term $1+\theta_{q+1}-\theta_{m-1}$. Now for the second product in (7.20) we have that (second2)

$$\prod_{\substack{0\le i,j\le m-2\\ j-i\ge m-q}}(1+t_i-t_j)\;\le\;\prod_{\substack{0\le i,j\le m-2\\ j-i\ge m-q}}(1+\theta_{i+1}-\theta_j)\;=\;\prod_{\substack{1\le i,j\le m-2\\ j-i\ge m-q-1}}(1+\theta_i-\theta_j). \tag{7.22}$$

If $q\ge 1$ then combining (7.21) and (7.22) gives

$$u_m^q(t_0,\ldots,t_{m-1})\le\Bigg(\prod_{\substack{1\le i,j\le m\\ 2\le j-i\le q+1}}(\theta_j-\theta_i)\Bigg)\Bigg(\prod_{\substack{1\le i,j\le m-1\\ m-3\ge j-i\ge m-q-1}}(1+\theta_i-\theta_j)\Bigg)=\widetilde{v}_m^{q+1}(\theta),$$

while if $q=0$ we have directly that

$$u_m^0(t_0,\ldots,t_{m-1})=1=\widetilde{v}_m^1(\theta).$$

This establishes the second half of (7.14).

We now prove (7.15). This follows directly by noting that

$$0\le u_m^{m-1}(t_1,\ldots,t_{m-1},0)\le u_m^{m-2}(t_1,\ldots,t_{m-1},0),$$

$$0\le u_m^{m-1}(t_0,\ldots,t_{m-1})\le u_m^{m-2}(t_0,\ldots,t_{m-1}),$$

and then applying (7.14) for $q=m-2$. With this the proof is complete. ■

Proposition 7.5. *Let $m\ge 2$. Then we have the pointwise bounds* (ubnd)

$$0\le(\mathbf{A}_i u_m^q)(\theta)\le u_m^{q+1}(\theta),\qquad 0\le(\mathbf{A}_i u_m)(\theta)\le u_m(\theta), \tag{7.23}$$

for $0\le q\le m-2$ and $i=0,1$, for $\theta\in T_m$. Thus for every $\varphi\in C(T_m)$ we have the pointwise bound (b3)

$$|(\mathbf{A}^k\varphi)(\theta)|\le 2^k u_m(\theta)\|\varphi\|, \tag{7.24}$$

for $k\ge m-1$ and $\theta\in T_m$.

Proof. Let $\varphi=u_m^q$ in (7.3) where $0\le q\le m-2$. The using (7.14) in Lemma 7.4, we have for every $\theta\in T_m$ that

$$\begin{aligned}0\le(\mathbf{A}_0u_m^q)(\theta)&=\int_{\theta_1}^{\theta_2}\!\!\cdots\int_{\theta_{m-1}}^{\theta_m}u_m^q(t_1,\ldots,t_{m-1},0)\,dt_{m-1}\cdots dt_1\\ &\le\int_{\theta_1}^{\theta_2}\!\!\cdots\int_{\theta_{m-1}}^{\theta_m}v_m^{q+1}(\theta)\,dt_{m-1}\cdots dt_1=w_m(\theta)v_m^{q+1}(\theta)=u_m^{q+1}(\theta).\end{aligned}$$

Similarly, if $1 \le q \le m-2$ we have that (a1q)

$$\begin{aligned} 0 \le (\mathbf{A}_1 u_m^q)(\theta) &= \int_{-1}^{\theta_1}\int_{\theta_1}^{\theta_2}\cdots\int_{\theta_{m-1}}^{\theta_m} u_m^q(t_0,\ldots,t_{m-1})\,dt_{m-1}\cdots dt_0 \\ &\le \int_{-1}^{\theta_1}\int_{\theta_1}^{\theta_2}\cdots\int_{\theta_{m-1}}^{\theta_m} \widetilde{v}_m^{q+1}(\theta)\,dt_{m-1}\cdots dt_0 \\ &= \left(\frac{1+\theta_1}{1+\theta_1-\theta_{m-1}}\right)\widetilde{w}_m(\theta)\widetilde{v}_m^{q+1}(\theta) \le \widetilde{w}_m(\theta)\widetilde{v}_m^{q+1}(\theta) = u_m^{q+1}(\theta). \end{aligned} \tag{7.25}$$

If $q = 0$ we again have (7.25) except with an inequality $\le$ in place of the final equal sign, as

$$\widetilde{w}_m(\theta)\widetilde{v}_m^1(\theta) = \widetilde{w}_m(\theta) \le w_m(\theta) = u_m^1(\theta).$$

This gives the first half of (7.23). For the second half of (7.23), involving $(\mathbf{A}_i u_m)(\theta)$, one argues similarly except using (7.15) instead of (7.14), where we recall that $u_m = u_m^{m-1}$. We omit the details.

It follows that $0 \le (\mathbf{A}u_m^q)(\theta) \le 2u_m^{q+1}(\theta)$ if $0 \le q \le m-2$, while $0 \le (\mathbf{A}u_m^{m-1})(\theta) \le 2u_m^{m-1}(\theta)$, for every $\theta \in T_m$. Thus (b1)

$$0 \le (\mathbf{A}^k u_m^0)(\theta) \le 2^k u_m^{\gamma(k)}(\theta), \qquad \gamma(k) = \min\{k,\, m-1\}, \tag{7.26}$$

for every $k \ge 1$. Also, as $\mathbf{A}$ is a positive operator with respect to the cone $C(T_m)^+$ in (7.5), we have the pointwise bound (b2)

$$|(\mathbf{A}^k\varphi)(\theta)| \le (\mathbf{A}^k|\varphi|)(\theta) \le [(\mathbf{A}^k u_m^0)(\theta)]\|\varphi\| \tag{7.27}$$

for every $\varphi \in C(T_m)$, where we recall that $u_m^0(\theta) \equiv 1$ identically. Combining (7.26) and (7.27) gives (7.24), as desired. ■

Related to the operator $\mathbf{A}$ is the operator $\mathbf{B}$, which we define as $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, where (bformx)

$$\begin{aligned} (\mathbf{B}_0\varphi)(\theta) &= \frac{1}{u_m(\theta)}\int_{\theta_1}^{\theta_2}\cdots\int_{\theta_{m-1}}^{\theta_m} u_m(t_1,\ldots,t_{m-1},0)\varphi(t_1,\ldots,t_{m-1},0)\,dt_{m-1}\cdots dt_1, \\ (\mathbf{B}_1\varphi)(\theta) &= \frac{1}{u_m(\theta)}\int_{-1}^{\theta_1}\int_{\theta_1}^{\theta_2}\cdots\int_{\theta_{m-1}}^{\theta_m} u_m(t_0,\ldots,t_{m-1})\varphi(t_0,\ldots,t_{m-1})\,dt_{m-1}\cdots dt_0. \end{aligned} \tag{7.28}$$

Formally, $\mathbf{B}$ is conjugate to $\mathbf{A}$ via the operator given by multiplication by u_m, and $\mathbf{B}$ will play a significant role in proving Theorem 7.3. In particular, obtaining the required equivalence $\mathbf{A}^k\varphi \sim u_m$ for large k is essentially the same as showing that $\mathbf{B}^k\psi \sim 1$ where $\varphi = u_m\psi$. Concerning the appropriate space on which $\mathbf{B}$ acts, we see that if $\varphi \in C(T_m)$ then the function $\mathbf{B}\varphi$ is continuous almost everywhere on T_m, specifically, it is continuous at each point $\theta \in O_m$ where (om)

$$O_m = \{\theta \in T_m \mid \theta_j < \theta_{j+1} \text{ for every } j \text{ satisfying } 1 \leq j \leq m-1\}. \tag{7.29}$$

However, as we shall see, $\mathbf{B}\varphi$ can have discontinuities in $T_m \setminus O_m$, and thus need not belong to $C(T_m)$. In light of the estimate (7.24), one might wish to consider $\mathbf{B}$ acting on the space $L^\infty(T_m)$ of bounded measurable functions. However, $\mathbf{B}\varphi$ is not well-defined for general $\varphi \in L^\infty(T_m)$ due to the zero entry in the final argument of φ in the formula (7.28) for $\mathbf{B}_0\varphi$. However, if we define (wm)

$$W_m = \{\varphi \in L^\infty(T_m) \mid \varphi \text{ is continuous at every point } \theta \in O_m\}, \tag{7.30}$$

where O_m is as in (7.29), then $W_m \subseteq L^\infty(T_m)$ is a closed subspace, and one easily sees that $\mathbf{B}$ is well-defined as an operator on W_m with range in W_m, that is, $\mathbf{B} \in \mathcal{L}(W_m)$. We have have the following result.

Proposition 7.6. *Let $m \geq 2$. Then $\mathbf{B}_0$ and $\mathbf{B}_1$ in* (7.28)*, and thus $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, define bounded linear operators on the space W_m with $\|\mathbf{B}_0\|, \|\mathbf{B}_1\| \leq 1$ and $\|\mathbf{B}\| \leq 2$. Here W_m is defined by* (7.29), (7.30) *with the norm inherited from $L^\infty(T_m)$.*

Proof. The proof is very similar to the proof of Proposition 7.5, and entails using the bounds (7.15) of Lemma 7.4 to estimate the integrals (7.28) just as before. We omit the details. ■

It is natural to ask what is the minimal closed invariant subspace $Y \subseteq W_m$ for the operator $\mathbf{B}$ which contains $C(T_m)$. Such Y would be given by letting (y)

$$Y = \overline{\bigcup_{n=0}^{\infty} Y_n}, \qquad \text{where} \qquad Y_0 = C(T_m), \qquad Y_{n+1} = \mathbf{B}Y_n + C(T_m) \qquad \text{for } n \geq 0, \tag{7.31}$$

where we note that $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y \subseteq W_m$. In the next section, as part of our efforts to prove Theorem 7.3, we show that if $m = 2$ then $Y = C(T_2)$, while if $m = 3$ then $Y = C(T_3) \oplus V$ where V is a certain two-dimensional subspace of W_m.

8 u_0-Positivity for $m = 2$ and $m = 3$

Let us now specialize to the cases $m = 2$ and $m = 3$, as in Theorem 7.3. We retain all the conventions and notation of the previous section. In working toward the proof of Theorem 7.3, our analysis here is largely concerned with the operator $\mathbf{B}$.

We first consider the case $m = 2$. Then $u_2(\theta_1, \theta_2) = \theta_2 - \theta_1$, and so (71)

$$(\mathbf{B}_0\varphi)(\theta) = -\fint_{\theta_1}^{\theta_2} t_1\varphi(t_1, 0)\, dt_1, \qquad (\mathbf{B}_1\varphi)(\theta) = \int_{-1}^{\theta_1}\fint_{\theta_1}^{\theta_2} (t_1 - t_0)\varphi(t_0, t_1)\, dt_1\, dt_0, \tag{8.1}$$

as in (7.28). For convenience of notation, we denote the so-called average integral by

$$\fint_a^b f(x)\, dx = \frac{1}{b-a}\int_a^b f(x)\, dx, \qquad \text{with} \qquad \fint_a^a f(x)\, dx = f(a).$$

Note that for locally integrable f, the above average integral is continuous as a function of a and b for $a \neq b$. It is also continuous where $a = b$ provided that f is continuous at this point. It thus follows that $\mathbf{B}_0$ and $\mathbf{B}_1$, and thus $\mathbf{B}$, are bounded linear operators on the space $C(T_2)$, and so $Y = C(T_2)$ for the space Y in (7.31). One sees moreover that $\mathbf{B}_0$, $\mathbf{B}_1$, and $\mathbf{B}$ are positive operators on $C(T_2)$ with respect to the cone $C(T_2)^+$. (Keep in mind that $t_1 \leq 0$ for the integrand in the formula for $\mathbf{B}_0$.)

We have the following result.

Proposition 8.1. *Let $\varphi \in C(T_2)^+ \setminus \{0\}$. Then for every $k \geq 3$ there exists $M_k > 0$ such that $(\mathbf{B}^k\varphi)(\theta) \geq M_k$ for every $\theta \in T_2$. Thus the operator $\mathbf{B}$ with $m = 2$ and acting on $C(T_2)$ is u_0-positive with respect to the cone $C(T_2)^+$, where $u_0(\theta) \equiv 1$ identically on T_2.*

Proof. It is clear that u_0-positivity follows from the existence of the lower bounds M_k, as we clearly have the pointwise upper bounds $|(\mathbf{B}^k\varphi)(\theta)| \leq \|\mathbf{B}^k\varphi\|$. Also, it is sufficient to prove the existence only of M_3, as the constants M_k for $k \geq 4$ follow directly by induction, using the positivity of $\mathbf{B}$. Indeed, having obtained M_j for $3 \leq j \leq k$, we obtain a lower bound M_{k+1} for $|(\mathbf{B}^{k+1})(\theta)|$ by applying $\mathbf{B}^3$ to the function $\mathbf{B}^{k-2}\varphi \in C(T_2)^+ \setminus \{0\}$.

To show that M_3 exists, it is enough, due to the continuity of $\mathbf{B}^3\varphi$, to show that if $\varphi \in C(T_2)^+ \setminus \{0\}$ then we have strict positivity $(\mathbf{B}^3\varphi)(\theta) > 0$ for every $\theta \in T_2$. To this end let

$$L = \{\theta \in T_2 \mid \theta_2 = 0\} = [-1, 0] \times \{0\},$$

which is the upper boundary of the set $T_2 \subseteq \mathbf{R}^2$. Then it is enough to prove that the following three facts hold for every $\varphi \in C(T_2)^+$.

(1) If $\varphi(\theta) > 0$ for some $\theta \in T_2$, then $(\mathbf{B}\varphi)(\widetilde{\theta}) > 0$ for some $\widetilde{\theta} \in L$;

(2) if $\varphi(\theta) > 0$ for some $\theta \in L$, then $(\mathbf{B}\varphi)(\widetilde{\theta}) > 0$ for every $\widetilde{\theta} \in L$; and

(3) if $\varphi(\theta) > 0$ for every $\theta \in L$, then $(\mathbf{B}\varphi)(\widetilde{\theta}) > 0$ for every $\widetilde{\theta} \in T_2$.

The proofs of properties (1) through (3) follow easily from the formulas (8.1). With $\varphi \in C(T_2)^+$, if $\varphi(\theta) > 0$ for some $\theta = (\theta_1, \theta_2) \in T_2$ we may assume without loss that $-1 < \theta_1 < \theta_2 \leq 0$. Then $(\mathbf{B}_1\varphi)(\theta_1, 0) > 0$ holds, in particular because the integrand $(t_1 - t_0)\varphi(t_0, t_1)$ in (8.1), which is nonnegative throughout the range $-1 \leq t_0 \leq \theta_1 \leq t_1 \leq 0$, is strictly positive at $(t_0, t_1) = (\theta_1, \theta_2)$. With this, (1) is established.

Now suppose that $\varphi(\theta) > 0$ for some $\theta = (\theta_1, 0) \in L$. Then $(\mathbf{B}_0\varphi)(\widetilde{\theta}_1, 0) > 0$ for every $\widetilde{\theta}_1$ satisfying $-1 \leq \widetilde{\theta}_1 \leq \theta_1$ and $\widetilde{\theta}_1 \neq 0$, and $(\mathbf{B}_1\varphi)(\widetilde{\theta}_1, 0) > 0$ for every $\widetilde{\theta}_1$ satisfying $\theta_1 \leq \widetilde{\theta}_1 \leq 0$ and $\widetilde{\theta}_1 \neq -1$. In any case, $(\mathbf{B}\varphi)(\widetilde{\theta}) > 0$ for every $\widetilde{\theta} = (\widetilde{\theta}_1, 0) \in L$. This establishes (2).

Finally suppose that $\varphi(\theta) > 0$ for every $\theta \in L$. Then $(\mathbf{B}_0\varphi)(\widetilde{\theta}) > 0$ for every $\widetilde{\theta} \in T_2$ except $\widetilde{\theta} = (0, 0)$. However, $(\mathbf{B}_1\varphi)(0, 0) > 0$ for this point. With this, (3) is established and the result is proved. ■

Let us now consider the case $m = 3$, so $\theta = (\theta_1, \theta_2, \theta_3) \in T_3$. Here

$$u_3(\theta) = (\theta_2 - \theta_1)(\theta_3 - \theta_2)(\theta_3 - \theta_1)(1 + \theta_1 - \theta_2).$$

We introduce the functions (nu)

$$\nu_0(\theta) = \frac{-(\theta_2 + \theta_3)}{1 + \theta_1 - \theta_2}, \qquad \nu_1(\theta) = \frac{1 + \theta_1}{1 + \theta_1 - \theta_2}, \tag{8.2}$$

which will play a key role in our analysis. Observe that due to the ordering of the θ_j in the definition (4.7) of T_3, the functions ν_0 and ν_1 are well-defined and continuous everywhere in T_3 except at the point $\theta = (-1, 0, 0)$. Further, we have the bounds $0 \leq \nu_0(\theta_1, \theta_2, \theta_3) \leq 2$ and $0 \leq \nu_1(\theta_1, \theta_2, \theta_3) \leq 1$ throughout $T_3 \setminus \{(-1, 0, 0)\}$, so $\nu_0, \nu_1 \in W_3$, where we recall the definition of W_m in (7.30).

After a short calculation one sees from (7.28) that (bm3)

$$\begin{aligned}
(\mathbf{B}_0\varphi)(\theta) &= \fint_{\theta_1}^{\theta_2} \fint_{\theta_2}^{\theta_3} \Phi_0(t_1, t_2, \theta_1, \theta_2, \theta_3)\varphi(t_1, t_2, 0)\, dt_2\, dt_1, \\
(\mathbf{B}_1\varphi)(\theta) &= \nu_1(\theta)(\widetilde{\mathbf{B}}_1\varphi)(\theta), \qquad \text{where} \\
(\widetilde{\mathbf{B}}_1\varphi)(\theta) &= \fint_{-1}^{\theta_1} \fint_{\theta_1}^{\theta_2} \fint_{\theta_2}^{\theta_3} \Phi_1(t_0, t_1, t_2, \theta_1, \theta_3)\varphi(t_0, t_1, t_2)\, dt_2\, dt_1\, dt_0,
\end{aligned} \tag{8.3}$$

for any $\varphi \in W_3$, and where the kernels Φ_0 and Φ_1 are given by (phiker)

$$\begin{aligned} \Phi_0(t_1, t_2, \theta_1, \theta_2, \theta_3) &= \left(\frac{t_2 - t_1}{\theta_3 - \theta_1}\right) t_1 \left(\frac{t_2}{1 + \theta_1 - \theta_2}\right)(1 + t_1 - t_2), \\ \Phi_1(t_0, t_1, t_2, \theta_1, \theta_3) &= (t_1 - t_0)\left(\frac{t_2 - t_1}{\theta_3 - \theta_1}\right)(t_2 - t_0)(1 + t_0 - t_1). \end{aligned} \tag{8.4}$$

Note that we have grouped like terms in the kernels (8.4), so that each ratio in these formulas is at most +1 in absolute value. In particular, we have that

$$0 \le t_2 - t_1 \le \theta_3 - \theta_1, \qquad 0 \le -t_2 \le 1 + \theta_1 - \theta_2,$$

and so (phibnd)

$$0 \le \Phi_0(t_1, t_2, \theta_1, \theta_2, \theta_3) \le 1, \qquad 0 \le \Phi_1(t_0, t_1, t_2, \theta_1, \theta_3) \le 1, \tag{8.5}$$

as long as $-1 \le t_0 \le \theta_1 \le t_1 \le \theta_2 \le t_2 \le \theta_3 \le 0$. This confirms the conclusion of Proposition 7.6 in the case $m = 3$, in particular that $\mathbf{B}_0$, $\mathbf{B}_1$, and $\mathbf{B}$ define bounded linear operators on W_3. However, in contrast to the case $m = 2$ above, we shall see that here $\mathbf{B}$ is not an operator on $C(T_3)$, as $\mathbf{B}\varphi$ is not in general a continuous function on T_3 even if φ is continuous there. Instead, the following result holds.

Theorem 8.2. *Let $V \subseteq W_3$ denote the two-dimensional vector space spanned by the functions ν_0 and ν_1 in* (8.2)*, and let $C_{0,V} \subseteq C_V \subseteq W_3$ be defined as* (cv)

$$C_V = C(T_3) \oplus V, \qquad C_{0,V} = C_0(T_3) \oplus V, \qquad C_0(T_3) = \{\varphi \in C(T_3) \mid \varphi(-1, 0, 0) = 0\}. \tag{8.6}$$

Then the space C_V is invariant under the operators $\mathbf{B}_0$, $\mathbf{B}_1$, and thus $\mathbf{B}$, and moreover, the ranges of these operators on C_V are contained in $C_{0,V}$. More precisely, if $\varphi \in C_V$ then (49)

$$\begin{aligned} &(\mathbf{B}\varphi)(\theta) = Q_0 \nu_0(\theta) + Q_1 \nu_1(\theta) + \psi(\theta), \\ &Q_0 = \frac{1}{2} \int_{-1}^{0} t^2 (1 + t) \varphi(t, 0, 0)\, dt, \qquad Q_1 = \int_{-1}^{0} t^2 (1 + t) \varphi(-1, t, 0)\, dt, \end{aligned} \tag{8.7}$$

where $\psi \in C_0(T_3)$. Further, we have $Y = C_V$ for the space Y defined in (7.31)*.*

Remark. The above theorem implies that although $\mathbf{B}\varphi$ need not be continuous even if φ is continuous, the discontinuities of $\mathbf{B}\varphi$ can only be of a special form and located at the specific point $(-1, 0, 0)$ on the boundary of T_3. The analogous issue for $m \geq 4$, namely a description or classification of the possible discontinuities that can arise for iterates $\mathbf{B}^k\varphi$ where $\varphi \in C(T_m)$, or more generally a characterization of the space Y, should be relevant to the conjecture stated earlier, as well as being an interesting question in its own right.

A number of preliminary results are needed before proving Theorem 8.2. We begin by examining the continuity properties of $\mathbf{B}_0\varphi$ and $\widetilde{\mathbf{B}}_1\varphi$ in T_3 for $\varphi \in C_V$. For such φ, it is clear from the form (8.4) of the kernels Φ_i and from the formulas (8.2) for ν_0 and ν_1 that the only possible points $\theta \in T_3$ at which $\mathbf{B}_0\varphi$ is discontinuous are where either $\theta_3 - \theta_1 = 0$ or where $1 + \theta_1 - \theta_2 = 0$, and that the only possible points of discontinuity of $\widetilde{\mathbf{B}}_1\varphi$ are where $\theta_3 - \theta_1 = 0$. Note that $\theta_3 - \theta_1 = 0$ for $\theta \in T_3$ if and only if $\theta = (\theta_*, \theta_*, \theta_*)$ for some $\theta_* \in [-1, 0]$. Also note that $1 + \theta_1 - \theta_2 = 0$ for $\theta \in T_3$ if and only if $\theta = (-1, 0, 0)$. The following lemma describes these continuity properties of $\mathbf{B}_0\varphi$ and $\widetilde{\mathbf{B}}_1\varphi$ at these points.

Lemma 8.3. *Let $\varphi \in C_V$. Then the only possible point $\theta \in T_3$ of discontinuity of $\mathbf{B}_0\varphi$ is $\theta = (-1, 0, 0)$, while $\widetilde{\mathbf{B}}_1\varphi$ is continuous throughout T_3, that is, $\widetilde{\mathbf{B}}_1\varphi \in C(T_3)$. Further,* (btrip)

$$\begin{aligned} (\mathbf{B}_0\varphi)(\theta_*, \theta_*, \theta_*) &= \frac{\theta_*^2 \varphi(\theta_*, \theta_*, 0)}{2}, \\ (\widetilde{\mathbf{B}}_1\varphi)(\theta_*, \theta_*, \theta_*) &= \frac{1}{2} \fint_{-1}^{\theta_*} (\theta_* - t)^2 (1 + t - \theta_*) \varphi(t, \theta_*, \theta_*)\, dt, \end{aligned} \tag{8.8}$$

for every $\theta \in [-1, 0]$.

Proof. From the remarks preceeding the statement of the lemma, all that is necessary is to prove continuity of $\mathbf{B}_0\varphi$ and $\widetilde{\mathbf{B}}_1\varphi$ at each point of the form $(\theta_*, \theta_*, \theta_*)$ in T_3. We present only the proof for $\mathbf{B}_0$, as the proof for $\widetilde{\mathbf{B}}_1$ is similar. With $\varphi \in C_V$ fixed, let $\gamma_1 = \theta_2 - \theta_1$ and $\gamma_2 = \theta_3 - \theta_2$, which are nonnegative quantities for $\theta \in T_m$. Making the change of variables $t_1 = \theta_1 + \tau_1\gamma_1$ and $t_2 = \theta_1 + \gamma_1 + \tau_2\gamma_2$ in (8.3), we obtain

$$\begin{aligned} (\mathbf{B}_0\varphi)(\theta) &= \int_0^1 \int_0^1 \Phi_0(\theta_1 + \tau_1\gamma_1,\, \theta_1 + \gamma_1 + \tau_2\gamma_2,\, \theta_1,\, \theta_1 + \gamma_1,\, \theta_1 + \gamma_1 + \gamma_2) \\ &\qquad \times\ \varphi(\theta_1 + \tau_1\gamma_1,\, \theta_1 + \gamma_1 + \tau_2\gamma_2,\, 0)\, d\tau_2\, d\tau_1 \\ &= \int_0^1 \int_0^1 \left(\frac{(1 - \tau_1)\gamma_1 + \tau_2\gamma_2}{\gamma_1 + \gamma_2} \right) S(\tau_1, \tau_2, \theta_1, \gamma_1, \gamma_2) \varphi(\theta_1 + \tau_1\gamma_1,\, \theta_1 + \gamma_1 + \tau_2\gamma_2,\, 0)\, d\tau_2\, d\tau_1 \end{aligned}$$

where

$$S(\tau_1,\tau_2,\theta_1,\gamma_1,\gamma_2) = \frac{(\theta_1+\tau_1\gamma_1)(\theta_1+\gamma_1+\tau_2\gamma_2)(1+(\tau_1-1)\gamma_1-\tau_2\gamma_2)}{1-\gamma_1}.$$

This formula is valid throughout T_3 as long as the coordinates θ_j for $j = 1, 2, 3$ are not all equal and $\theta \neq (-1, 0, 0)$, equivalently as long as $\gamma_1 + \gamma_2 > 0$ and $\gamma_1 \neq 1$. Upon letting $\theta = (\theta_1, \theta_2, \theta_3)$ approach a given point $(\theta_*, \theta_*, \theta_*)$ in T_3 (say, along a sequence), one sees that γ_1 and γ_2 approach 0, hence $S(\tau_1,\tau_2,\theta_1,\gamma_1,\gamma_2)$ approaches θ_*^2 and $\varphi(\theta_1+\tau_1\gamma_1, \theta_1+\gamma_1+\tau_2\gamma_2, 0)$ approaches $\varphi(\theta_*, \theta_*, 0)$, uniformly in the range of integration. The fact that

$$\int_0^1\int_0^1 \left(\frac{(1-\tau_1)\gamma_1+\tau_2\gamma_2}{\gamma_1+\gamma_2}\right) d\tau_2\, d\tau_1 = \frac{1}{2},$$

with an integrand which is bounded uniformly for nonnegative γ_1 and γ_2, implies that

$$(\mathbf{B}_0\varphi)(\theta) \to \frac{\theta_*^2\varphi(\theta_*,\theta_*,0)}{2} = (\mathbf{B}_0\varphi)(\theta_*,\theta_*,\theta_*),$$

where the above equality may be taken as the defnition of $(\mathbf{B}_0\varphi)(\theta_*,\theta_*,\theta_*)$. This gives the first equation in (8.8). We omit the proof of the second equation in (8.8), which is similar. With this, the result is proved. ■

Remark. Although $\gamma_1 \to 0$ and $\gamma_2 \to 0$ in the above proof, there is no assumption about the relative rates at which these quantities converge. Consequently, the ratio $((1-\tau_1)\gamma_1+\tau_2\gamma_2)/(\gamma_1+\gamma_2)$ in the integrand above need not have a pointwise limit in (τ_1,τ_2) as $\gamma_1,\gamma_2 \to 0$.

The next two lemmas give partial information on continuity properties of $\mathbf{B}_0\varphi$ near $(-1, 0, 0)$.

Lemma 8.4. *Let $\varphi \in C_V$ and suppose that $\varphi(\theta_1, 0, 0) \equiv 0$ identically for $\theta_1 \in (-1, 0]$. Then $(\mathbf{B}_0\varphi)(\theta)$ is continuous at each $\theta \in T_3$, that is, $\mathbf{B}_0\varphi \in C(T_3)$. Moreover, $(\mathbf{B}_0\varphi)(-1, 0, 0) = 0$, and thus $\mathbf{B}_0\varphi \in C_0(T_3)$.*

Proof. From Lemma 8.3, the only point at which $\mathbf{B}_0\varphi$ can fail to be continuous is $\theta = (-1, 0, 0)$. Now fix ε satisfying $0 < \varepsilon < 1$. Then if $\theta \in T_3 \setminus \{(-1, 0, 0)\}$ is such that $\theta_1 \leq -1+\varepsilon \leq \theta_2$ and $\theta_1 \neq \theta_2$, we have from the formula (8.3) that

$$(\mathbf{B}_0)(\theta) = \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{-1+\varepsilon}\!\!\!\!\!\!\!-\!\!\!\!\int_{\theta_2}^{\theta_3} \Phi_0(t_1,t_2,\theta_1,\theta_2,\theta_3)\varphi(t_1,t_2,0)\,dt_2\,dt_1$$

$$+\frac{1}{\theta_2-\theta_1}\int_{-1+\varepsilon}^{\theta_2}\!\!\!-\!\!\!\!\!\!\int_{\theta_2}^{\theta_3} \Phi_0(t_1,t_2,\theta_1,\theta_2,\theta_3)\varphi(t_1,t_2,0)\,dt_2\,dt_1.$$

We have thus the bounds

$$\frac{1}{\theta_2-\theta_1}\left|\int_{\theta_1}^{-1+\varepsilon}\fint_{\theta_2}^{\theta_3}\Phi_0(t_1,t_2,\theta_1,\theta_2,\theta_3)\varphi(t_1,t_2,0)\,dt_2\,dt_1\right| \le \left(\frac{-1+\varepsilon-\theta_1}{\theta_2-\theta_1}\right)\|\varphi\| \le \frac{\varepsilon\|\varphi\|}{\theta_2-\theta_1},$$

$$\frac{1}{\theta_2-\theta_1}\left|\int_{-1+\varepsilon}^{\theta_2}\fint_{\theta_2}^{\theta_3}\Phi_0(t_1,t_2,\theta_1,\theta_2,\theta_3)\varphi(t_1,t_2,0)\,dt_2\,dt_1\right| \le \sup_{-1+\varepsilon\le t_1\le\theta_2\le t_2\le 0}|\varphi(t_1,t_2,0)| = \delta_\varepsilon(\theta_2),$$

following from (8.5), where the above equality serves as the definition of $\delta_\varepsilon(\theta_2)$. Note that the point $(-1,0,0)$ is excluded from the region over which the above supremum is taken, and so, with ε fixed, $\delta_\varepsilon(\theta_2)$ depends continuously on θ_2 near $\theta_2=0$ and with $\delta_\varepsilon(0)=0$ from the assumptions on φ. Letting $\theta\to(-1,0,0)$ (say, along a sequence), we obtain

$$\limsup_{\theta\to(-1,0,0)}|(\mathbf{B}_0\varphi)(\theta)| \le \varepsilon\|\varphi\|$$

from the above. As ε can be chosen arbitrarily small, this implies the result. ■

Lemma 8.5. *Let $\varphi\in C_V$ and suppose that $\varphi(\theta_1,\theta_2,0)\equiv\varphi(\theta_1,0,0)$ identically for every $(\theta_1,\theta_2)\in T_2$ with $\theta_1\ne -1$. Then* (nnu1)

$$(\mathbf{B}_0\varphi)(\theta) = Q\nu_0(\theta)+\psi(\theta), \qquad Q=\frac{1}{2}\int_{-1}^{0}t^2(1+t)\varphi(t,0,0)\,dt, \tag{8.9}$$

for some $\psi\in C_0(T_3)$.

Proof. We first claim that (pwr)

$$\fint_{\theta_2}^{\theta_3}(t_2-t_1)t_1t_2(1+t_1-t_2)\,dt_2 = \frac{-1}{2}\Big(t_1^2(1+t_1)(\theta_2+\theta_3)+R(t_1,\theta_2,\theta_3)\Big), \tag{8.10}$$

where the polynomial R satisfies (rest)

$$R(t_1,\theta_2,\theta_3) = O(t_1(\theta_2^2+\theta_3^2)) \tag{8.11}$$

near the origin in $\mathbf{R}^3$. Indeed, this can be readily verified by direct calculation, by expanding the integrand in (8.10) in powers of t_2 about the origin. We omit the details. We next observe that (nu1)

$$(\mathbf{B}_0\varphi)(\theta) = \frac{-1}{2(\theta_3-\theta_1)(1+\theta_1-\theta_2)}\fint_{\theta_1}^{\theta_2}\Big(t_1^2(1+t_1)(\theta_2+\theta_3)+R(t_1,\theta_2,\theta_3)\Big)\varphi(t_1,0,0)\,dt_1, \tag{8.12}$$

which follows immediately from (8.3), (8.4), and (8.10), using the assumption on φ. Denoting

$$\widetilde{\psi}(\theta) = \frac{1}{2(\theta_3-\theta_1)}\int_{\theta_1}^{\theta_2} t_1^2(1+t_1)\varphi(t_1,0,0)\,dt_1,$$

$$\overline{\psi}(\theta) = \frac{-1}{2(\theta_3-\theta_1)(1+\theta_1-\theta_2)}\int_{\theta_1}^{\theta_2} R(t_1,\theta_2,\theta_3)\varphi(t_1,0,0)\,dt_1,$$

we have that $(\mathbf{B}_0\varphi)(\theta) = \nu_0(\theta)\widetilde{\psi}(\theta) + \overline{\psi}(\theta)$.

Certainly $\widetilde{\psi}$ is continuous in a neighborhood of the point $(-1,0,0)$ in T_3. Also, $\overline{\psi}$ is continuous in some neighborhood of $(-1,0,0)$, in fact with $\overline{\psi}(-1,0,0)=0$, in light of the estimate (8.11). Letting $\psi(\theta) = \nu_0(\theta)[\widetilde{\psi}(\theta) - Q] + \overline{\psi}(\theta)$ where $Q = \widetilde{\psi}(-1,0,0)$, we have that (8.9) holds. Also, ψ is continuous in some neighborhood of $(-1,0,0)$ in T_3, and $\psi(-1,0,0)=0$, where the boundedness of ν_0 near that point is used. Further, $\mathbf{B}_0\varphi$ and ν_0 are continuous at every point of $T_3\setminus\{(-1,0,0)\}$, using Lemma 8.3, so ψ is also continuous there, by (8.9). Thus $\psi \in C_0(T_3)$, as claimed. ■

Proposition 8.6. *Let $\varphi \in C_V$. Then $\mathbf{B}_0\varphi$ has the form* (8.9) *where $\psi \in C_0(T_3)$, with Q as in that formula.*

Proof. With $\varphi \in C_V$, write $\varphi(\theta) = \widetilde{\varphi}(\theta) + \overline{\varphi}(\theta)$ where

$$\widetilde{\varphi}(\theta_1,\theta_2,\theta_3) = \varphi(\theta_1,\theta_2,\theta_3) - \varphi(\theta_1,\theta_3,\theta_3), \qquad \overline{\varphi}(\theta_1,\theta_2,\theta_3) = \varphi(\theta_1,\theta_3,\theta_3).$$

Then $\widetilde{\varphi}$ and $\overline{\varphi}$ satisfy the conditions of Lemmas 8.4 and 8.5, respectively. Further, $\overline{\varphi}(\theta_1,0,0) \equiv \varphi(\theta_1,0,0)$ identically on $(-1,0]$. The result follows immediately. ■

Proof of Theorem 8.2. Let $\varphi \in C_V$. We have that $(\mathbf{B}_0\varphi)(\theta) = Q_0\nu_0(\theta) + \psi_0(\theta)$ with Q_0 as in (8.7) and where $\psi_0 \in C_0(T_3)$, by Proposition 8.6. Also, by Lemma 8.3 we have that $\widetilde{\mathbf{B}}_1\varphi \in C(T_3)$, and from (8.3) and (8.4) we have that $(\widetilde{\mathbf{B}}_1\varphi)(-1,0,0) = Q_1$ for Q_1 as in (8.7). Thus upon letting $\psi_1(\theta) = \nu_1(\theta)[(\widetilde{\mathbf{B}}_1\varphi)(\theta) - Q_1]$, we have from (8.3) that $(\mathbf{B}_1\varphi)(\theta) = Q_1\nu_1(\theta) + \psi_1(\theta)$, so upon letting $\psi(\theta) = \psi_0(\theta) + \psi_1(\theta)$ we have (8.7). The fact that $\psi \in C_0(T_3)$, or equivalently, that $\psi_1 \in C_0(T_3)$, follows directly from the definition of ψ_1 using the continuity of $\widetilde{\mathbf{B}}_1\varphi$ and the choice of Q_1.

To prove the final claim in the statement of the theorem, it is enough to show that every pair of numbers $Q_0, Q_1 \in \mathbf{R}$ as in (8.7) can be achieved for some $\varphi \in C(T_3)$. However, this follows easily from the explicit formulas (8.7) for Q_0 and Q_1. ■

Lemma 8.7. *Let* $\varphi \in C_{0,V}$ *have the form*

$$\varphi(\theta) = Q_0\nu_0(\theta) + Q_1\nu_1(\theta) + \psi(\theta)$$

for $\theta \in T_3 \setminus \{(-1,0,0)\}$ *where* $\psi \in C_0(T_3)$. *Assume that* $\varphi(\theta) > 0$ *for every* $\theta \in T_3 \setminus \{(-1,0,0)\}$ *and also that* $Q_0 > 0$ *and* $Q_1 > 0$. *Then there exists* $M > 0$ *such that* (79m)

$$\varphi(\theta) \geq M \tag{8.13}$$

for every $\theta \in T_3 \setminus \{(-1,0,0)\}$.

Proof. Denote $\theta_0 = (-1,0,0)$. Then for any r satisfying $0 < r \leq 1$, let

$$\delta(r) = \inf_{|\theta-\theta_0|\geq r} \varphi(\theta), \qquad \varepsilon(r) = \sup_{0\leq|\theta-\theta_0|\leq r} |\psi(\theta)|,$$

where $|\cdot|$ denotes the euclidean distance in $\mathbf{R}^3$ and $\theta \in T_3$. Since φ is continuous and positive throughout $T_m \setminus \{\theta_0\}$, it follows that $\delta(r)$ is positive and depends continuously on r. Also, since ψ is continuous in T_3 and $\psi(\theta_0) = 0$, we have that $\varepsilon(r)$ also depends continuously on r, and that $\lim_{r\to 0}\varepsilon(r) = 0$. (We note that it need not be the case that the function ψ is nonnegative everywhere.) Therefore,

$$\begin{aligned}\inf_{0<|\theta-\theta_0|\leq r} \varphi(\theta) &\geq \inf_{0<|\theta-\theta_0|\leq r} \Big(Q_0\nu_0(\theta) + Q_1\nu_1(\theta)\Big) - \varepsilon(r) \\ &= \inf_{0<|\theta-\theta_0|\leq r} \left(\frac{-Q_0(\theta_2+\theta_3) + Q_1(1+\theta_1)}{1+\theta_1-\theta_2}\right) - \varepsilon(r) \geq \min\{2Q_0, Q_1\} - \varepsilon(r).\end{aligned}$$

By choosing r sufficiently small that $\varepsilon(r) < \min\{2Q_0, Q_1\}$, one sees immediately that the desired inequality (8.13) holds throughout $T_m \setminus \{\theta_0\}$ with $M = \min\{\delta(r), \min\{2Q_0, Q_1\} - \varepsilon(r)\}$. ∎

Define the set

$$C_V^+ = \{\varphi \in C_V \mid \varphi(\theta) \geq 0 \text{ for every } \theta \in T_3 \setminus \{(-1,0,0)\}\},$$

which is a closed, convex cone in C_V. The following result is the analog of Proposition 8.1 for $m = 3$.

Proposition 8.8. *Let* $\varphi \in C_V^+ \setminus \{0\}$. *Then for every* $k \geq 5$ *there exists* $M_k > 0$ *such that* $(\mathbf{B}^k\varphi)(\theta) \geq M_k$ *for every* $\theta \in T_3 \setminus \{(-1,0,0)\}$. *Thus the operator* $\mathbf{B}$ *with* $m = 3$ *and acting on* C_V *is* u_0*-positive with respect to the cone* C_V^+, *where* $u_0(\theta) \equiv 1$ *identically on* T_3.

Proof. Just as in the proof of Proposition 8.1, it is sufficient here only to prove the existence of M_5. Toward this end, let us first define the sets

$$\begin{aligned} L_0 &= T_3 \setminus \{(-1,0,0)\}, \\ L_1 &= \{\theta \in T_3 \setminus \{(-1,0,0)\} \mid \theta_3 = 0\}, \\ L_2 &= \{\theta \in T_3 \setminus \{(-1,0,0)\} \mid \theta_2 = \theta_3 = 0\}, \\ L_3 &= \{(0,0,0)\}, \end{aligned}$$

observing that $L_3 \subseteq L_2 \subseteq L_1 \subseteq L_0$. Also define

$$L_\delta = \{\theta \in T_3 \setminus \{(-1,0,0)\} \mid \theta_2 \in [-\delta, 0) \text{ and } \theta_3 = 0\}$$

for any $\delta > 0$. We claim the following facts hold for every $\varphi \in C_V^+$.

(1) If $\varphi(\theta) > 0$ for some $\theta \in L_0$, then $(\mathbf{B}\varphi)(\widetilde{\theta}) > 0$ for some $\widetilde{\theta} \in L_1$;

(2) if $\varphi(\theta) > 0$ for some $\theta \in L_1$, then $(\mathbf{B}\varphi)(\widetilde{\theta}) > 0$ for some $\widetilde{\theta} \in L_2$;

(3) if $\varphi(\theta) > 0$ for some $\theta \in L_2$, then there exists $\delta > 0$ such that $(\mathbf{B}\varphi)(\widetilde{\theta}) > 0$ for every $\widetilde{\theta} \in L_\delta$;

(4) if there exists $\delta > 0$ such that $\varphi(\theta) > 0$ for every $\theta \in L_\delta$, then $(\mathbf{B}\varphi)(\widetilde{\theta}) > 0$ for every $\widetilde{\theta} \in L_1 \setminus L_2$; and

(5) if $\varphi(\theta) > 0$ for every $\theta \in L_1 \setminus L_2$, then $(\mathbf{B}\varphi)(\widetilde{\theta}) > 0$ for every $\widetilde{\theta} \in L_0 \setminus L_2$.

Additionally, we claim the following facts hold for every $\varphi \in C_V^+$.

(3′) If $\varphi(\theta) > 0$ for some $\theta \in L_2$, then $(\mathbf{B}\varphi)(\widetilde{\theta}) > 0$ for $\widetilde{\theta} = (0,0,0)$, that is, for $\widetilde{\theta} \in L_3$; and

(5′) if $\varphi(\theta) > 0$ for every $\theta \in L_1 \setminus L_2$, then $(\mathbf{B}\varphi)(\widetilde{\theta}) > 0$ for every $\widetilde{\theta} \in L_2 \setminus L_3$.

If one accepts the above facts then it is immediate from (1)–(5) that if $\varphi \in C_V^+ \setminus \{0\}$ then $(\mathbf{B}^5\varphi)(\theta) > 0$ for every $\theta \in L_0 \setminus L_2$. (Note that if $\varphi \in C_V^+ \setminus \{0\}$ then $\varphi(\theta) > 0$ for some $\theta \in L_0$.) It is also immediate from (1)–(4) and (5′) that if $\varphi \in C_V^+ \setminus \{0\}$ then $(\mathbf{B}^5)(\theta) > 0$ for every $\theta \in L_2 \setminus L_3$ and thus for every $\theta \in (L_0 \setminus L_2) \cup (L_2 \setminus L_3) = L_0 \setminus L_3 = L_0 \setminus \{(0,0,0)\}$. Finally, one sees that if $\varphi \in C_V^+ \setminus \{0\}$ then also $\mathbf{B}^2\varphi \in C_V^+ \setminus \{0\}$, and using (1), (2), and (3′) one concludes that $(\mathbf{B}^5\varphi)(\theta) > 0$ at $\theta = (0,0,0)$. One therefore concludes that if $\varphi \in C_V^+ \setminus \{0\}$, then $(\mathbf{B}^5)(\theta) > 0$ for every $\theta \in L_0 = T_3 \setminus \{(-1,0,0)\}$.

Now fixing any $\varphi \in C_V^+ \setminus \{0\}$, write

$$(\mathbf{B}^5\varphi)(\theta) = Q_0\nu_0(\theta) + Q_1\nu_1(\theta) + \psi(\theta)$$

with $\psi \in C_0(T_3)$ as per Theorem 8.2. Then

$$Q_0 = \frac{1}{2}\int_{-1}^{0} t^2(1+t)[(\mathbf{B}^4\varphi)(t,0,0)]\,dt, \qquad Q_1 = \int_{-1}^{0} t^2(1+t)[(\mathbf{B}^4\varphi)(-1,t,0)]\,dt.$$

As $\mathbf{B}^2\varphi \in C_V^+ \setminus \{0\}$, one has from (1) and (2) that $(\mathbf{B}^4\varphi)(\theta) > 0$ for some $\theta \in L_2$, that is, for some $\theta = (t,0,0)$ with $t \in (-1,0]$. Thus $Q_0 > 0$. Similarly, as $\mathbf{B}\varphi \in C_V^+ \setminus \{0\}$, one has from (1)–(3) that there exists $\delta > 0$ such that $(\mathbf{B}^4\varphi)(\theta) > 0$ for every $\theta \in L_\delta$, and in particular for every $\theta = (-1,t,0)$ with $t \in [-\delta, 0)$. Thus $Q_1 > 0$. With this, the existence of a uniform lower bound M_5 for $\mathbf{B}^5\varphi$ follows directly from Lemma 8.7.

There remains to establish the properties (1)–(5) and (3′) and (5′). For the most part, these follow rather straightforwardly from the formulas (8.3), (8.4) for $\mathbf{B}_0\varphi$ and $\mathbf{B}_1\varphi$. Let us write (8.3) as (bm33)

$$\begin{aligned}(\mathbf{B}_0\varphi)(\widetilde{\theta}) &= \fint_{\widetilde{\theta}_1}^{\widetilde{\theta}_2}\fint_{\widetilde{\theta}_2}^{\widetilde{\theta}_3} \Phi_0(t_1,t_2,\widetilde{\theta}_1,\widetilde{\theta}_2,\widetilde{\theta}_3)\varphi(t_1,t_2,0)\,dt_2\,dt_1,\\ (\widetilde{\mathbf{B}}_1\varphi)(\widetilde{\theta}) &= \nu_1(\widetilde{\theta})\fint_{-1}^{\widetilde{\theta}_1}\fint_{\widetilde{\theta}_1}^{\widetilde{\theta}_2}\fint_{\widetilde{\theta}_2}^{\widetilde{\theta}_3} \Phi_1(t_0,t_1,t_2,\widetilde{\theta}_1,\widetilde{\theta}_3)\varphi(t_0,t_1,t_2)\,dt_2\,dt_1\,dt_0,\end{aligned} \tag{8.14}$$

using the variable $\widetilde{\theta} = (\widetilde{\theta}_1,\widetilde{\theta}_2,\widetilde{\theta}_3)$. We recall that the arguments of these integrals are nonnegative throughout the range of integration, and so it is enough to prove for each of the indicated $\widetilde{\theta}$ in the above claimed properties, that either $(\mathbf{B}_0\varphi)(\widetilde{\theta}) > 0$ or $(\mathbf{B}_1\varphi)(\widetilde{\theta}) > 0$. Generally, this will be done by exhibiting a point (t_1,t_2) or (t_0,t_1,t_2) in the range of integration at which the integrand is strictly positive. As we are taking average integrals, it will not matter if the upper and lower limits of an integral are equal.

To prove (1), we assume that $\varphi(\theta) > 0$ for some $\theta = (\theta_1,\theta_2,\theta_3) \in L_0$, and without loss we may assume that (spaced)

$$-1 < \theta_1 < \theta_2 < \theta_3 < 0 \tag{8.15}$$

as φ is continuous on L_0. Now letting $\widetilde{\theta} = (\widetilde{\theta}_1,\widetilde{\theta}_2,\widetilde{\theta}_3) = (\theta_1,\theta_2,0) \in L_1$, one sees directly that $(\mathbf{B}_1\varphi)(\widetilde{\theta}) > 0$. In particular, the relevant integrand in (8.14) is strictly positive at the point $(t_0,t_1,t_2) = (\theta_1,\theta_2,\theta_3)$ which lies within the range of integration, as we have from (8.4) that

$$\Phi_1(\theta_1,\theta_2,\theta_3,\theta_1,0)\varphi(\theta_1,\theta_2,\theta_3) = (\theta_2-\theta_1)\left(\frac{\theta_3-\theta_2}{-\theta_1}\right)(\theta_3-\theta_1)(1+\theta_1-\theta_2)\varphi(\theta_1,\theta_2,\theta_3) > 0.$$

Observe that the assumptions (8.15) are used in drawing this conclusion. Additionally, $\nu_1(\widetilde{\theta}) > 0$ as $\theta_1 > -1$. With this (1) is established.

The proof of (2) is similar. We assume that $\varphi(\theta) > 0$ for some $\theta = (\theta_1, \theta_2, 0) \in L_1$, and without loss $-1 < \theta_1 < \theta_2 < \theta_3 = 0$. Letting $\widetilde{\theta} = (\theta_1, 0, 0) \in L_2$, one sees that the integrand of the second integral in (8.14) is positive at $(t_0, t_1, t_2) = (\theta_1, \theta_2, 0)$ and also $\nu_1(\widetilde{\theta}) > 0$ as before. Thus again $(\mathbf{B}_1\varphi)(\widetilde{\theta}) > 0$, and (2) is proved.

The proof of (3) is slightly different from the proofs of (1) and (2). First, assuming that $\varphi(\theta) > 0$ for some $\theta \in L_2$, we may assume that $\theta = (\theta_1, 0, 0)$ where $-1 < \theta_1 < 0$. Further, by continuity, there exists $\delta > 0$ such that $\varphi(\theta_1, \gamma, 0) > 0$ for every $\gamma \in [-\delta, 0]$, and where also $\theta_1 < -\delta$. Now let any point $\widetilde{\theta} \in L_\delta$ be given, that is, $\widetilde{\theta} = (\widetilde{\theta}_1, \widetilde{\theta}_2, 0)$ where $-1 \le \widetilde{\theta}_1 \le \widetilde{\theta}_2 < 0$ and also $-\delta \le \widetilde{\theta}_2 < 0$. Two cases now arise. First, suppose that $\widetilde{\theta}_1 \ge \theta_1$. Then as in the proofs of (1) and (2), one shows that $(\mathbf{B}_1\varphi)(\widetilde{\theta}) > 0$ by noting that the point $(t_0, t_1, t_2) = (\theta_1, \widetilde{\theta}_2, 0)$ lies in the domain of integration and the relevant integrand is positive there, and again that $\nu_1(\widetilde{\theta}) > 0$. The fact that (thinq)

$$\theta_1 < -\delta \le \widetilde{\theta}_2 < 0, \tag{8.16}$$

in particular, is used here. For the second case we assume that $\widetilde{\theta}_1 \le \theta_1$, and here we show that $(\mathbf{B}_0\varphi)(\widetilde{\theta}) > 0$. Indeed, the relevant integrand is strictly positive at $(t_1, t_2) = (\theta_1, \widetilde{\theta}_2)$, again because of (8.16). This establishes (3). **NEED TO FINISH PROOF.** ∎

Proof of Theorem 7.3. Given any $\varphi \in C(T_m)^+ \setminus \{0\}$, then by (7.24) of Proposition 7.5 we have the upper bound $\mathbf{A}^k\varphi \le 2^k \|\varphi\| u_m$ (here the order is with respect to the cone $C(T_m)^+$) for every $k \ge m-1$, and in particular for every $k \ge 3$ if $m = 2$ and for every $k \ge 5$ if $m = 3$.

To obtain a lower bound for $\mathbf{A}^k\varphi$, fix $\psi \in C(T_m)^+ \setminus \{0\}$ satisfying $\psi \le \varphi$ and such that the support of the function ψ is contained in the set O_m. Then upon defining

$$\zeta(\theta) = \frac{\psi(\theta)}{u_m(\theta)} \qquad \text{for } \theta \in T_m,$$

we have that $\zeta \in C(T_m)^+ \setminus \{0\}$. Moreover, we have directly from the formulas (7.3) and (7.28) defining $\mathbf{A}$ and $\mathbf{B}$ that

$$(\mathbf{B}^k\zeta)(\theta) = \frac{(\mathbf{A}^k\psi)(\theta)}{u_m(\theta)} \le \frac{(\mathbf{A}^k\varphi)(\theta)}{u_m(\theta)}$$

for every $k \ge 1$ and for $\theta \in O_m$. From this it follows, by Proposition 8.1 in the case $m = 2$, and by Proposition 8.8 in the case $m = 3$, that we have the lower bounds (ab)

$$M_k u_m(\theta) \leq (\mathbf{A}^k \varphi)(\theta) \tag{8.17}$$

for every $k \geq 3$ if $m = 2$ and for every $k \geq 5$ if $m = 3$. The bounds (8.17) are valid for θ in the interior of T_m, and thus for every $\theta \in T_m$ as the functions involved are continuous in T_m.

We conclude that $\mathbf{A}^k \varphi \sim u_m$ for every $k \geq 3$ if $m = 2$ and for every $k \geq 5$ if $m = 2$, as desired. ■

Proof of Theorem 7.1. This follows directly from Proposition 7.2 and Theorem 7.3. ■

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