ADMISSIBLE ARRAYS AND A NONLINEAR GENERALIZATION OF PERRON–FROBENIUS THEORY

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ABSTRACT

Let $K^n = \{x \in \mathbb{R}^n : x_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant n\}$ and suppose that $f: K^n \longrightarrow K^n$ is nonexpansive with respect to the ℓ_1-norm and $f(0) = 0$. It is known that for every $x \in K^n$ there exists a periodic point $\xi = \xi_x \in K^n$ (so $f^p(\xi) = \xi$ for some minimal positive integer $p = p_\xi$) and $f^k(x)$ approaches $\{f^j(\xi) : 0 \leqslant j < p\}$ as k approaches infinity. What can be said about $P^*(n)$, the set of positive integers p for which there exists a map f as above and a periodic point $\xi \in K^n$ of f of minimal period p? If f is linear (so that f is a nonnegative, column stochastic matrix) and $\xi \in K^n$ is a periodic point of f of minimal period p, then, by using the Perron–Frobenius theory of nonnegative matrices, one can prove that p is the least common multiple of a set S of positive integers the sum of which equals n. Thus the paper considers a nonlinear generalization of Perron–Frobenius theory. It lays the groundwork for a precise description of the set $P^*(n)$. The idea of admissible arrays on n symbols is introduced, and these arrays are used to define, for each positive integer n, a set of positive integers $Q(n)$ determined solely by arithmetical and combinatorial constraints. The paper also defines by induction a natural sequence of sets $P(n)$, and it is proved that $P(n) \subset P^*(n) \subset Q(n)$. The computation of $Q(n)$ is highly nontrivial in general, but in a sequel to the paper $Q(n)$ and $P(n)$ are explicitly computed for $1 \leqslant n \leqslant 50$, and it is proved that $P(n) = P^*(n) = Q(n)$ for $n \leqslant 50$, although in general $P(n) \neq Q(n)$. A further sequel to the paper (with Sjoerd Verduyn Lunel) proves that $P^*(n) = Q(n)$ for all n. The results in the paper generalize earlier work by Nussbaum and Scheutzow and place it in a coherent framework.

1. *Admissible arrays, periodic points and lower semi-lattices*

If D is a subset of $\mathbb{R}^n$, a map $f: D \longrightarrow \mathbb{R}^n$ is called *nonexpansive with respect to the ℓ_1-norm* or *ℓ_1-nonexpansive* if, for all $x, y \in D$, one has

$$\|f(x) - f(y)\|_1 \leqslant \|x - y\|_1,$$

where

$$\|z\|_1 := \sum_{i=1}^{n} |z_i| \quad \text{and} \quad z = (z_1, z_2, \ldots, z_n).$$

If $D \subset \mathbb{R}^n$ is closed and $f: D \longrightarrow D$ is ℓ_1-nonexpansive and there exists $\eta \in D$ such that $\sup\{\|f^j(\eta)\|_1 : j \geqslant 1\} < \infty$, then results of Akcoglu and Krengel [**1**] imply that, for every $x \in D$, there exists a positive integer $p_x = p$ and a point $\xi_x = \xi \in D$ with

$$\lim_{j\to\infty} f^{jp}(x) = \xi \quad \text{and} \quad f^p(\xi) = \xi. \tag{1.1}$$

Here f^k denotes the composition of f with itself k times. Related results for 'polyhedral norms' have been obtained by D. Weller [**16**], R. D. Nussbaum [**8**], R. Sine [**15**], R. N. Lyons and R. D. Nussbaum [**5**], P. Martus [**6**] and S.-K. Lo [**4**].

In general, if D is a topological space and $g: D \to D$ is a map, then we say that $\xi \in D$ is a periodic point of g of minimal period p if $g^p(\xi) = \xi$ and $g^j(\xi) \neq \xi$ for $0 < j$

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$< p$. If ξ has minimal period p and $g^m(\xi) = \xi$, then it is well known that $p \mid m$. As is suggested by equations (1.1), the periodic points of an ℓ_1-nonexpansive map $f: D \subset \mathbb{R}^n \longrightarrow D$ play a central role in the understanding of the dynamics of the discrete dynamical system $x \to f^k(x)$, $k \geqslant 0$.

For general sets $D \subset \mathbb{R}^n$, very little is known about the possible periods of periodic points of ℓ_1-nonexpansive maps $f: D \to D$. This is related to the fact that f may not have an extension $F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which is ℓ_1-nonexpansive (see [**17**]). However, if

$$K^n := \{x \in \mathbb{R}^n : x_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant n\}, \tag{1.2}$$

$f: K^n \longrightarrow K^n$ is ℓ_1-nonexpansive and $f(0) = 0$, then Akcoglu and Krengel [**1**] have proved that the minimal period p of any periodic point of f satisfies $p \leqslant n!$, and Scheutzow [**13**] has shown that $p \leqslant \operatorname{lcm}(1, 2, \ldots, n)$, where $\operatorname{lcm}(1, 2, \ldots, n)$ denotes the least common multiple of the integers $1, 2, \ldots, n$. In [**9**, **11**], Nussbaum established various other constraints on possible periods p, defined a function $\phi(n)$ with $p \leqslant \phi(n)$, computed $\phi(n)$ for $n \leqslant 32$, and proved that $\phi(n)$ is a best possible upper bound for p for $n \leqslant 32$.

Our goal in this section is to introduce the idea of an admissible array on n symbols, and to use admissible arrays to give generalizations of the constraints on the minimal period p which were obtained in [**9**]. First, we need to introduce some notation and recall some further results from the literature. If K^n is given by expression (1.2), K^n induces a partial ordering on $\mathbb{R}^n$ by

$$x \leqslant y \quad \text{if and only if } x_i \leqslant y_i \text{ for } 1 \leqslant i \leqslant n,$$

where x_i and y_i denote the coordinates of x and y, respectively. We write $x < y$ if $x \leqslant y$ and $x \neq y$, and we write $x \ll y$ if $x_i < y_i$ for $1 \leqslant i \leqslant n$. We use the notation $x \nleqslant y$ to mean that it is false that $x \leqslant y$, and we say that $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ are *incomparable* or *not comparable* if $x \nleqslant y$ and $y \nleqslant x$. A map $f: D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is *order-preserving* if $f(x) \leqslant f(y)$ for all $x, y \in D$ with $x \leqslant y$. If $f_i(x)$ denotes the i-coordinate of $f(x)$, then f is called *integral-preserving* if

$$\sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} x_i$$

for all $x \in D$. If $D = K^n$ or $D = \mathbb{R}^n$, and $f: D \longrightarrow D$ is integral-preserving, then results of Crandall and Tartar [**3**] imply that f is order-preserving if and only if f is nonexpansive with respect to the ℓ_1-norm.

DEFINITION 1.1. Let $u = (1, 1, \ldots, 1) \in \mathbb{R}^n$. If $f: K^n \longrightarrow K^n$, we write $f \in \mathscr{F}(n)$ if and only if the following hold:

(i) $f(\lambda u) = \lambda u$ for all $\lambda \geqslant 0$.
(ii) The map f is order-preserving.
(iii) The map f is integral-preserving.

DEFINITION 1.2. If $f: K^n \longrightarrow K^n$, we write $f \in \mathscr{G}(n)$ if and only if the following hold:

(i) $f(0) = 0$.
(ii) The map f is nonexpansive with respect to the ℓ_1-norm.

The results of Crandall and Tartar [**3**] imply that $\mathscr{F}(n) \subset \mathscr{G}(n)$. Note that both $\mathscr{F}(n)$ and $\mathscr{G}(n)$ are closed under composition of functions.

DEFINITION 1.3. If p is a positive integer, we write $p \in \hat{P}(n)$ if and only if there exist $f \in \mathscr{F}(n)$ and a periodic point $\xi \in K^n$ of f of minimal period p. We write $p \in P^*(n)$ if and only if there exist $f \in \mathscr{G}(n)$ and a periodic point $\xi \in K^n$ of f of minimal period p.

Because $\mathscr{F}(n) \subset \mathscr{G}(n)$, we know that $\hat{P}(n) \subset P^*(n)$. If S_n denotes the symmetric group on n symbols and σ is a permutation in S_n, then σ induces a linear map $\hat{\sigma}: \mathbb{R}^n \longrightarrow \mathbb{R}^n, \sigma \mid K^n \in \mathscr{F}(n)$, and it is easy to see that $\xi = (1, 2, \ldots, n)$ is a periodic point of minimal period p equal to the order of σ in the finite group S_n. Thus $\hat{P}(n)$ contains the set of all orders of elements of S_n. However, as Theorem 1.1 shows, $\hat{P}(n)$ is, in general, larger than the set of orders of elements of S_n. In Theorem 1.1 and throughout this paper, $\operatorname{lcm}(S)$ denotes the least common multiple of a set of integers S, and $\gcd(S)$ denotes the greatest common divisor of S.

THEOREM 1.1 [**9**, Section 3]. *If* $p_1 \in \hat{P}(n_1)$ *and* $p_2 \in \hat{P}(n_2)$, *then* $\operatorname{lcm}(p_1, p_2) \in \hat{P}(n_1 + n_2)$. *If* $p_i \in \hat{P}(m)$ *for* $1 \leqslant i \leqslant r$, *then* $r \operatorname{lcm}(p_1, p_2, \ldots, p_r) \in \hat{P}(rm)$.

By using Theorem 1.1, and recalling that $2 \in \hat{P}(3)$ and $3 \in \hat{P}(3)$, we see that $12 = 2 \operatorname{lcm}(2,3) \in \hat{P}(6)$, but every element of S_6 has order $p \leqslant 6$.

We leave to the reader the verification of the fact that $P^*(1) = \{1\}$, that $P^*(n) \subset P^*(n+1)$ for $n \geqslant 1$, and that, if $p \in P^*(n)$ and $d \mid p$, then $d \in P^*(n)$. By using Theorem 1.1 and the fact that $\hat{P}(1) = \{1\}$, one can also see that $\hat{P}(n) \subset \hat{P}(n+1)$ for all $n \geqslant 1$ and that, if $p \in \hat{P}(n)$ and $d \mid p$, then $d \in \hat{P}(n)$. (To see that if $p \in P^*(n)$ (respectively $\hat{P}(n)$) and $p = dm$ for positive integers d and m, then $d \in P^*(n)$ (respectively $\hat{P}(n)$), note that, if $f \in \mathscr{G}(n)$ (respectively $\mathscr{F}(n)$) and $f^p(\xi) = \xi$, then $f^m = g \in \mathscr{G}(n)$ (respectively $\mathscr{F}(n)$) and $g^d(\xi) = \xi$).

DEFINITION 1.4. We define inductively, for each $n \geqslant 1$, a collection of positive integers $P(n)$ by $P(1) = \{1\}$ and, for $n > 1$, $p \in P(n)$ if and only if one of the following holds:

(i) $p = \operatorname{lcm}(p_1, p_2)$, where $p_1 \in P(n_1)$, $p_2 \in P(n_2)$ and n_1 and n_2 are positive integers with $n = n_1 + n_2$.

(ii) $n = rm$ for integers $r > 1$ and $m \geqslant 1$ and $p = r \operatorname{lcm}(p_1, p_2, \ldots, p_r)$, where $p_i \in P(m)$ for $1 \leqslant i \leqslant r$.

By using Definition 1.4(i) with $n_1 = n - 1$ and $n_2 = 1$, we see that $P(n-1) \subset P(n)$, and property (ii) with $r = n$ shows that $n \in P(n)$, so $\{1, 2, \ldots, n\} \subset P(n)$. In [**12**], the sets $P(n)$ have been computed explicitly for $n \leqslant 50$. By using Theorem 1.1, we also see that

$$P(n) \subset \hat{P}(n) \subset P^*(n)$$

so $P(n)$ provides a 'lower bound' for $\hat{P}(n)$ and $P^*(n)$.

We now use results in [**9**, **13**] to obtain an 'upper bound' for $P^*(n)$. If $x, y \in \mathbb{R}^n$, we define $x \wedge y$ and $x \vee y$ in the standard way:

$$x \wedge y := z \in \mathbb{R}^n \quad \text{and} \quad x \vee y = w \in \mathbb{R}^n,$$

where $z_i = \min\{x_i, y_i\}$ and $w_i = \max\{x_i, y_i\}$ for $1 \leqslant i \leqslant n$. If $V \subset \mathbb{R}^n$, then V is called a *lower semilattice* if $x \wedge y \in V$ whenever $x \in V$ and $y \in V$. We call V a *lattice* if $x \wedge y \in V$ and $x \vee y \in V$ whenever $x \in V$ and $y \in V$. We always denote the cardinality of a set V by $|V|$.

A *finite lower semilattice* (respectively *finite lattice*) is a lower semilattice (respectively lattice) with finite cardinality. If $A \subset \mathbb{R}^n$, then there is a minimal (in the sense of set inclusion) lower semilattice $V \supset A$ and a minimal lattice $Y \supset A$. We call V the lower semilattice generated by A, and Y the lattice generated by A. If $|A| < \infty$, then it follows that $|V| < \infty$ and $|Y| < \infty$. If V is a lower semilattice, then a map h: $V \longrightarrow V$ is called a *lower semilattice homomorphism of* V if

$$h(x \wedge y) = h(x) \wedge h(y) \quad \text{for all } x, y \in V.$$

If Y is a lattice, then a map $h: Y \longrightarrow Y$ is a lattice homomorphism if $h(x \wedge y) = h(x) \wedge h(y)$ and $h(x \vee y) = h(x) \vee h(y)$ for all $x, y \in Y$. If $W \subset \mathbb{R}^n$ is a lower semilattice (respectively lattice), $h: W \longrightarrow W$ is a semilattice (respectively lattice) homomorphism of W and $\xi \in W$ is a periodic point of minimal period p of h, then let V denote the finite lower semilattice (respectively lattice) generated by $A = \{h^j(\xi): 0 \leqslant j < p\}$. It then follows that $h(V) \subset V$ and $h^p(x) = x$ for all $x \in V$. In particular, $h|V$ is a lower semilattice homomorphism (respectively lattice homomorphism), $h|V$ is one–one and onto V, and $(h|V)^{-1} = h^{p-1}|V$ is also a semilattice (respectively lattice) homomorphism of V.

The relevance of these ideas to our situation is indicated by the following theorems.

THEOREM 1.2 [**13**]. *Suppose that* $f \in \mathscr{G}(n)$ (*see Definition* 1.2) *and that* $\xi \in K^n$ *is a periodic point of* f *of minimal period* p. *Let* $A = \{f^j(\xi): 0 \leqslant j < p\}$ *and let* V *denote the finite lower semilattice generated by* A. *Then* $f(V) \subset V$, $g := f|V$ *is a lower semilattice homomorphism of* V, $f^p(x) = x$ *for all* $x \in V$, *and* $g^{-1} = f^{p-1}|V$ *is a lower semilattice homomorphism of* V.

In Theorem 1.3, recall that a norm $\|\cdot\|$ on $\mathbb{R}^n$ is called *strictly monotonic* if $\|x\| < \|y\|$ whenever $0 \leqslant x < y$. The ℓ_p-norms are strictly monotonic for $1 \leqslant p < \infty$; the ℓ_∞ norm is not strictly monotonic

THEOREM 1.3 [**10**, Proposition 2.1]. Suppose that $f: K^n \longrightarrow K^n$ *is an order-preserving map with* $f(0) = 0$ *and that* f *is nonexpansive with respect to a strictly monotonic norm* $\|\cdot\|$ (*so* $\|f(x) - f(y)\| \leqslant \|x - y\|$ *for all* $x, y \in K^n$). *Assume that* $\xi \in K^n$ *is a periodic point of* f *of period* p, *let* $A = \{f^j(\xi): 0 \leqslant j < p\}$, *and define* L *to be the lattice generated by* A. *It then follows that* $f(L) \subset L$, $f|L$ *is a lattice homomorphism*, $f^p(x) = x$ *for all* $x \in L$, *and* $(f|L)^{-1} = f^{p-1}|L$ *is a lattice homomorphism.*

DEFINITION 1.5. If $f: D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we write $f \in \mathscr{H}(n)$ (respectively $f \in \mathscr{J}(n)$) if and only if D is a lower semilattice (respectively lattice), $f(D) \subset D$, and f is a lower semilattice homomorphism (respectively lattice homomorphism) of D.

DEFINITION 1.6. If p is a positive integer, we write $p \in Q^*(n)$ (respectively $p \in \tilde{Q}(n)$) if and only if there exist $f \in \mathscr{H}(n)$ (respectively $f \in \mathscr{J}(n)$) and a periodic point ξ of f of minimal period p.

Obviously, $\tilde{Q}(n) \subset Q^*(n)$, and Theorem 1.2 implies that $P^*(n) \subset Q^*(n)$. Theorem 1.3 implies that $\hat{P}(n) \subset \tilde{Q}(n)$.

We need to recall some further definitions concerning lower semilattices and lattices. If W is a lower semilattice in $\mathbb{R}^n$ and $A \subset W$, then A is said to be *bounded* *above in* W (respectively *bounded below in* W) if there exists $b \in W$ with $b \geqslant a$ for all $a \in A$ (respectively $b \leqslant a$ for all $a \in A$); b is called an *upper bound for* A *in* W (respectively *lower bound for* A *in* W). If W is a *finite* lower semilattice in $\mathbb{R}^n$, then any nonempty set $A \subset W$ is bounded below in W. Furthermore, there exists a lower bound β for A in W such that $b \leqslant \beta$ if b is any other lower bound for A in W. Clearly, such a β is unique, and we write

$$\beta = \inf{}_W(A).$$

If A is bounded above in W, we define $B = \{b \in W \mid b \geqslant a \text{ for all } a \in A\}$ and define

$$\sup{}_W(A) = \inf{}_W(B).$$

If W is finite lower semilattice and $g: W \longrightarrow W$ is a one–one map, then elementary group theory implies that g^p is the identity on W for some positive integer p. If, in addition, g is a lower semilattice homomorphism, then g and $g^{-1} = g^{p-1}$ preserve the partial ordering on W, and using these facts one can see that, for any set $S \subset W$ which is bounded above in W,

$$g(\sup{}_W(S)) = \sup{}_W(g(S)). \tag{1.3}$$

In particular, if $S = \{x^j \in W: 1 \leqslant j \leqslant m\}$ and if $x^j \in W$ is a periodic point of g of minimal period p_j, then it follows from equation (1.3) that $x = \sup{}_W(S)$ is a periodic point of g of period $q = \operatorname{lcm}(p_1, p_2, \ldots, p_m)$ (although q need not be the minimal period of x).

If $W \subset \mathbb{R}^n$ is a finite lower semilattice and $x \in W$, then we define $h_W(x)$, the *height of* x *in* W, by

$$\begin{aligned} h_W(x) = \sup\{k \geqslant 0 : &\text{there exist } y^0, y^1, \ldots, y^k \in W \text{ with } y^k = x \\ &\text{and } y^j < y^{j+1} \text{ for } 0 \leqslant j < k\}. \end{aligned} \tag{1.4}$$

If there does not exist $u \in W$ with $u < x$, we define $h_W(x) = 0$. One can easily see that there is a unique element $x \in W$ with $h_W(x) = 0$ (the *minimal element of* W). If $x \in W$, then we define $S_x = \{u \in W \mid u < x\}$, and we say that x is *irreducible in* W if S_x is empty or if

$$x > z := \sup{}_W(S_x). \tag{1.5}$$

If x is irreducible in W, S_x is nonempty and $z := \sup{}_W(S_x)$, then we define $I_W(x)$ by

$$I_W(x) = \{i \mid x_i > z_i\}. \tag{1.6}$$

If x is the minimal element of W, then we define $I_W(x) = \{i : 1 \leqslant i \leqslant n\}$. By mathematical induction on the height of points $\xi \in W$, one can prove that, for all $\xi \in W$,

$$\xi = \sup{}_W\{x \mid x \in W, x \leqslant \xi \text{ and } x \text{ is irreducible in } W\}. \tag{1.7}$$

LEMMA 1.1 (compare [**13**]). *Let* V *be a finite lower semilattice in* $\mathbb{R}^n$ *and let* f: $V \longrightarrow V$ *be a one–one map which is a lower semilattice homomorphism. If* $y \in V$ *and* $f^j(y) \neq y$, *then* y *and* $f^j(y)$ *are incomparable and* $h_V(y) = h_V(f^j(y))$, *where* $h_V(\cdot)$ *is given by equation* (1.4). *If* y *is irreducible in* V, *then* $f^j(y)$ *is irreducible in* V. *If* $\eta \in V$ *and* $\zeta \in V$ *and* η and ζ *are incomparable and* η *and* ζ *are irreducible in* V, *then it follows that*

$$I_V(\eta) \cap I_V(\zeta) = \varnothing. \tag{1.8}$$

If y *is irreducible in* V *and* y *is a periodic point of minimal period* p, *then* $1 \leqslant p \leqslant n$.

Proof. To prove that y and $f^j(y)$ are incomparable under the given assumption, it suffices to prove that $h_V(y) = h_V(f^j(y))$. However, we have already observed that f and f^{-1} are lower semilattice homomorphisms, so f and f^{-1} are order-preserving maps of V and f^j and f^{-j} are order-preserving. The equality of $h_V(y)$ and $h_V(f^j(y))$ now follows directly from equation (1.4).

If y is irreducible and $h_V(y) = 0$, then $h_V(f^j(y)) = 0$. However, this implies that y and $f^j(y)$ both equal the unique minimal element of V, contrary to the assumption that $y \neq f^j(y)$. Thus we can assume that $h_V(y) > 0$ and $S_y \neq \emptyset$ and $S_{f^j(y)} \neq \emptyset$, where

$$S_z := \{x \in V : x < z\}.$$

Because f^j and f^{-j} are order-preserving, we see that

$$f^j(S_y) = S_{f^j(y)}$$

so equation (1.3) and the irreducibility of y imply that

$$\sup{}_V(S_{f^j(y)}) = f^j(\sup{}_V(S_y)) < f^j(y),$$

which is precisely the assertion that $f^j(y)$ is irreducible in V.

If η and ζ are as in the statement of Lemma 1.1, but equation (1.8) is false, select $i \in I_V(\eta) \cap I_V(\zeta)$. Because η and ζ are incomparable, we have

$$\eta \wedge \zeta < \eta \quad \text{and} \quad \eta \wedge \zeta < \zeta.$$

By the definition of 'irreducible', we have

$$(\eta \wedge \zeta)_i = \eta_i \wedge \zeta_i < \eta_i \quad \text{and} \quad \eta_i \wedge \zeta_i < \zeta_i,$$

which gives a contradiction.

If y is irreducible in V and has minimal period p, our previous remarks imply that the points $f^j(y), 0 \leqslant j < p$, are irreducible in V and are incomparable. Equation (1.8) implies that $I_V(f^j(y))$ is disjoint from $I_V(f^k(y))$ for $0 \leqslant j < k < p$. Since $I_V(f^j(y))$ is a nonempty subset of $\{i : 1 \leqslant i \leqslant n\}$ for $0 \leqslant j < p$, we conclude that $p \leqslant n$. □

Our next proposition is a technical result which plays an important role in our construction of admissible arrays.

PROPOSITION 1.1. *Let W be a lower semilattice in $\mathbb{R}^n$ and let $g : W \to W$ be a lower semilattice homomorphism. Assume that $\xi \in W$ is a periodic point of g of minimal period p. Let V denote the lower semilattice generated by $\{g^j(\xi) : j \geqslant 0\}$ and let $g \mid V = f$, so f is a lower semilattice automorphism of V onto V and $f^p(x) = x$ for all $x \in V$. Then there exist elements $y^i \in V$ for $1 \leqslant i \leqslant m$, with the following properties*:

(i) *$y^i \leqslant \xi$ for $1 \leqslant i \leqslant m$.*

(ii) *The element y^i is an irreducible element of V and a periodic point of f with minimal period p_i, $1 \leqslant p_i \leqslant n$.*

(iii) $p = \operatorname{lcm}(p_1, p_2, p_3, \ldots, p_m)$.

(iv) *$h_V(y^i) \leqslant h_V(y^{i+1})$ for $1 \leqslant i < m$, where $h_V(\cdot)$ is the height function given by equation* (1.4).

(v) *For $1 \leqslant i < j \leqslant m$, the sets $\{f^k(y^i) : k \geqslant 0\}$ and $\{f^k(y^j) : k \geqslant 0\}$ are disjoint.*

(vi) *For $1 \leqslant i < j \leqslant m$, the elements y^i and y^j are not comparable.*

Proof. By using equation (1.7) we see that there are irreducible elements z^i, $1 \leqslant i \leqslant \mu$, in V with

$$\xi = \sup{}_V\{z^i : 1 \leqslant i \leqslant \mu\}. \tag{1.9}$$

We can assume that μ is minimal in the sense that no collection of irreducible elements with fewer than μ elements will satisfy equation (1.9). If z^i_k denotes the k-coordinate of z^i, then it follows from (1.9) and the minimality of μ that for each i, $1 \leqslant i \leqslant \mu$, there exists j, $1 \leqslant j \leqslant n$, such that

$$z^i_j > \sup\{z^k_j : 1 \leqslant k \leqslant \mu,\ k \neq i\}. \tag{1.10}$$

By relabelling, we can assume that

$$h_V(z^i) \leqslant h_V(z^{i+1}) \quad 1 \leqslant i < \mu. \tag{1.11}$$

Expression (1.10) implies that $\mu \leqslant n$ and that z^i and z^j are not comparable for $1 \leqslant i < j \leqslant \mu$. We have assumed that z^i is irreducible, so if q_i denotes the minimal period of z^i, then Lemma 1.1 implies that $1 \leqslant q^i \leqslant n$. Because $f^p(x) = x$ for all $x \in V$, we know that $q_i \mid p$ for $1 \leqslant i \leqslant \mu$, and, if $q = \operatorname{lcm}(q_1, q_2, \dots, q_\mu)$, then $q \mid p$. On the other hand, we know that

$$f^q(\xi) = f^q(\sup{}_V\{z^i : 1 \leqslant i \leqslant \mu\}) = \sup{}_V\{f^q(z^i) : 1 \leqslant i \leqslant \mu\} = \xi$$

so $f^q(\xi) = \xi$ and $p \mid q$. It follows that

$$\operatorname{lcm}(q_1, \dots, q_\mu) = p. \tag{1.12}$$

We have verified all the properties listed in Proposition 1.1 except for property (v). Define $y^1 = z^1$. If $y^i = z^{\sigma(i)}$ for $1 \leqslant i \leqslant k$, where $\sigma(i) < \sigma(i+1)$ for $1 \leqslant i < k$, then define $y^{k+1} = z^{\sigma(k+1)}$, where $\sigma(k+1) = s$ is the first index s with $\sigma(k) < s \leqslant \mu$ such that the orbit $\{f^\nu(y^{k+1}) : \nu \geqslant 0\}$ is disjoint from the orbits $\{f^\nu(y^i) : \nu \geqslant 0\}$ for $1 \leqslant i \leqslant k$. If no index $s = \sigma(k+1)$ exists, then we stop with the elements y^i, $1 \leqslant i \leqslant k$. The elements y^i, $1 \leqslant i \leqslant m \leqslant \mu$, constructed in this way clearly satisfy properties (i) and (ii) of Proposition 1.1. By construction, y^i is a periodic point of f of period $p_i = q_{\sigma(i)}$. Expressions (1.10) and (1.11) remain true for y^i, $1 \leqslant i \leqslant m$, so properties (iv) and (vi) of Proposition 1.1 are satisfied. Our construction insures that property (v) is satisfied. Moreover, if we note that whenever $\{f^\nu(z^i) : \nu \geqslant 0\}$ and $\{f^\nu(z^j) : \nu \geqslant 0\}$ have nonempty intersection then $q_i = q_j$, we see that

$$\operatorname{lcm}(\{p_i : 1 \leqslant i \leqslant m\}) = \operatorname{lcm}(\{q_i : 1 \leqslant i \leqslant \mu\}) = p$$

so property (iii) remains true. □

Proposition 1.2 provides the motivation for the definition of admissible arrays which is given later.

Proposition 1.2. *Let W, g, V and f be as in Proposition* 1.1. *Assume that y^i, $1 \leqslant i \leqslant m$, are elements of V with the following properties*:

(i) *The element y^i is an irreducible element of V for $1 \leqslant i \leqslant m$.*
(ii) *$h_V(y^i) \leqslant h_V(y^{i+1})$ for $1 \leqslant i < m$, where $h_V(\cdot)$ is the height function of equation* (1.4).
(iii) *For $1 \leqslant i < j \leqslant m$, the sets $\{f^k(y^i) : k \geqslant 0\}$ and $\{f^k(y^j) : k \geqslant 0\}$ are disjoint.*
(iv) *For $1 \leqslant i < j \leqslant m$, the elements y^i and y^j are not comparable.*

Let p_i denote the minimal period of y^i as a periodic point of f. For $0 \leqslant j < p_i$, select $a_{ij} \in I_V(f^j(y^i))$ and define a_{ij} for general $j \in \mathbb{Z}$ by making the map $j \longrightarrow a_{ij}$ periodic of period p_i. Define $\Sigma = \{i \in \mathbb{Z} : 1 \leqslant i \leqslant n\}$, a set with n elements. Then the semi-infinite array a_{ij}, $1 \leqslant i \leqslant m$, $j \in \mathbb{Z}$, has the following properties:

(a) *$a_{ij} \in \Sigma$ for $1 \leqslant i \leqslant m$, $j \in \mathbb{Z}$, and $a_{ij} \neq a_{ik}$ for $0 \leqslant j < k < p_i$ and $1 \leqslant i \leqslant m$. The map $j \longrightarrow a_{ij}$ is periodic of minimal period p_i, with $1 \leqslant p_i \leqslant n$.*

(b) *If $1 \leqslant m_1 < m_2 < \ldots < m_{r+1} \leqslant m$ is any increasing sequence of $(r+1)$ integers between 1 and m, and if*

$$a_{m_i s_i} = a_{m_{i+1} t_i} \tag{1.13}$$

for $1 \leqslant i \leqslant r$, then

$$\sum_{i=1}^{r} (t_i - s_i) \not\equiv 0 \mod (\rho), \tag{1.14}$$

where $\rho = \gcd(\{p_{m_i} : 1 \leqslant i \leqslant r+1\})$, that is, the greatest common divisor of $\{p_{m_i} : 1 \leqslant i \leqslant r+1\}$.

Proof. Lemma 1.1 implies that $1 \leqslant p_i \leqslant n$ and that $I_V(f^j(y^i)) \cap I_V(f^k(y^i))$ is empty for $0 \leqslant j < k < p_i$. This gives property (a) of the numbers a_{ij}.

Next suppose that equation (1.13) is satisfied. If $\rho = \gcd(p_{m_1}, p_{m_2}, \ldots, p_{m_{r+1}})$, then elementary number theory implies that there are integers $A_1, A_2, \ldots, A_{r+1}$ with

$$\rho = \sum_{i=1}^{r+1} A_i p_{m_i}. \tag{1.15}$$

Assume, by way of contradiction, that for s_i and t_i as in equation (1.13),

$$\sum_{i=1}^{r} (t_i - s_i) \equiv 0 \pmod \rho. \tag{1.16}$$

Equations (1.15) and (1.16) imply that there are integers $B_1, B_2, \ldots, B_{r+1}$ with

$$\sum_{i=1}^{r} (t_i - s_i) = \sum_{i=1}^{r+1} B_i p_{m_i}. \tag{1.17}$$

By assumption (i), we know that $f^{s_i}(y^{m_i}) := \eta$ and $f^{t_i}(y^{m_{i+1}}) := \zeta$ are irreducible elements of V, and $\eta \neq \zeta$ because of assumption (iii). By (1.13), $I_V(\eta)$ and $I_V(\zeta)$ have nonempty intersection, so Lemma 1.1 implies that η and ζ are comparable. However, we see from assumption (ii) that

$$h_V(\eta) \leqslant h_V(\zeta)$$

so we conclude that

$$\eta < \zeta. \tag{1.18}$$

Because f^{-s_i} is order-preserving on V and $f^{-B_i p_{m_i}}(y^{m_i}) = y^{m_i}$, we deduce from expression (1.18) that

$$y^{m_i} < f^{(t_i - s_i) - B_i p_{m_i}}(y^{m_{i+1}}). \tag{1.19}$$

If we apply expression (1.19) repeatedly and recall that f^j is order-preserving on V for all integers j, then we obtain

$$y^{m_1} < f^{\nu}(y^{m_{r+1}}), \quad \nu := \sum_{i=1}^{r} (t_i - s_i) - \sum_{i=1}^{r} B_i p_{m_i}.$$

Because $f^{\mu}(y^{m_{r+1}}) = y^{m_{r+1}}$, where $\mu = -B_{r+1} p_{m_{r+1}}$, we conclude that

$$y^{m_1} < f^{\nu+\mu}(y^{m_{r+1}}), \quad \nu + \mu := \sum_{i=1}^{r} (t_i - s_i) - \sum_{i=1}^{r+1} B_i p_{m_i} = 0. \tag{1.20}$$

Expression (1.20) implies that $y^{m_1} < y^{m_{r+1}}$, which contradicts assumption (iv). $\square$

We now wish to define an admissible array on n symbols as a semi-infinite collection of numbers (a_{ij}) which satisfies properties (a) and (b) of Proposition 1.2. It is convenient to give a slightly more general definition.

DEFINITION 1.7. Suppose that $(L, <)$ is a finite, totally ordered set and that Σ is a finite set with n elements. let $\mathbb{Z}$ denote the integers, and for each $i \in L$ suppose that $\phi_i : \mathbb{Z} \to \Sigma$ is a map. We shall say that $\{\phi_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ is an *admissible array on n symbols* if the maps ϕ_i satisfy the following conditions:

(i) For each $i \in L$, the map $\phi_i : \mathbb{Z} \longrightarrow \Sigma$ is periodic of minimal period p_i, where $1 \leqslant p_i \leqslant n$. Furthermore, for $1 \leqslant j < k \leqslant p_i$ we have $\phi_i(j) \neq \phi_i(k)$.

(ii) If $m_1 < m_2 < \ldots < m_{r+1}$ is any increasing sequence of $(r+1)$ elements of L and if

$$\phi_{m_i}(s_i) = \phi_{m_{i+1}}(t_i) \quad 1 \leqslant i \leqslant r, \tag{1.21}$$

then it follows that

$$\sum_{i=1}^{r} (t_i - s_i) \not\equiv 0 \mod (\rho), \tag{1.22}$$

where $\rho = \gcd(p_{m_1}, p_{m_2}, \ldots, p_{m_{r+1}})$.

Note that the concept of an admissible array on n symbols depends on the ordering $<$ on L. Usually, L is a finite subset of the integers with the usual ordering. In fact, suppose that $(L, <)$ is a finite, totally ordered set, Σ is a set with n elements and $\{\phi_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ is an admissible array on n symbols. Let $(L_1, <_1)$ be a totally ordered set with $|L| = |L_1|$ and let Σ_1 be a set with $|\Sigma_1| = |\Sigma|$. Suppose that $\sigma : L_1 \to L$ is an order-preserving, one–one map (such a map always exists) and $\theta : \Sigma \to \Sigma_1$ is a one–one map. For $i \in L_1$, define $\hat{\phi}_i : \mathbb{Z} \to \Sigma_1$, by

$$\hat{\phi}_i(j) = \theta(\phi_{\sigma(i)}(j)).$$

The reader can check that $\{\hat{\phi}_i : \mathbb{Z} \to \Sigma_1 : i \in L_1\}$ is also an admissible array on n symbols. By this observation, if $|L| = m$, and $\{\phi_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ is an admissible array on n symbols, then we can assume, if we wish, that $L = \{j \in \mathbb{Z} : 1 \leqslant j \leqslant m\}$ with the usual ordering and $\Sigma = \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$.

If $\{\phi_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ is an admissible array on n symbols and $L_o \subset L$ with the ordering inherited from L, then we call $\{\phi_i : \mathbb{Z} \longrightarrow \Sigma \mid i \in L_o\}$ a *subarray of* $\{\phi_i : \mathbb{Z} \longrightarrow \Sigma \mid i \in L\}$. One can check that $\{\phi_i : \mathbb{Z} \longrightarrow \Sigma \mid i \in L_o\}$ is an admissible array on n symbols.

If the p_i, $i \in L$, are as in Definition 1.7, then we are interested in the possible numbers $\operatorname{lcm}(\{p_i \mid i \in L\})$ which can arise from different admissible arrays on n symbols.

DEFINITION 1.8. Suppose that $\{q_i \mid 1 \leqslant i \leqslant m\} = S$ is a set of positive integers with $1 \leqslant q_i \leqslant n$ for $1 \leqslant i \leqslant m$ and $q_i \neq q_j$ for $1 \leqslant i < j \leqslant m$. We shall say that S is *array-admissible for n* if there exist a totally ordered set $(L, <)$ with $|L| = m$, an admissible array on n symbols $\{\phi_i : \mathbb{Z} \longrightarrow \Sigma \mid i \in L\}$ such that ϕ_i has minimal period p_i, and a one–one map σ of $\{i \in \mathbb{Z} : 1 \leqslant i \leqslant m\}$ onto L such that $q_i = p_{\sigma(i)}$.

DEFINITION 1.9. $Q(n) = \{\operatorname{lcm}(S) : S \text{ is array-admissible for } n\}$.

REMARK 1.1. Let $\theta(n)$ denote the number of primes $r \leqslant n$. In computing $Q(n)$, it suffices to consider sets S which are array-admissible for n and which satisfy $|S| \leqslant \theta(n)$. In order to see this, suppose that $q \in Q(n)$ and let $\{\phi_i : \mathbb{Z} \longrightarrow \Sigma \mid i \in L\}$ be an admissible array on n symbols such that ϕ_i is periodic of period $p_i \leqslant n$ and $q = \operatorname{lcm}(\{p_i \mid i \in L\})$ and $p_i \neq p_j$ for $i \neq j$. If λ is a prime factor of q, then we know that $\lambda \leqslant n$. If $t = t(\lambda)$ is the largest integer such that $\lambda^t \mid q$, then there exists $i = i(\lambda) \in L$ such that $\lambda^t \mid p_{i(\lambda)}$. If we define $L_o = \{i(\lambda) \mid \lambda \text{ is a prime factor of } q\}$, then we know that $|L_o| \leqslant \theta(n)$, $\{\phi_i : \mathbb{Z} \longrightarrow \Sigma \mid i \in L_o\}$, is an admissible array on n symbols and $q = \operatorname{lcm}(\{p_i \mid i \in L_o\})$.

We have defined the sets $\hat{P}(n)$ and $P^*(n)$ (Definition 1.3), $P(n)$ (Definition 1.4) $\tilde{Q}(n)$ and $Q^*(n)$ (Definition 1.6) and $Q(n)$ (Definition 1.9). Theorem 1.4 summarizes what we have proved about these sets.

THEOREM 1.4. *For every positive integer n we have $\tilde{Q}(n) \subset Q^*(n)$ and*

$$P(n) \subset \hat{P}(n) \subset P^*(n) \subset Q^*(n) \subset Q(n). \tag{1.23}$$

Proof. By using Theorem 1.1 and Theorem 1.2, we see that it only remains to prove that $Q^*(n) \subset Q(n)$. If $p \in Q^*(n)$, then there exist a finite lower semilattice V and a lower semilattice automorphism $f: V \longrightarrow V$ which has a periodic point ξ of minimal period p. Furthermore, as in Proposition 1.1, there are irreducible elements $y^i \in V$, $1 \leqslant i \leqslant m$, which satisfy properties (i)–(vi) in Proposition 1.1, and we can assume that V is generated by $\{f^j(\xi) \mid j \geqslant 0\}$. Define $L = \{i \in \mathbb{Z} : 1 \leqslant i \leqslant m\}$ with the natural ordering and $\Sigma = \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$ and select $a_{ij} \in I_V(f^j(y^i))$ as in Proposition 1.2. Proposition 1.2 implies that if $\theta_i : \mathbb{Z} \longrightarrow \Sigma$ is defined by $\theta_i(j) = a_{ij}$, so that θ_i is periodic of period p_i with $1 \leqslant p_i \leqslant n$, then $\{\theta_i : \mathbb{Z} \longrightarrow \Sigma \mid i \in L\}$ is an admissible array. Property (iii) in Proposition 1.1 implies that

$$p = \operatorname{lcm}(\{p_i : 1 \leqslant i \leqslant m\}). \tag{1.24}$$

By possibly taking a subarray, we can also assume that $p_i \neq p_j$ for $1 \leqslant i < j \leqslant m$ and that equation (1.24) remains true. Thus we see that $p \in Q(n)$ and $Q^*(n) \subset Q(n)$. $\square$

REMARK 1.2. If $f \in \mathscr{G}(n)$, the point $\xi \in K^n$ is a periodic point of f, and L is the lattice generated by $A := \{f^j(\xi) : j \in \mathbb{Z}\}$, then it is not necessarily true that $f(L) \subset L$ or that $f|L$ is a lattice homomorphism. Nevertheless, the first author has shown in separate work that $P^*(n) \subset \tilde{Q}(n)$.

Theorem 1.4 raises many natural questions. Basically, one can ask whether any of the inclusions in Theorem 1.4 can be replaced by equalities, and if not, to what extent various sets differ.

QUESTION 1.1. *Is it true that $\hat{P}(n) = P^*(n)$ for all $n \geqslant 1$?*

QUESTION 1.2. *Is it true that $\hat{P}(n) = \tilde{Q}(n)$ for all $n \geqslant 1$ or that $P^*(n) = \tilde{Q}(n)$ for all $n \geqslant 1$?*

QUESTION 1.3. *Is it true that $\hat{P}(n) = Q^*(n)$ for all $n \geqslant 1$ or that $P^*(n) = Q^*(n)$ for all $n \geqslant 1$?*

QUESTION 1.4. *Is it true that $\hat{P}(n) = Q(n)$ for all $n \geqslant 1$ or that $P^*(n) = Q(n)$ for all $n \geqslant 1$?*

QUESTION 1.5. *Is it true that $P(n) = \hat{P}(n)$ for all $n \geqslant 1$?*

Note added in proof. Since this paper was submitted in May 1995, several of these questions have been answered. It is proved in **[18]** that $Q^*(n) = \tilde{Q}(n) = Q(n) = P^*(n)$ for all $n \geqslant 1$, and it is proved in **[12]** that $P(n) = \hat{P}(n) = Q(n)$ for $1 \leqslant n \leqslant 50$, but that $P(78) \neq Q(78)$.

2. *Properties of admissible arrays*

If, for a given n, one can prove that $P(n) = Q(n)$, then Theorem 1.4 implies that

$$P(n) = \hat{P}(n) = P^*(n) = \tilde{Q}(n) = Q^*(n) = Q(n).$$

In subsequent work, the first author and Sjoerd Verduyn Lunel **[12]** have taken this approach and proved that $P(n) = Q(n)$ for $n \leqslant 50$. The difficulty in proving such a result is that the definition of $Q(n)$ is indirect and clumsy to work with. In the remainder of this paper we establish further theorems about admissible arrays and about sets S which are array-admissible for n. These theorems, together with other ideas developed in [**12**], have allowed it to be proved that $P(n) = Q(n)$ for $1 \leqslant n \leqslant 50$ and have also yielded further progress on questions raised in Section 1.

THEOREM 2.1. *Let $\hat{L} = \{i \in \mathbb{Z} : 1 \leqslant i \leqslant m+1\}$ with the usual ordering and let Σ denote a set with n elements. Assume that $\{\hat{\theta}_i : \mathbb{Z} \to \Sigma \mid i \in \hat{L}\}$ is an admissible array on n symbols. If $\hat{B}_i := \{\hat{\theta}_i(j) : j \in \mathbb{Z}\}$, then assume that $\hat{B}_i \cap \hat{B}_{i+1} \neq \varnothing$ for $1 \leqslant i \leqslant m$ and write $\hat{p}_i = |\hat{B}_i|$ (so that $\hat{\theta}_i$ is periodic with minimal period $\hat{p}_i$). Let $r_1 > 1$ and $r_2 \geqslant 1$ be integers, and define $r = r_1 r_2$. Assume that, for $1 \leqslant i \leqslant m+2$,*

$$\gcd(\hat{p}_{i-1}, \hat{p}_i) \mid r. \tag{2.1}$$

(*We use the convention that $\hat{p}_o = 1$ and $\hat{p}_{m+2} = 1$ in expression* (2.1)). *Assume also that there exists an integer k, $1 \leqslant k \leqslant m+1$, such that*

$$\gcd(\hat{p}_{k-1}, \hat{p}_k) \mid r_1 \quad \textit{and} \quad \gcd(\hat{p}_{k+1}, \hat{p}_k) \mid r_1. \tag{2.2}$$

Then it follows that

$$m+1 = |L| \leqslant r_1 r_2 - r_2 + 1. \tag{2.3}$$

Proof. Because $\hat{B}_i \cap \hat{B}_{i+1} \neq \varnothing$, there exist integers s_i and t_i with

$$\hat{\theta}_i(s_i) = \hat{\theta}_{i+1}(t_i) \quad 1 \leqslant i \leqslant m.$$

Define $\delta_i = s_i - t_i$ and note that expression (2.1) and the definition of an admissible array imply that

$$\sum_{i=\lambda}^{v} \delta_i \not\equiv 0 \pmod{r} \quad 1 \leqslant \lambda \leqslant v \leqslant m. \tag{2.4}$$

Note that we can associate to any admissible array on n symbols $\{\theta_i : \mathbb{Z} \longrightarrow \Sigma \mid i \in L\}$ a *reversed array on n symbols* by reversing the ordering on L. Specifically, if $\leqslant$ denotes the ordering on L, define a new ordering $\leqslant'$ on L by $a \leqslant' b$ if and only if $b \leqslant a$. If L' denotes the set L with the new ordering $\leqslant'$, it is clear that $\{\theta_i : \mathbb{Z} \longrightarrow \Sigma \mid i \in L'\}$ is also an admissible array on n symbols, which we call the reversed array. If k is as in the statement of Theorem 2.1, we see that, possibly by replacing the original array with the reversed array, we can assume without loss of generality that $k > 1$. (Here we also note that Theorem 2.1 is obviously true if $m = 0$, so we can assume that $m > 0$).

By virtue of the above remarks we shall assume that $k > 1$ and $m > 0$. For $1 \leqslant \lambda \leqslant m$, we define integers η_λ by

$$\eta_\lambda = \sum_{i=1}^{\lambda} \delta_i. \tag{2.5}$$

We claim that, for $1 \leqslant v \leqslant m$ and $v \neq k-1$,

$$\eta_{k-1} \not\equiv \eta_v \pmod{r_1}. \tag{2.6}$$

To prove equation (2.6), first note that for $1 \leqslant v < k-1$ we have

$$\eta_{k-1} - \eta_v = \sum_{i=v+1}^{k-1} \delta_i.$$

Our assumptions imply that

$$\gcd(\hat{p}_{v+1}, \hat{p}_{v+2}, \ldots, \hat{p}_k) \mid r_1 \tag{2.7}$$

so the definition of admissible arrays and expression (2.7) imply that

$$\eta_{k-1} - \eta_v \not\equiv 0 \pmod{r_1} \quad 1 \leqslant v < k-1.$$

If $k-1 < v \leqslant m$, we have

$$\eta_v - \eta_{k-1} = \sum_{i=k}^{v} \delta_i. \tag{2.8}$$

Our assumptions imply that

$$\gcd(p_k, p_{k+1}, \ldots, p_{v+1}) \mid r_1 \tag{2.9}$$

so, by using the definition of an admissible array and expression (2.9), we conclude that

$$\eta_v - \eta_{k-1} \not\equiv 0 \pmod{r_1} \quad k-1 < v \leqslant m.$$

Thus we have established equation (2.6).

Define $k_1 = k-1$ and let π be the natural map of $\mathbb{Z}$ onto $\mathbb{Z}/(r)$ which takes an integer j to its equivalence class mod r. Define $\Gamma \subset \mathbb{Z}$ by

$$\Gamma = \{\eta_i : 1 \leqslant i \leqslant m,\ i \neq k_1\} \cup \{\eta_{k_1} + jr_1 : 0 \leqslant j < r_2\}.$$

By using equations (2.4) and (2.6), we see that Γ has $m-1+r_2$ elements and that $\pi \mid \Gamma$ is one–one. We further claim that $\pi(\eta) \neq 0$ for $\eta \in \Gamma$. If $\pi(\eta) = 0$ for some $\eta \in \Gamma$, then either (i) $\eta_i \equiv 0 \pmod{r}$ for some $i \neq k_1$, or (ii) $\eta_{k_1} + jr_1 \equiv 0 \pmod{r}$ for some j, $0 \leqslant j < r_2$. In the first case, we find that $\eta_i \equiv 0 \pmod{r}$, which contradicts equation (2.4). In the second case, we see that

$$\eta_{k_1} \equiv 0 \pmod{r_1}.$$

We deduce that

$$\eta_{k_1} = \sum_{i=1}^{k-1} \delta_i \equiv 0 \pmod{r_1}. \tag{2.10}$$

However, we know that

$$\gcd(p_1, \ldots, p_k) \mid r_1.$$

Thus equation (2.10) is impossible, so we have proved that $\pi(\eta) \neq 0$ for $\eta \in \Gamma$.

It follows that π is a one–one map of Γ into $(\mathbb{Z}/(r) - \{0\})$, a set with $r-1$ elements. We conclude that

$$|\Gamma| = m - 1 + r_2 \leqslant r - 1$$

and Theorem 2.1 is proved.

Theorem 2.2 is a straightforward consequence of Theorem 2.1, but it is often easier to apply.

THEOREM 2.2. *Suppose that* $(L, <)$ *is a finite, totally ordered set, that* Σ *is a set with n elements, and that* $\{\phi_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ *is an admissible array on n symbols. Let* p_i *denote the minimal period of* ϕ_i *(so that* $1 \leqslant p_i \leqslant n$*), and assume that* $p_i \neq p_j$ *for all* $i, j \in L$ *with* $i \neq j$*. Let* $S = \{p_i \mid i \in L\}$*, and, for each* $p_i \in S$*, define* $B_{p_i} \subset \Sigma$ *by*

$$B_{p_i} = \{\phi_i(j) \mid j \in \mathbb{Z}\}. \tag{2.11}$$

Then the following conditions are satisfied:

(A1) $|B_p| = p$ *for all* $p \in S$ *and* $B_p \cap B_q = \varnothing$ *for all* $p, q \in S$ *such that* $p \neq q$ *and* $\gcd(p, q) = 1$.

(B1) *There does* not *exist a set* $R \subset S$ *with the following properties*:
   (a) $|R| = r+1$*, where* $r > 1$*, and* $\gcd(p, q) \mid r$ *for all* $p, q \in R$ *with* $p \neq q$.
   (b) $B_p \cap B_q \neq \varnothing$ *for all* $p, q \in R$.

(C1) *There does* not *exist a set* $R \subset S$ *with the following properties*:
   (a) $|R| = r_1 r_2 - r_2 + 2$*, where* $r_1 > 1$ *and* $r_2 \geqslant 1$ *are integers.*
   (b) *If* $r := r_1 r_2$*, then* $\gcd(p, q) \mid r$ *for all* $p, q \in R$ *with* $p \neq q$.
   (c) *There exists* $\hat{p} \in R$ *with* $\gcd(p, \hat{p}) \mid r_1$ *for all* $p \in R - \{\hat{p}\}$.
   (d) $B_p \cap B_q \neq \varnothing$ *for all* $p, q \in R$.

Proof. The fact that $|B_p| = p$ for all $p \in S$ is part of the definition of an admissible array. If $i, j \in L$ and $i < j$ and $B_{p_i} \cap B_{p_j} \neq \varnothing$, then there exist integers s and t with $\phi_i(s) = \phi_j(t)$. By the definition of an admissible array, we obtain

$$t - s \not\equiv 0 \pmod{\rho} \quad \rho := \gcd(p_i, p_j).$$

This equation is impossible if $\rho = 1$, so, if $\gcd(p_i, p_j) = 1$ for $i \neq j$, then it must be that $B_{p_i} \cap B_{p_j} = \varnothing$. This proves condition (A1).

Note that condition (B1) is a special case of condition (C1) with $r_2 = 1$ and $r = r_1 r_2 = r_1$ in condition (C1). Thus it suffices to prove condition (C1). Assume, by way of contradiction, that a set R as in condition (C1) exists. Let $L_1 = \{i \in L : p_i \in R\}$, and define $|L_1| = m+1$. If $\hat{L} = \{i \in \mathbb{Z} \mid 1 \leqslant i \leqslant m+1\}$, then let $\sigma : \hat{L} \to L_1$ be a one–one, order-preserving map. Define $\hat{\theta}_i = \phi_{\sigma(i)}$, so that $\{\hat{\theta}_i : \mathbb{Z} \to \Sigma \mid i \in L\}$ is an admissible array on n symbols. Define $\hat{B}_i$ by

$$\hat{B}_i = \{\hat{\theta}_i(j) \mid j \in \mathbb{Z}\} = B_{p_{\sigma(i)}}.$$

Condition (C1) insures that

$$\hat{B}_i \cap \hat{B}_j \neq \varnothing \quad \text{for all } i, j \in \hat{L}.$$

If we define $\hat{p}_i = |\hat{B}_i| = p_{\sigma(i)}$, then condition (C1) implies that expression (2.1) is satisfied. If $k \in \hat{L}$ is selected so that $\hat{p}_k = \hat{p}$ (where $\hat{p}$ is as in condition (C1)), then condition (C1) implies that expressions (2.2) are satisfied. It follows that all the hypotheses of Theorem 2.1 are satisfied, so

$$|R| = |L_1| = |\hat{L}| = m+1 \leqslant r_1 r_2 - r_2 + 1.$$

However, we assumed that $|R| = r_1 r_2 - r_2 + 2$, a contradiction.

Given a set $S \subset \{j \in \mathbb{Z} \mid 1 \leqslant j \leqslant n\}$, we want to find verifiable conditions which insure that S is *not* array-admissible for n. We now show how Theorem 2.1 and Theorem 2.2 can be applied to obtain such conditions.

DEFINITION 2.1. A set $S \subset \{1, 2, \ldots, n\}$ satisfies *condition* A *for the integer n* if S does *not* contain a subset Q such that the following hold:

(i) $\gcd(\alpha, \beta) = 1$ for all $\alpha, \beta \in Q$ with $\alpha \neq \beta$.
(ii) $\sum_{\alpha \in Q} \alpha > n$.

COROLLARY 2.1. *Assume that $S \subset \{1, 2, \ldots, n\}$ is array-admissible for n. Then S satisfies condition A for the integer n.*

Proof. By definition, there exists a finite, totally ordered set $(L, <)$, a set Σ with n elements, and an admissible array on n symbols $\{\phi_i : \mathbb{Z} \longrightarrow \Sigma \mid i \in L\}$ such that ϕ_i is periodic of minimal period p_i and $S = \{p_i : i \in L\}$. Assume, by way of contradiction, that S contains a set Q as in Definition 2.1. If B_{p_i} is defined by equation (2.11), then Theorem 2.2(A1) implies that $|B_p| = p$ and $B_p \cap B_q = \emptyset$ for all $p, q \in Q$ with $p \neq q$. It follows that

$$\left| \bigcup_{p \in Q} B_p \right| = \sum_{p \in Q} \left| B_p \right| > n.$$

This is a contradiction, because $\bigcup_{p \in Q} B_p \subset \Sigma$ and $|\Sigma| = n$.

In order to state our next results, we need to define certain covering properties of collections of finite sets. As usual, if Σ is a set, then 2^Σ denotes the collection of all subsets of Σ.

DEFINITION 2.2. Suppose that n and r are positive integers with $r + 1 \leqslant n$, the set Σ is a set with n elements, $S \subset \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$ is a set with $|S| \geqslant r + 1$, and $S_o \subset S$ is a set with $|S_o| \leqslant r + 1$. Suppose that $\Gamma : S \to 2^\Sigma$ is a map such that

$$|\Gamma(p)| = \mathrm{p} \tag{2.12}$$

for all $p \in S$. We say that Γ has the $(n, r; S_o)$ *covering property* if there exists a set T with $|T| = r + 1$, $S_o \subset T \subset S$ and

$$\Gamma(p) \cap \Gamma(q) \neq \emptyset \quad \text{for all } p, q \in T.$$

We say that S has the *absolute* $(n, r; S_o)$ *covering property* if, whenever Σ is a set with n elements and $\Gamma : S \to 2^\Sigma$ is a map which satisfies equation (2.12), then Γ has the $(n, r; S_o)$ covering property. If S_o is empty, we talk about the (n, r) *covering property* rather than the $(n, r; \emptyset)$ covering property.

We shall not study here the general question of when a set $S \subset \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$ has the absolute $(n, r; S_o)$ covering property. Proposition 2.1 gives an example of a sufficient condition for the absolute (n, r) covering property.

PROPOSITION 2.1. *Suppose that $S = \{p_i : 1 \leqslant i \leqslant m\}$ is a collection of positive integers p_i with $1 \leqslant p_i \leqslant n$ for $1 \leqslant i \leqslant m$ and $p_i \neq p_j$ for $1 \leqslant i < j \leqslant m$. Let r be a positive integer and assume that*

$$\sum_{i=1}^{m} p_i > rn. \tag{2.13}$$

Let Σ be a set with n elements and let $\Gamma : S \to 2^\Sigma$ be a map such that $|\Gamma(p)| = p$ for all $p \in S$. Then there exist $r + 1$ integers $1 \leqslant i_1 < i_2 < \ldots < i_{r+1} \leqslant m$ (so that $r + 1 \leqslant m$) with

$$\bigcap_{k=1}^{r+1} \Gamma(p_{i_k}) \neq \emptyset. \tag{2.14}$$

In particular, S satisfies the absolute (n, r) covering property.

Proof. Let χ_i denote the characteristic function of $\Gamma(p_i)$, so that $\chi_i(x) = 1$ if $x \in \Gamma(p_i)$ and $\chi_i(x) = 0$ if $x \notin \Gamma(p_i)$. Assume that Proposition 2.1 is false. Then, for every $x \in \Sigma$, we have

$$\sum_{i=1}^{m} \chi_i(x) \leqslant r.$$

It follows that

$$\sum_{i=1}^{m} p_i = \sum_{i=1}^{m} \sum_{x \in \Sigma} \chi_i(x) = \sum_{x \in \Sigma} \sum_{i=1}^{m} \chi_i(x) \leqslant rn,$$

and this contradicts expression (2.13).

We can now give another useful condition on sets of integers $S \subset \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$.

DEFINITION 2.3. A set $S \subset \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$ satisfies *condition* C′ *for* n if S does *not* contain disjoint subsets Q and R with the following properties:

(i) $\gcd(\alpha, \beta) = 1$ for all $\alpha \in Q$ and $\beta \in Q \cup R$ with $\alpha \neq \beta$.

(ii) There are integers $r_1 > 1$ and $r_2 \geqslant 1$ such that $\gcd(\alpha, \beta) \mid r$, $r = r_1 r_2$, for all $\alpha, \beta \in R$ with $\alpha \neq \beta$.

(iii) There exists $\gamma_o \in R$ such that $\gcd(\alpha, \gamma_o) \mid r_1$ for all $\alpha \in R$, $\alpha \neq \gamma_o$.

(iv) The subset R has the absolute $(n^*, r_1 r_2 - r_2 + 1; \{\gamma_o\})$ covering property, where $n^* := n - \sum_{\alpha \in Q} \alpha$.

We allow Q or R to be empty in Definition 2.3. If R is empty, conditions (ii) and (iii) are vacuous, and we interpret condition (iv) as meaning that

$$n < \sum_{\alpha \in Q} \alpha.$$

Thus condition C′ gives condition A (Definition 2.1) by taking $R = \varnothing$. We have preferred to state condition A separately, however.

If Q is empty, condition (i) in Definition 2.3, is vacuous, and we interpret $n^* = n$ in condition (iv).

Condition C′ may seem unnatural, but we see below that if a set $S \subset \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$ does contain subsets Q and R as in Definition 2.3, then S is not array-admissible for n. Furthermore, we see that condition C′ implies a number of simpler conditions which insure that S is not array-admissible for n.

COROLLARY 2.2. *Assume that $S \subset \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$ is array-admissible for n. Then S satisfies condition* C′ *for n* (*see Definition* 2.3).

Proof. Assume, by way of contradiction, that there exist disjoint subsets Q and R of S as in Definition 2.3. Because S is array-admissible for n, there exists an admissible array $\{\phi_i : \mathbb{Z} \longrightarrow \Sigma \mid i \in L\}$ such that $|\Sigma| = n$, ϕ_i is periodic of minimal period p_i for $i \in L$, the set $S = \{p_i : i \in L\}$, and $p_i \neq p_j$ for $i \neq j$. For each $p_i \in S$, we let $B_{p_i} \subset \Sigma$ be given by

$$B_{p_i} = \{\phi_i(j) : j \in \mathbb{Z}\}$$

so $|B_{p_i}| = p_i$. We define $B_Q = \bigcup_{q \in Q} B_q$. By using Theorem 2.2(A1) we see that $|B_Q| = \sum_{q \in Q} q$ and $B_Q \cap B_p = \varnothing$ for all $p \in R$. If we define $\Sigma_1 = \Sigma - B_Q$, it follows that $|\Sigma_1| =$

n^* and $B_p \subset \Sigma_1$ for all $p \in R$. We define $m = r_1 r_2 - r_2 + 1$. Because R has the absolute $(n^*, m; \{\gamma_o\})$ covering property, there exists a subset $R_1 \subset R$ such that $|R_1| = m+1$, $\gamma_o \in R_1$ and $B_p \cap B_q \neq \varnothing$ for all $p, q \in R_1$. However, the existence of R_1 contradicts Theorem 2.2(C1).

Definition 2.4 gives a condition which is essentially a special case of condition C' but which is adequate for many applications.

DEFINITION 2.4. A subset $S \subset \{1, 2, \dots, n\}$ satisfies *condition* B′ *for n* if S does *not* contain disjoint subsets Q and R which satisfy the following properties:

(i) $\gcd(\alpha, \beta) = 1$ for all $\alpha \in Q$ and $\beta \in Q \cup R$ with $\alpha \neq \beta$.

(ii) There exists an integer $r > 1$ such that $\gcd(\alpha, \beta) \mid r$ for all $\alpha, \beta \in R$ with $\alpha \neq \beta$.

(iii) R has the absolute (n^*, r) covering property, where $n^* = n - \sum_{\alpha \in Q} \alpha$.

If Q is empty, then Definition 2.4(i) is vacuous and $n^* = n$. If R is empty, property (ii) is vacuous and we interpret property (iii) as meaning that $n < \sum_{\alpha \in Q} \alpha$.

COROLLARY 2.3. *Assume that* $S \subset \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$ *is array-admissible for n. Then S satisfies condition B′ for n* (*see Definition* 2.4).

Proof. Assume, by way of contradiction, that there exist disjoint subsets Q and R of S as in Definition 2.4. The proof now proceeds exactly as in the proof of Corollary 2.2, except that, at the last stage, Theorem 2.2(B1) is contradicted. The details are left to the reader.

If S is an array-admissible set of integers, then we have derived constraints on S by using facts about admissible arrays. In [**9**], admissible arrays were never defined. However, various *ad hoc* constraints on sets of integers were obtained. We show that all of the constraints obtained in [**9**] are special cases of conditions A, B′ and C′. Condition A itself has already been introduced in [**9**], but we need to recall other definitions from [**9**]. We request the reader's indulgence for a collection of complicated definitions.

DEFINITION 2.5 (compare [**9**]). A set $S \subset \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$ is said to satisfy *condition* B *for n* if S does not contain disjoint subsets Q and R with the following properties:

(i) $\gcd(\alpha, \beta) = 1$ for all $\alpha \in Q$ and $\beta \in Q \cup R$ with $\alpha \neq \beta$.

(ii) The subset R has $r+1$ elements, $r \geqslant 1$, and $\gcd(\alpha, \beta) \mid r$ for all $\alpha, \beta \in R$ with $\alpha \neq \beta$.

(iii) For all $\alpha, \beta \in R$ with $\alpha \neq \beta$, $\alpha + \beta > n^* := n - \sum_{\gamma \in Q} \gamma$.

Our condition B is a slight generalization of [**9**, condition B].

DEFINITION 2.6 (compare [**9**, p. 362]). We say that $S \subset \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$ satisfies *condition* C *for n* if there do *not* exist disjoint subsets Q and R of S which satisfy the following:

(i) $\gcd(\alpha, \beta) = 1$ for all $\alpha \in Q$ and $\beta \in Q \cup R$ with $\alpha \neq \beta$.

(ii) There are integers $r_1 > 1$ and $r_2 \geqslant 1$ such that $\gcd(\alpha, \beta) \mid r$, $r = r_1 r_2$, for all $\alpha, \beta \in R$ with $\alpha \neq \beta$.

(iii) There exists $\gamma_o \in R$ such that $\gcd(\alpha, \gamma_o) \mid r_1$ for all $\alpha \in R$ with $\alpha \neq \gamma_o$.
(iv) $|R| \geqslant r_1 r_2 - r_2 + 2$ and $\alpha + \beta > n^* := n - \sum_{q \in Q} q$ for all $\alpha, \beta \in R$ with $\alpha \neq \beta$.

Condition C in Definition 2.6 is a direct generalization of [**9**, condition C, p. 362]. In [**9**] it is assumed that $r_1 = r_2 = \rho$.

For the reader's convenience, we also recall [**9**, condition *D*].

Definition 2.7 [**9**]. A set $S \subset \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$ satisfies *condition* D *for n* if S does *not* contain a set R with the following properties:

(i) $|R| = m + r - 1$, where $m \geqslant 2$ and $r \geqslant 2$, and $\gcd(p, q) \mid r$ for all $p, q \in R$ with $p \neq q$.

(ii) There exist disjoint subsets R_1 and R_2 of R with $R_1 \cup R_2 = R$, $|R_1| = m$ and $|R_2| = r - 1$, $\sum_{p \in R_1} p > n$, and $p + q > n$ for all $p \in R$ and $q \in R_2$.

At the risk of straining the reader's patience, we give a final definition which is in the same spirit.

Definition 2.8. Suppose that $S \subset \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$. We say that S satisfies *condition* E *for n* if S does not contain disjoint subsets Q and R with the following properties:

(i) $\gcd(\alpha, \beta) = 1$ for all $\alpha \in Q$ and $\beta \in Q \cup R$ with $\alpha \neq \beta$.
(ii) There is an integer $r \geqslant 1$ such that $\gcd(\alpha, \beta) \mid r$ for all $\alpha, \beta \in R$ with $\alpha \neq \beta$.
(iii) $\sum_{\beta \in R} \beta > r n^*$, where $n^* := n - \sum_{\alpha \in Q} \alpha$.

The motivation for Definitions 2.3–2.8 is provided by Theorem 2.3.

Theorem 2.3. *Assume that* $S \subset \{j \in \mathbb{Z} : 1 \leqslant j \leqslant n\}$ *is array-admissible for n. Then S satisfies conditions* A, B′, C′, B, C, D *and* E *for n.*

Proof. We have already proved that S satisfies conditions A, B′ and C′ (Corollaries 2.1–2.3). To prove that S satisfies condition B, it suffices to prove that if S does not satisfy condition B, then it does not satisfy condition B′. Thus assume that S does not satisfy condition B, and let Q and R be as in Definition 2.5. To show that S does not satisfy condition B′, it suffices to prove that R has the absolute (n^*, r) covering property. Thus let Σ_1 be a set with n^* elements and let $\Gamma : R \longrightarrow 2^{\Sigma_1}$ be a map such that $|\Gamma(\alpha)| = \alpha$ for all $\alpha \in R$. We assume that $\alpha + \beta > n^*$ for all $\alpha, \beta \in R$ with $\alpha \neq \beta$, so we must have $\Gamma(\alpha) \cap \Gamma(\beta) \neq \emptyset$, and we are done.

In order to prove that S satisfies condition C, it suffices to prove that if it does not satisfy condition C, then it does not satisfy condition C′. Thus assume that S does not satisfy condition C, and let Q and R be as in Definition 2.6. Comparison with Definition 2.3 shows that we obtain a contradiction if we prove that R has the absolute $(n^*, r_1 r_2 - r_2 + 1; \{\gamma_o\})$ covering property. Using Definition 2.6(iv), select a set $R_1 \subset R$ such that $|R_1| = r_1 r_2 - r_2 + 2$ and $\gamma_o \in R_1$. Let Σ_1 be a set with n^* elements and let $\Gamma : R \to 2^{\Sigma_1}$ be any map such that $|\Gamma(\alpha)| = \alpha$ for all $\alpha \in R$. It is assumed that $\alpha + \beta > n^*$ for all $\alpha, \beta \in R$, $\alpha \neq \beta$, so necessarily $\Gamma(\alpha) \cap \Gamma(\beta) \neq \emptyset$ for all $\alpha, \beta \in R_1$. This proves that R has the absolute $(n^*, r_1 r_2 - r_2 + 1; \{\gamma_o\})$ covering property.

In order to prove that S satisfies condition *D*, it suffices to prove that if S does not satisfy condition D, then it does not satisfy condition B′. Thus assume that S does not satisfy condition D, and let R, R_1, R_2, m and r be as in Definition 2.7. Take Q to be the empty set. Referring to Definition 2.4, we see that to obtain a contradiction it suffices to prove that R has the absolute (n, r) covering property. Let Σ be a set with n elements and, for each $\alpha \in R$, let $\Gamma(\alpha) \subset \Sigma$ be a set with α elements. Because we assume that

$$\sum_{\alpha \in R_1} \alpha > n,$$

there exist $\alpha_1, \alpha_2 \in R_1$ with $\alpha_1 \neq \alpha_2$ and $\Gamma(\alpha_1) \cap \Gamma(\alpha_2) \neq \emptyset$. Because $\alpha + \beta > n$ for all $\alpha \in R$ and $\beta \in R_2$ with $\alpha \neq \beta$, we have $\Gamma(\alpha) \cap \Gamma(\beta) \neq \emptyset$ for all such α and β. If $S = \{\alpha_1, \alpha_2\} \cup R_2$, then S has $r+1$ elements and $\Gamma(\alpha) \cap \Gamma(\beta) \neq \emptyset$ for all $\alpha, \beta \in S$. This shows that R has the absolute (n, r) covering property.

It remains to prove that S satisfies condition E. We argue by contradiction and suppose that S does not satisfy condition E, so that there exist disjoint subsets Q and R of S as in Definition 2.8. We know that S satisfies condition B′, and, comparing Definition 2.8 and Definition 2.4, we see that, in order to obtain a contradiction, it suffices to prove that R has the absolute (n^*, r) covering property. However, Proposition 2.1 implies that R has the absolute (n^*, r) covering property.

REMARK 2.1. Theorems 2.1–2.3 play an important role in [**12**], where, among other results, it is proved that $P(n) = Q(n)$ for $1 \leqslant n \leqslant 50$. The sets $P(n)$ are relatively easy to determine (with the aid of a computer), and we know that $P(n) \subset Q(n)$ for all n. If $S \subset \{j \in \mathbb{Z} \mid 1 \leqslant j \leqslant n\}$, then we need only check whether S is array-admissible for n when $\mathrm{lcm}(S) \notin P(n)$, and Theorems 2.1–2.3 provide a way of showing that most such sets are not array-admissible.

TABLE 1. Factorization of largest element of $Q(n)$ for $n \leqslant 50$.

n	Largest element of $Q(n)$	n	Largest element of $Q(n)$
2	2	26	$2^4 \cdot 3 \cdot 5 \cdot 13$
3	3	27	$2^4 \cdot 3^2 \cdot 5 \cdot 7$
4	2^2	28	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
5	$2 \cdot 3$	29	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
6	$2^2 \cdot 3$	30	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
7	$2^2 \cdot 3$	31	$2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
8	$2^3 \cdot 3$	32	$2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
9	$2^3 \cdot 3$	33	$2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
10	$2^2 \cdot 3 \cdot 5$	34	$2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
11	$2^2 \cdot 3 \cdot 5$	35	$2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
12	$2^3 \cdot 3 \cdot 5$	36	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
13	$2^3 \cdot 3 \cdot 5$	37	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
14	$2^3 \cdot 3 \cdot 7$	38	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
15	$2^2 \cdot 3^2 \cdot 5$	39	$2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 17$
16	$2^4 \cdot 3 \cdot 7$	40	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$
17	$2^2 \cdot 3 \cdot 5 \cdot 7$	41	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$
18	$2^2 \cdot 3 \cdot 5 \cdot 7$	42	$2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
19	$2^3 \cdot 3 \cdot 5 \cdot 7$	43	$2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
20	$2^4 \cdot 3 \cdot 5 \cdot 7$	44	$2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
21	$2^4 \cdot 3 \cdot 5 \cdot 7$	45	$2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
22	$2^4 \cdot 3 \cdot 5 \cdot 7$	46	$2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
23	$2^4 \cdot 3 \cdot 5 \cdot 7$	47	$2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
24	$2^4 \cdot 3 \cdot 5 \cdot 11$	48	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
25	$2^4 \cdot 3 \cdot 5 \cdot 11$	49	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
		50	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$

REMARK 2.2. In [**9**] and elsewhere, the first author has computed the largest element of $P^*(n)$ by hand for $n \leqslant 32$. As already noted, a computer-assisted calculation of $Q(n)$ for $n \leqslant 50$ has been obtained in [**12**]. For the reader's interest, we provide Table 1, which shows the factorization of the largest element of $Q(n)$ for $n \leqslant 50$. We refer the reader to [**12**] for further details of the computation of $Q(n)$ and the theorems which facilitate the computation.

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