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Math 640
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Homework 15

Problem 5

Throughout this document, for a family $\mathcal{F}$ of $\{\pm 1\}$-sequences, let

$$\mathrm{wt}\left(\mathcal{F}\right) = \sum_{w \in \mathcal{F}} t^{\mathrm{Happy}(w)}.$$

Generating Function for $P\left(n, t\right)$

We will prove the following:

Proposition 1. *Let*

$$p\left(t, q\right) = \sum_{n=0}^{\infty} P\left(n, t\right).$$

Then,

$$p\left(t, q\right) = \frac{2}{\sqrt{1 - 4t^2 q} + \sqrt{1 - 4q}}.$$

We will use the following lemmas:

Lemma 1. *Let $\mathcal{C}_n$ denote the family of $\{\pm 1\}$-sequences with n 1's and n -1's whose partial sums are all nonnegative. Let $C\left(n, t\right) = \mathrm{wt}\left(\mathcal{C}_n\right)$, and let*

$$c\left(t, q\right) = \sum_{n=0}^{\infty} C\left(n, t\right).$$

Then,

$$c\left(t, q\right) = \frac{1 - \sqrt{1 - 4t^2 q}}{2t^2 q}.$$

Proof. Let $\mathcal{C} = \bigcup_{n=0}^{\infty} \mathcal{C}_n$. Every sequence in $\mathcal{C}$ is either empty or of the form

$$\left[1, \text{sequence in } \mathcal{C}, -1, \text{sequence in } \mathcal{C}\right].$$

The empty sequence gives a contribution of 1 to $c\left(t, q\right)$, and the 1 and -1 introduce a factor of q and a factor of t^2 since we are nonnegative. So, $c\left(t, q\right)$ satisfies

$$c\left(t, q\right) = 1 + t^2 q c\left(t, q\right)^2.$$

Solving the quadratic equation gives the desired function. □

Lemma 2. *Let $\overline{\mathcal{C}}_n$ denote the family of $\{\pm 1\}$-sequences with n 1's and n -1's whose partial sums are all nonpositive. Let $\overline{C}(n,t) = \mathrm{wt}\left(\overline{\mathcal{C}}_n\right)$, and let*

$$\overline{c}(t,q) = \sum_{n=0}^{\infty} \overline{C}(n,t).$$

Then,

$$\overline{c}(t,q) = \frac{1-\sqrt{1-4q}}{2q}.$$

Proof. Let $\overline{\mathcal{C}} = \bigcup_{n=0}^{\infty} \overline{\mathcal{C}}_n$. Every sequence in $\overline{\mathcal{C}}$ is either empty or of the form

$$\left[-1, \text{sequence in } \overline{\mathcal{C}}, 1, \text{sequence in } \overline{\mathcal{C}}\right].$$

The empty sequence gives a contribution of 1 to $\overline{c}(t,q)$, and the 1 and -1 introduce a factor of q. But, no t's are introduced, since we are nonpositive. So, $\overline{c}(t,q)$ satisfies

$$\overline{c}(t,q) = 1 + q\overline{c}(t,q)^2.$$

Solving the quadratic equation gives the desired function. □

We will now prove Proposition 1.

Proof. Let $\mathcal{P}_n$ be the set of $\{\pm 1\}$-sequences with n 1's and n -1's, and let $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$. We see that $P(n,t) = \mathrm{wt}(\mathcal{P}_n)$. Every sequence in $\mathcal{P}$ is one of the following:

- Empty.
- Of the form $[1, \text{sequence in } \mathcal{C}, -1, \text{sequence in } \mathcal{P}]$.
- Of the form $\left[-1, \text{sequence in } \overline{\mathcal{C}}, 1, \text{sequence in } \mathcal{P}\right]$.

By similar reasoning as used in the lemmas, $p(t,q)$ therefore satisfies

$$p(t,q) = 1 + qt^2 c(t,q)\, p(t,q) + q\overline{c}(t,q)\, p(t,q).$$

Solving for $p(t,q)$, substituting in the functions for $c(t,q)$ and $\overline{c}(t,q)$ and simplifying gives the desired answer. □

Proof of Conjecture for $P(n,t)$

Proof. Our conjecture is equivalent to

$$p(t,q) = \sum_{n=0}^{\infty} C(n)\, q^n \sum_{i=0}^{n} t^{2i}.$$

But,

$$\sum_{i=0}^{n} t^{2i} = \frac{t^{2n+2}-1}{t^2-1},$$

so it suffices to show that

$$p(t,q) = \sum_{n=0}^{\infty} C(n)\, q^n \left(\frac{t^{2n+2}-1}{t^2-1}\right).$$

This is equal to

$$\frac{1}{t^2-1}\left(\sum_{n=0}^{\infty} C(n)\, q^n t^{2n+2} - \sum_{n=0}^{\infty} C(n)\, q^n\right).$$

Maple knows that this is equal to

$$\frac{1}{t^2-1}\left(\frac{1-\sqrt{1-4t^2q}}{2q} - \frac{1-\sqrt{1-4q}}{2q}\right).$$

This is equal to the formula from Proposition 1. (This can be seen easily by rationalizing the denominator of the latter using the conjugate, or Maple can prove it.) □

Generating Function for $Q(n,t)$

We will prove the following:

Proposition 2. *Let*

$$r(t,q) = \sum_{n=0}^{\infty} Q(n,t).$$

Then,

$$r(t,q) = \frac{2 + \frac{4tq}{1-2tq+\sqrt{1-4t^2q^2}} + \frac{4q}{1-2q+\sqrt{1-4q^2}}}{\sqrt{1-4t^2q^2}+\sqrt{1-4q^2}}.$$

We will use the following lemmas:

Lemma 3. *Let $\mathcal{U}_n$ denote the family of $\{\pm 1\}$-sequences of length n whose partial sums are all nonnegative. Let $U(n,t) = \mathrm{wt}(\mathcal{U}_n)$, and let*

$$u(t,q) = \sum_{n=0}^{\infty} U(n,t).$$

Then,

$$u(t,q) = \frac{1}{1-tq-t^2q^2c(t,q^2)}.$$

Proof. Let $\mathcal{U} = \bigcup_{n=0}^{\infty} \mathcal{U}_n$. Every sequence in $\mathcal{U}$ is one of the following:

- Empty.
- Of the form $[1, \text{sequence in } \mathcal{U}]$.
- Of the form $[1, \text{sequence in } \mathcal{C}, -1, \text{sequence in } \mathcal{U}]$.

By similar reasoning as used earlier, along with the fact that before a sequence of length $2n$ gave a contribution of q^n but now it contributes q^{2n}, $u(t,q)$ therefore satisfies

$$u(t,q) = 1 + tqu(t,q) + t^2q^2c\left(t,q^2\right)u(t,q).$$

Solving for $u(t,q)$ gives the desired function. □

Lemma 4. *Let $\overline{\mathcal{U}}_n$ denote the family of $\{\pm 1\}$-sequences of length n whose partial sums are all nonpositive. Let $\overline{U}(n,t) = \mathrm{wt}\left(\overline{\mathcal{U}}_n\right)$, and let*

$$\overline{u}(t,q) = \sum_{n=0}^{\infty} \overline{U}(n,t).$$

Then,

$$\overline{u}(t,q) = \frac{1}{1 - q - q^2\overline{c}(t,q^2)}.$$

Proof. Let $\overline{\mathcal{U}} = \bigcup_{n=0}^{\infty} \overline{\mathcal{U}}_n$. Every sequence in $\overline{\mathcal{U}}$ is one of the following:

- Empty.
- Of the form $\left[-1, \text{sequence in } \overline{\mathcal{U}}\right]$.
- Of the form $\left[-1, \text{sequence in } \overline{\mathcal{C}}, -1, \text{sequence in } \overline{\mathcal{U}}\right]$.

By similar reasoning as used earlier, along with the fact that before a sequence of length $2n$ gave a contribution of q^n but now it contributes q^{2n}, $\overline{u}(t,q)$ therefore satisfies

$$\overline{u}(t,q) = 1 + q\overline{u}(t,q) + q^2\overline{c}\left(t,q^2\right)\overline{u}(t,q).$$

Solving for $\overline{u}(t,q)$ gives the desired function. □

We will now prove Proposition 2.

Proof. Let $\mathcal{Q}_n$ be the set of $\{\pm 1\}$-sequences of length n, and let $\mathcal{Q} = \bigcup_{n=0}^{\infty} \mathcal{Q}_n$. We see that $Q(n,t) = \mathrm{wt}(\mathcal{Q}_n)$. Every sequence in $\mathcal{Q}$ is one of the following:

- Empty.
- Of the form $[1, \text{sequence in } \mathcal{U}]$.

- Of the form $\left[-1, \text{sequence in } \overline{\mathcal{U}}\right]$.
- Of the form $[1, \text{sequence in } \mathcal{C}, -1, \text{sequence in } \mathcal{Q}]$.
- Of the form $\left[-1, \text{sequence in } \overline{\mathcal{C}}, 1, \text{sequence in } \mathcal{Q}\right]$.

By similar reasoning as used in the lemmas, $r(t,q)$ therefore satisfies

$$r(t,q) = 1 + tqu(t,q) + q\overline{u}(t,q) + t^2q^2c\left(t,q^2\right)r(t,q) + q^2\overline{c}\left(t,q^2\right)r(t,q).$$

Solving for $r(t,q)$, substituting in the functions for $u(t,q)$, $\overline{u}(t,q)$, $c(t,q)$, and $\overline{c}(t,q)$ and simplifying (a bit) gives the desired answer. $\square$