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Math 640
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Homework 11

Problem 4

We will prove the following:

Lemma 1 (Etheridge, Lemma 1.3.2)**.** *Suppose that the risk-free dollar interest rate is r. Let S_0 be the initial stock price, and assume that the price will either be S_0u or S_0d at time T. Also, assume that $d < e^{rT} < u$. Then, the price of European option with payoff $C(S_T)$ (for the buyer) at time T is*

$$\left(\frac{1-de^{-rT}}{u-d}\right)C(S_0u)+\left(\frac{ue^{-rT}-1}{u-d}\right)C(S_0d).$$

Moreover, the seller of the option can construct a portfolio whose value at time T is exactly $(S_T + C(S_T))$ by using the money received for the option to buy

$$\frac{C(S_0u)-C(S_0d)}{S_0u-S_0d}$$

units of stock at time zero and holding the remainder at the interest rate r.

Proof. Let P denote the price of the option. There are three conditions that need to be met:

- The seller must invest some money, x_2, in the stock and hold some money, x_1, at the interest rate r. We allow the seller to borrow money as well, so x_1 can be negative. But, the seller must invest all the money he or she receives from the buyer (since failure to do so would lose out on free interest), so we have $x_1 + S_0x_2 = P$. The value of the seller's portfolio at time T is then $e^{rT}x_1 + S_Tx_2$.

- If the stock price is S_0u at time T, we want the value of seller's portfolio to be zero. Otherwise, either the buyer or the seller would have an opportunity for arbitrage (in combination with the next item). This corresponds to the equation $e^{rT}x_1 + S_0ux_2 - C(S_0u) = 0$, since the payoff for the buyer is a cost for the seller.

- If the stock price is S_0d at time T, we also want the value of seller's portfolio to be zero. Otherwise, either the buyer or the seller would have an opportunity for arbitrage (in combination with the previous item). This corresponds to the equation $e^{rT}x_1 + S_0dx_2 - C(S_0d) = 0$, since the payoff for the buyer is a cost for the seller.

So, we have the following system of three linear equations, in the unknowns x_1, x_2, and P:

$$\begin{cases} x_1 + S_0 x_2 = P \\ e^{rT} x_1 + S_0 u x_2 = C(S_0 u) \\ e^{rT} x_1 + S_0 d x_2 = C(S_0 d). \end{cases}$$

Subtracting the third equation from the second yields

$$(S_0 u - S_0 d)\, x_2 = C(S_0 u) - C(S_0 d).$$

Solving for x_2 gives

$$x_2 = \frac{C(S_0 u) - C(S_0 d)}{S_0 u - S_0 d},$$

thereby proving the second part of the lemma (since x_2 is the amount of stock the seller buys).

We now continue to solve the system. Substituting into the second equation gives

$$e^{rT} x_1 + S_0 u \left(\frac{C(S_0 u) - C(S_0 d)}{S_0 u - S_0 d} \right) = C(S_0 u).$$

Solving for x_1 gives (noting that the S_0's cancel in the complicated term):

$$x_1 = e^{-rT} C(S_0 u) - e^{-rT} u \left(\frac{C(S_0 u) - C(S_0 d)}{u - d} \right).$$

So, $P = x_1 + S_0 x_2$, meaning that

$$\begin{aligned} P &= e^{-rT} C(S_0 u) - e^{-rT} u \left(\frac{C(S_0 u) - C(S_0 d)}{u - d} \right) + \frac{C(S_0 u) - C(S_0 d)}{u - d} \\ &= \frac{e^{-rT} C(S_0 u)(u - d)}{u - d} - e^{-rT} u \left(\frac{C(S_0 u) - C(S_0 d)}{u - d} \right) + \frac{C(S_0 u) - C(S_0 d)}{u - d} \\ &= C(S_0 u) \left(\frac{u e^{-rT} - d e^{-rT} - u e^{-rT} + 1}{u - d} \right) + C(S_0 d) \left(\frac{u e^{-rT} - 1}{u - d} \right) \\ &= \left(\frac{1 - d e^{-rT}}{u - d} \right) C(S_0 u) + \left(\frac{u e^{-rT} - 1}{u - d} \right) C(S_0 d). \end{aligned}$$

as required. $\square$