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Math 640
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Homework 7

Problem 3

We will prove the following result.

Theorem 1. *The unique polynomial of degree less than n interpolating the points $(x_1, y_1), \ldots, (x_n, y_n)$ is*

$$f(x) = \sum_{j=1}^{n} y_j \prod_{i \neq j} \frac{x - x_i}{x_j - x_i}.$$

We will use the following lemma:

Lemma 1. *A polynomial of degree less than n that vanishes at n distinct places must be the zero polynomial.*

Proof. Let $f(x)$ be a polynomial of degree less than n that vanishes at n distinct places $x_1, \ldots, x_n$. Then

$$\prod_{i=1}^{n} (x - x_i)$$

divides f. Since f is degree less than n, this means that f is identically 0, as required. □

We will now prove the desired result.

Proof. Let $f(x)$ be as in the theorem statement. It is easy to see that $f(x_i) = y_i$ for all i, since all terms in the sum become zero except for one of them, which is y_i. So, all that remains is to prove uniqueness. Let $g(x)$ be any polynomial of degree less than n such that $g(x_i) = y_i$ for all i. Then, $(g - f)(x)$ is zero at all of the x_i, and it has degree less than n. Hence, $g - f \equiv 0$, so $g = f$, as required. □