Nathan Fox
Math 640
April 28, 2013

Homework 25

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Theorem 1. *We have*

$$e^q = \prod_{i=1}^{\infty} \left(1 - q^i\right)^{-\frac{\mu(i)}{i}},$$

where μ denotes the classical Möbius function from number theory.

Proof. Taking logarithms of both sides, it will suffice to show that

$$q = \sum_{i=1}^{\infty} -\frac{\mu(i)}{i} \log\left(1 - q^i\right).$$

Recall that

$$\log(1 - q) = \sum_{j=1}^{\infty} -\frac{q^j}{j}.$$

Hence, it will suffice to show that

$$q = \sum_{i=1}^{\infty} -\frac{\mu(i)}{i} \sum_{j=1}^{\infty} -\frac{q^{ij}}{j}.$$

We have

$$\sum_{i=1}^{\infty} -\frac{\mu(i)}{i} \sum_{j=1}^{\infty} -\frac{q^{ij}}{j} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu(i) \frac{q^{ij}}{ij}.$$

The coefficient on q^n in this sum equals

$$\sum_{ij=n} \frac{\mu(i)}{n} = \frac{1}{n} \sum_{i|n} \mu(i).$$

The primary identity satisfied by the Möbius function is that

$$\sum_{i|n} \mu(i) = \begin{cases} 1 & n = 1 \\ 0 & n > 1. \end{cases}$$

Hence, we have

$$\sum_{ij=n} \frac{\mu(i)}{n} = \begin{cases} 1 & n = 1 \\ 0 & n > 1. \end{cases}$$

This implies that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu(i) \frac{q^{ij}}{ij} = q,$$

as required. □