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Math 640
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Homework 22

1

Let $w(i)$ denote the probability of winning a Gambler's ruin game, with a loaded coin with probability of winning p if your current capital is i and the exit capital is N. We have the recurrence

$$w(i) = pw(i+1) + (1-p)w(i-1).$$

We can rewrite this as

$$(1-p)w(i-1) - w(i) + pw(i+1) = 0.$$

The corresponding equation is

$$(1-p)x^{-1} - 1 + px = 0,$$

which is equivalent to

$$(1-p) - x + px^2 = 0.$$

The solutions to this equation are

$$\alpha = \frac{1+\sqrt{1+4p(p-1)}}{2p}$$

and

$$\beta = \frac{1-\sqrt{1+4p(p-1)}}{2p}.$$

Hence, we have $w(i)\, c_1\alpha^i + c_2\beta^i$ for some c_1, c_2. Using the values $w(0) = 0$ and $w(N) = 1$, we obtain the equations

$$\begin{cases} c_1 + c_2 = 0 \\ c_1\alpha^N + c_2\beta^N = 1. \end{cases}$$

Solving yields $c_1 = \frac{1}{\alpha^N - \beta^N}$ and $c_2 = \frac{1}{\beta^N - \alpha^N}$. Hence, we have

$$\begin{aligned} w(i) &= \frac{\left(\frac{1+\sqrt{1+4p(p-1)}}{2p}\right)^i}{\left(\frac{1+\sqrt{1+4p(p-1)}}{2p}\right)^N - \left(\frac{1-\sqrt{1+4p(p-1)}}{2p}\right)^N} + \frac{\left(\frac{1-\sqrt{1+4p(p-1)}}{2p}\right)^i}{\left(\frac{1-\sqrt{1+4p(p-1)}}{2p}\right)^N - \left(\frac{1+\sqrt{1+4p(p-1)}}{2p}\right)^N} \\ &= \frac{\left(\frac{1+\sqrt{1+4p(p-1)}}{2p}\right)^i - \left(\frac{1-\sqrt{1+4p(p-1)}}{2p}\right)^i}{\left(\frac{1+\sqrt{1+4p(p-1)}}{2p}\right)^N - \left(\frac{1-\sqrt{1+4p(p-1)}}{2p}\right)^N} \end{aligned}$$

3

Let $d(i)$ denote the expected duration of a Gambler's ruin game, with a loaded coin with probability of winning p if your current capital is i and the exit capital is N. We have the recurrence

$$d(i) = 1 + pd(i+1) + (1-p)\,d(i-1).$$

We can rewrite this as

$$(1-p)\,d(i-1) - d(i) + pd(i+1) = -1.$$

The corresponding equation is

$$(1-p)\,x^{-1} - 1 + px = 0,$$

which is equivalent to

$$(1-p) - x + px^2 = 0.$$

The solutions to this equation are

$$\alpha = \frac{1+\sqrt{1+4p(p-1)}}{2p}$$

and

$$\beta = \frac{1-\sqrt{1+4p(p-1)}}{2p}.$$

Hence, we have $d(i)\,c_1\alpha^i + c_2\beta^i + PS(i)$ for some c_1, c_2 and a particular solution PS. One can check that a particular solution to this recurrence is

$$PS(i) = \frac{i}{1-2p},$$

obtained by guessing a solution of the form $ai+b$ for some constant a. So, we have $d(i)\,c_1\alpha^i + c_2\beta^i + \frac{i}{1-2p}$ for some c_1, c_2. Using the values $d(0) = 0$ and $d(N) = 0$, we obtain the equations

$$\begin{cases} c_1 + c_2 = 0 \\ c_1\alpha^N + c_2\beta^N = \frac{N}{2p-1}. \end{cases}$$

Solving yields $c_1 = \frac{N}{(2p-1)(\alpha^N - \beta^N)}$ and $c_2 = \frac{N}{(2p-1)(\beta^N - \alpha^N)}$. Hence, we have

$$
\begin{aligned}
d\left(i\right) = & \frac{N\left(\frac{1+\sqrt{1+4p(p-1)}}{2p}\right)^i}{(2p-1)\left(\left(\frac{1+\sqrt{1+4p(p-1)}}{2p}\right)^N - \left(\frac{1-\sqrt{1+4p(p-1)}}{2p}\right)^N\right)} \\
& + \frac{N\left(\frac{1-\sqrt{1+4p(p-1)}}{2p}\right)^i}{(2p-1)\left(\left(\frac{1-\sqrt{1+4p(p-1)}}{2p}\right)^N - \left(\frac{1+\sqrt{1+4p(p-1)}}{2p}\right)^N\right)} + \frac{i}{1-2p} \\
= & \frac{N\left(\left(\frac{1+\sqrt{1+4p(p-1)}}{2p}\right)^i - \left(\frac{1-\sqrt{1+4p(p-1)}}{2p}\right)^i\right)}{(2p-1)\left(\left(\frac{1+\sqrt{1+4p(p-1)}}{2p}\right)^N - \left(\frac{1-\sqrt{1+4p(p-1)}}{2p}\right)^N\right)} + \frac{i}{1-2p}.
\end{aligned}
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