Homework 14

4

Proposition 1. *We have*
$$V(2, p) = \frac{p(p-1)}{p+1}.$$

Proof. The proof is by induction on p. First, we know that $V(2,1) = 0$, which matches the formula. Now, assume that we know that
$$V(2, p-1) = \frac{(p-1)(p-2)}{p}.$$

Recall that Boyce proves that
$$V(1, p) = \frac{p^2}{p-1}.$$

So, we have

$$
\begin{aligned}
V(2, p) &= \left(\frac{2}{p+2}\right)(-1 + V(1,p)) + \left(\frac{p}{p+2}\right)(1 + V(2, p-1)) \\
&= \left(\frac{2}{p+2}\right)\left(-1 + \frac{p^2}{p-1}\right) + \left(\frac{p}{p+2}\right)\left(1 + \frac{(p-1)(p-2)}{p}\right) \\
&= \left(\frac{2}{p+2}\right)\left(\frac{p^2 - p - 1}{p-1}\right) + \left(\frac{p}{p+2}\right)\left(\frac{p^2 - 2p + 2}{p}\right) \\
&= \frac{2p(p^2 - p - 1) + p(p+1)(p^2 - 2p + 2)}{p(p+1)(p+2)} \\
&= \frac{2p^3 - 2p^2 - 2p + (p^2 + p)(p^2 - 2p + 2)}{p(p+1)(p+2)} \\
&= \frac{2p^3 - 2p^2 - 2p + p^4 - 2p^3 + 2p^2 + p^3 - 2p^2 + 2p}{p(p+1)(p+2)} \\
&= \frac{p^4 + p^3 - 2p^2}{p(p+1)(p+2)} \\
&= \frac{p^2(p+2)(p-1)}{p(p+1)(p+2)} \\
&= \frac{p(p-1)}{p+1},
\end{aligned}
$$

as required. $\square$