

Turán theorems for unavoidable patterns

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ABSTRACT. We prove Turán-type theorems for two related Ramsey problems raised by Bollobás and by Fox and Sudakov. First, for $t \geq 3$, we show that any two-colouring of the complete graph on n vertices that is δ -far from being monochromatic contains an *unavoidable t -colouring* when $\delta \gg n^{-1/t}$, where an unavoidable t -colouring is any two-colouring of a clique of order $2t$ in which one colour forms either a clique of order t or two disjoint cliques of order t . Next, for $t \geq 3$, we show that any tournament on n vertices that is δ -far from being transitive contains an *unavoidable t -tournament* when $\delta \gg n^{-1/\lceil t/2 \rceil}$, where an unavoidable t -tournament is the blow-up of a cyclic triangle obtained by replacing each vertex of the triangle by a transitive tournament of order t . Conditional on a well-known conjecture about bipartite Turán numbers, both our results are sharp up to implied constants and hence determine the order of magnitude of the corresponding off-diagonal Ramsey numbers.

1. INTRODUCTION

The starting point of Ramsey theory, namely Ramsey's theorem [15], is the assertion that given any natural number $t \in \mathbb{N}$, every two-colouring (of the edges, here and elsewhere) of the complete graph K_n on n vertices contains a monochromatic copy of K_t for all large enough $n \in \mathbb{N}$; the asymptotic behaviour of the smallest such integer, namely the Ramsey number $\mathcal{R}(t)$, has been the subject of intense scrutiny (see [7, 5, 16, 3], for example) through the past seventy or so years.

A priori, one cannot expect to find any non-monochromatic patterns in a given two-colouring of a complete graph, since the colouring in question might itself be monochromatic. In the light of this, Bollobás [4] asked what non-monochromatic patterns one is guaranteed to find in any two-colouring of K_n that is δ -far from being monochromatic for some $\delta > 0$, where a two-colouring of K_n is said to be *δ -far from being monochromatic* if each colour in the colouring, henceforth red and blue, appears on at least δn^2 edges. Call a two-colouring of K_{2t} an *unavoidable t -colouring* if one colour class forms either a clique of order t or two disjoint cliques of order t . It is not hard to see that the only non-monochromatic patterns one could hope to find, even in a given two-colouring of K_n that is far from being monochromatic, are precisely such unavoidable colourings (since the given colouring might itself be of this form).

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Confirming Bollobás's prediction, Cutler and Montagh [4] showed that for any $t \in \mathbb{N}$ and $\delta > 0$, there exists a least integer $\mathcal{C}(t, \delta) \in \mathbb{N}$ such that any two-colouring of K_n that is δ -far from being monochromatic contains an unavoidable t -colouring for all $n \geq \mathcal{C}(t, \delta)$. Turning to quantitative estimates, Fox and Sudakov [8] subsequently determined the order of growth of the Ramsey number $\mathcal{C}(t, \delta)$, showing that $\mathcal{C}(t, \delta) = (1/\delta)^{\Theta(t)}$.

Our first result, which serves as a warm up for our second result (which is the focus of this paper), pins down the off-diagonal growth rate of the Ramsey number $\mathcal{C}(t, \delta)$ for each $t \geq 3$ as $\delta \rightarrow 0$; bounds of a similar type, albeit with suboptimal exponents, were recently obtained independently by Caro, Hansberg and Montejano [2]. Our first result in its Turán-type formulation is as follows.

Theorem 1.1. *For each integer $t \geq 3$, there exists a $C = C(t) > 0$ such that any two-colouring of the complete graph on $n \geq C$ vertices that is $Cn^{-1/t}$ -far from being monochromatic contains an unavoidable t -colouring.*

A well-known conjecture of Kovári, Sós and Turán [13], see also [10], asserts that

$$\text{ex}(n, K_{a,b}) = \Omega(n^{2-1/a}) \quad (1)$$

for all $b \geq a \geq 2$. Conditional on the truth of (1) in the case of $a = b = t$, our first result is easily seen to be sharp up to the multiplicative factor $C(t)$ in its statement for each $t \geq 3$. Theorem 1.1 therefore determines, conditional on (1), the order of magnitude of $\mathcal{C}(t, \delta)$ when t is much smaller than $1/\delta$; indeed, it follows from our result that, for each fixed $t \geq 3$ and in the limit of $\delta \rightarrow 0$, we have

$$\mathcal{C}(t, \delta) = \Theta\left(\left(\frac{1}{\delta}\right)^t\right). \quad (2)$$

Next, we consider a closely related problem for tournaments. Recall that a tournament is a directed graph obtained by orienting an undirected complete graph, and a tournament is said to be transitive if there is a linear ordering of the vertices in which all the edges point in the same direction. The analogue of Ramsey's theorem for tournaments was proved by Erdős and Moser [6] who showed that for each $t \in \mathbb{N}$, every tournament on n vertices contains a transitive subtournament of order t for all sufficiently large $n \in \mathbb{N}$. As before, one cannot hope to find any non-transitive patterns in a general tournament, since the tournament in question might itself be transitive. Hence, in the spirit of Bollobás's question, Fox and Sudakov [8] asked what non-transitive tournaments one is guaranteed to find in any tournament that is δ -far from being transitive for some $\delta > 0$, where a tournament of order n is said to be δ -far from being transitive if the orientation of at least δn^2 of its edges need to be reversed in order to make it transitive. Consider the tournament with $3t$ vertices, which we call the *unavoidable t -tournament*, formed by taking three disjoint transitive tournaments each of order t on vertex sets V_1 , V_2 and V_3 , and directing edges from V_i to V_{i+1} for

each $i = 0, 1, 2$, with indices being taken modulo 3. As before, it is not hard to see that the only non-transitive patterns one could hope to find, even in a given tournament that is far from being transitive, are precisely such unavoidable tournaments (since the given tournament might itself be of this form). Fox and Sudakov [8] showed that for any $t \in \mathbb{N}$ and $\delta > 0$, there exists a least integer $\mathcal{D}(t, \delta) \in \mathbb{N}$ such that any tournament on n vertices that is δ -far from being transitive contains an unavoidable t -tournament for all $n \geq \mathcal{D}(t, \delta)$. Turning to quantitative estimates, Long [14] subsequently determined the order of growth of the Ramsey number $\mathcal{D}(t, \delta)$, showing in particular that $\mathcal{D}(t, \delta) = (1/\delta)^{\Theta(t)}$.

Our second result, our main contribution here, pins down the off-diagonal growth rate of the Ramsey number $\mathcal{D}(t, \delta)$ for each $t \geq 3$ as $\delta \rightarrow 0$. Our second theorem, also in its Turán-type formulation, is as follows.

Theorem 1.2. *For each integer $t \geq 3$, there exists a $C = C(t) > 0$ such that any tournament on $n \geq C$ vertices that is $Cn^{-1/\lceil t/2 \rceil}$ -far from being transitive contains an unavoidable t -tournament.*

Our second result is also tight up to the multiplicative factor $C(t)$ in its statement for each $t \geq 3$, again conditional on the truth of (1) in the case of $a = \lceil t/2 \rceil$ and $b = t$. Theorem 1.2 hence determines, conditional on (1), the order of magnitude of $\mathcal{D}(t, \delta)$ when t is much smaller than $1/\delta$; indeed, it follows from our result that, for each fixed $t \geq 3$ and in the limit of $\delta \rightarrow 0$, we have

$$\mathcal{D}(t, \delta) = \Theta\left(\left(\frac{1}{\delta}\right)^{\lceil t/2 \rceil}\right). \quad (3)$$

The rest of this paper is organised as follows. We gather some preliminary facts together in Section 2. We present the short proof of Theorem 1.1, as well as the simple construction demonstrating its sharpness, in Section 3. We postpone the more delicate proof of Theorem 1.2, as well as the construction demonstrating its sharpness, to Section 4. Our main results address unavoidable patterns of order $t \geq 3$, and the case of $t = 1$ is trivial; the exceptional case of $t = 2$ however demonstrates some anomalous behaviour, which we address in Section 5. Finally, we conclude with a discussion of open problems in Section 6.

2. PRELIMINARIES

Here, we collect together the conventions we adopt when dealing with graphs, both directed and undirected, as well as a few useful results that we shall rely on.

Let $G = (V, E)$ be an undirected graph. We write $v(G)$ and $e(G)$ respectively for the number of vertices and edges of G . We denote the neighbourhood of a vertex $x \in V(G)$ by $N_G(x)$, and we write $d_G(x) = |N_G(x)|$ for the degree of x in G . More generally, for

a set of vertices $X \subset V(G)$, the common neighbourhood $N_G(X)$ of X is defined to be the set of vertices adjacent to all the vertices in X . Here and elsewhere, we omit the subscripts specifying the graph when the graph in question is clear from the context. Finally, for a set $X \subset V(G)$ of vertices, we write $G[X]$ for the subgraph of G induced by X , and given two disjoint sets $X, Y \subset V(G)$, we write $G[X, Y]$ for the induced bipartite subgraph between the vertex classes X and Y in G .

For a fixed graph H , the Turán number $\text{ex}(n, H)$ is the maximum number of edges in an n -vertex graph with no subgraph isomorphic to H . It is known that the Kovári–Sós–Turán conjecture (1) is, if true, tight up to multiplicative constants. The following fact, which may be deduced using the technique of dependent random choice, demonstrates this and a bit more; see [9], for example.

Proposition 2.1. *For all $k, t \in \mathbb{N}$, there exists a $C = C(k, t) > 0$ such that any graph G on n vertices with $e(G) \geq Cn^{2-1/t}$ edges contains a set $S \subset V(G)$ of k vertices in which each subset $X \subset S$ of t vertices satisfies $|N(X)| \geq k$. \square*

It will be convenient to record the fact that bipartite Turán problems are ‘degenerate’ in the following sense.

Proposition 2.2. *For all $t \in \mathbb{N}$ and $\varepsilon > 0$, there exists a $C = C(t, \varepsilon) > 0$ such that every bipartite graph G between vertex classes X and Y with $|X|, |Y| \geq C$ and $e(G) \geq \varepsilon|X||Y|$ contains a copy of $K_{t,t}$. \square*

We also need the following simple asymmetric analogue of the above claim.

Proposition 2.3. *Let G be a bipartite graph between vertex classes X and Y , with $|X| \leq |Y|$ and $e(G) \geq \varepsilon|X||Y|$, where $0 < \varepsilon < 1/2$. Then, provided $|X| \geq 4\varepsilon^{-1}k$, there is a copy of $K_{k, (\varepsilon/2e)^{k+1}|Y|}$ in G .*

Proof. Let $X' \subset X$ be the set of vertices $x \in X$ with $d(x) \geq \varepsilon|Y|/2$, and note that $|X'| \geq |X|/2 \geq 2\varepsilon^{-1}k$. Choose a subset $X'' \subset X'$ size exactly $2\varepsilon^{-1}k$. By double counting, we see that the set Y' of $y \in Y$ with at least k neighbours in X'' satisfies $|Y'| \geq \varepsilon|Y|/2$. As there are at most

$$\binom{2\varepsilon^{-1}k}{k} \leq (2e/\varepsilon)^k$$

sets of size k within X'' , at least $(\varepsilon/2e)^k|Y'| \geq (\varepsilon/2e)^{k+1}|Y|$ vertices in Y are all joined to all the vertices of some k -set contained in X'' , and we are done. \square

Our notation when dealing with directed graphs mirrors our notation for undirected graphs: for instance, we write $N^+(\cdot)$ for out-neighbourhoods, $d^-(\cdot)$ for in-degrees, and so on; also, we shall simply write xy for an edge in a digraph directed from x to y . We shall require some simple properties of orderings of tournaments that minimise the

number of ‘backward edges’. Let $\sigma = (v_1, v_2, \dots, v_n)$ be an ordering of the vertex set of an n -vertex digraph D . A forward edge in σ is an edge of D of the form $v_i v_j$ with $i < j$, and a backward edge is any edge of D that is not a forward edge. Given $i < j$, the interval or segment $[i, j]$ in σ refers to the set of vertices $\{v_i, v_{i+1}, \dots, v_j\}$. Also, for $i < j$, we say that the distance between v_i and v_j is $j - i$, the length of an edge then being the distance between its endpoints. Finally, given two disjoint sets $X, Y \subset V(D)$, we say X precedes Y in σ , or $X < Y$ in short, if each vertex of X appears in σ before every vertex of Y .

The following proposition, see [14], follows easily from ‘switching’ arguments.

Proposition 2.4. *Let T be an n -vertex tournament and let $\sigma = (v_1, v_2, \dots, v_n)$ be an ordering of $V(T)$ that minimises the number of backward edges. Then for any $1 \leq i < j \leq n$, we have*

- (1) $|N^+(v_i) \cap [i+1, j]| \geq (j-i)/2$,
- (2) $|N^-(v_j) \cap [i, j-1]| \geq (j-i)/2$, and
- (3) σ minimises, when restricted to the interval $[i, j]$, the number of backward edges in $T[[i, j]]$. \square

We also require the following variant of a lemma due to Long [14] that says, roughly speaking, that a tournament either has many ‘long backward edges’ or a subtournament that is ‘further from being transitive’.

Lemma 2.5. *Let $\alpha > 0$, let T be an n -vertex tournament that is α -far from being transitive and suppose that σ is an ordering of $V(T)$ that minimises the number of backward edges. Then T either contains*

- (1) $\alpha n^2/1000$ backwards edges in σ each of length at least $n/50$, or
- (2) a subtournament of order at least $n/20$ that is 6α -far from being transitive.

Proof. Suppose that the number of backward edges in σ of length at least $n/50$ is at most $\alpha n^2/1000$, so writing F for the set of backward edges in σ of length at most $n/50$, we have $|F| \geq (999/1000)\alpha n^2$.

Let A be the initial segment of the first $n/20$ vertices in σ , and let B be the terminal segment of the final $n/20$ vertices in σ . If there are more than $6\alpha(n/20)^2$ backward edges within either A or B , then we are done by Proposition 2.4. Let $F' \subset F$ be the set of those edges with at most one endpoint in A or in B , and note that $|F'| \geq |F| - 12\alpha(n/20)^2 \geq (28/30)\alpha n^2$.

Now, each edge in F' has length at most $n/50$ and does not lie entirely within either A or B , so each edge of F' lies entirely within the interval $[n/40, 39n/40]$. Since we have excluded edges at the extremes of σ in F' , it is straightforward to check that a uniformly random interval of $n/20$ vertices contains both endpoints of an edge in F'

with probability at least $1/40$. Hence, there is an interval I of $n/20$ vertices in σ with at least $(1/40)(28/30)\alpha n^2$ backward edges within it. From Proposition 2.4, it follows that $T[I]$ is at least 9α -far from being transitive, as required. \square

We shall make use of Ramsey's theorem in its various guises; see [12], for instance.

Proposition 2.6. *For all $t \in \mathbb{N}$, there exist integers $\mathcal{R}(t)$, $\mathcal{B}(t)$ and $\mathcal{T}(t)$ such that the following hold. Every two-colouring of K_n with $n \geq \mathcal{R}(t)$ contains a monochromatic copy of K_t . Every two-colouring of $K_{n,n}$ with $n \geq \mathcal{B}(t)$ contains a monochromatic copy of $K_{t,t}$. Every tournament on $n \geq \mathcal{T}(t)$ vertices contains a transitive subtournament on t vertices.* \square

Finally, a word on asymptotic notation is also in order. We shall make use of standard asymptotic notation; the variable tending to infinity will always be n unless we explicitly specify otherwise. When convenient, we shall also make use of some notation (of Vinogradov) that might be considered non-standard: given functions $f(n)$ and $g(n)$, we write $f \ll g$ if $f = O(g)$ and $f \gg g$ if $g = O(f)$. Here, constants suppressed by the asymptotic notation may depend on fixed parameters such as t , but not on n or quantities depending on n such as δ .

3. COLOURINGS

In this section, we deal with unavoidable colourings. We start by presenting an extremal construction complementing Theorem 1.1.

Proposition 3.1. *For each integer $t \geq 3$ and all large enough $n \in \mathbb{N}$, there is a two-colouring of K_n not containing an unavoidable t -colouring that is δ_t -far from being monochromatic, where $\delta_t(n) = (\text{ex}(n, K_{t,t}) - 1)/2n^2$.*

In particular, if the conjectural bound (1) holds, we actually have $\delta_t(n) \gg n^{-1/t}$ in the above construction for each $t \geq 3$.

Proof of Proposition 3.1. Given $t \geq 3$, start with a graph G on $[n]$ with $m = \text{ex}(n, K_{t,t}) - 1$ edges that does not contain any copies of $K_{t,t}$, and pass to a bipartite subgraph H of G with at least $m/2$ edges. Now, colour the edges of the complete graph on $[n]$ by colouring all the edges of H red, and all the other edges blue. The construction ensures that there is no red clique on three vertices, and that there are no red copies of $K_{t,t}$. Since $m = o(n^2)$, clearly the number of both the red edges and the blue edges is at least $m/2$ provided n is sufficiently large, so the claim follows. \square

Having demonstrated its sharpness, we now give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us fix $C = C(t)$ to be large enough to support all of the estimates that follow, and suppose that we have a two-coloured complete graph G on

$n \geq C$ vertices in which the number of both the red and the blue edges is at least $Cn^{2-1/t}$. We assume that G is a counterexample to the result that does not contain an unavoidable t -colouring and thereby derive a contradiction.

We denote the graphs spanned by the red and blue edges of G by R and B respectively. We also assume, without loss of generality, that there are at least as many blue edges as there are red edges in G .

The first step in the proof is to show that we may remove a very small number of vertices from G so that there are no red copies of K_t in the resulting graph.

Claim 3.2. *For every $\varepsilon > 0$, there exists $C_1 = C_1(t, \varepsilon)$ so that we may find a set $S \subset V(G)$ of size at most C_1 such that in $G' = G[V(G) \setminus S]$, every vertex is incident to at least $(1 - 2\varepsilon)n'$ blue edges, where $n' = v(G')$.*

Proof. We fix C_1 to be large enough, with the benefit of hindsight, to support the argument that follows.

If the set S_r of vertices x with $d_R(x) \geq \varepsilon n$ has size at most C_1 , the claim follows by taking $S = S_r$. Hence, assume that $|S_r| \geq C_1$. As blue is the most common colour, we know that $e(B) \geq \binom{n}{2}/2$, so there are at least $n/8$ vertices x for which $d_B(x) \geq n/4$; let S_b be a set of C_1 such vertices disjoint from S_r .

We claim that there is an $m = m(t, \varepsilon)$ such that S_r does not contain any blue copies of K_m , and such that S_b does not contain any red copies of K_m . To see this, assume that m is sufficiently large and that $X \subset S_r$ induces a blue copy of K_m . Then, provided m is large enough, Proposition 2.2 implies that there is a subset $Y \subset X$ of order t for which the common neighbourhood $|N_R(Y)| \geq \mathcal{R}(t)$. Ramsey's theorem applied to $N_R(Y)$ now shows that there is an unavoidable t -colouring in G , a contradiction. Hence, we may assume that S_r does not induce any blue copies of K_m in G . The same argument, with the colours interchanged, allows us to assume that S_b does not induce any red copies of K_m in G .

Now, from the bipartite form of Ramsey's theorem applied to the complete bipartite graph between S_r and S_b , we may find $S'_r \subset S_r$ and $S'_b \subset S_b$, both of order $\mathcal{R}(m)$, such that the complete bipartite graph between S'_r and S'_b is monochromatic. Applying Ramsey's theorem to each of S'_r and S'_b (combined with our earlier observation), we find a red copy of K_m inside S'_r and a blue copy of K_m inside S'_b , which together yield an unavoidable t -colouring in G , a contradiction. \square

We apply the previous claim with $\varepsilon = 1/(10t)$ to pass to a two-coloured complete graph G' where all the vertices are incident to many blue edges, and as before, we denote the graphs spanned by the red and the blue edges of G' by R' and B' respectively.

Claim 3.3. *There are no red copies of K_t in G' .*

Proof. Suppose that X forms a red clique on t vertices in G' . Since $d_{B'}(x) \geq (1 - 2\varepsilon)n'$ for each $x \in X$, it follows that $|N_{B'}(X)| \geq n' - t(n'/10t) \geq n'/2 \geq \mathcal{R}(t)$. By applying Ramsey's theorem to $N_{B'}(X)$, we find an unavoidable t -colouring in G , a contradiction. \square

Observe that the number of red edges in G' is at least $C_2(n')^{2-1/t}$ for some $C_2 = C_2(t)$, since we have removed at most C_1 vertices and $C_1 n$ red edges in passing from G to G' . Provided C_2 is large enough, it is easy to see that there are many red copies of $K_{t,t}$ in G' .

The second step in the proof is to find a reasonably ‘well distributed’ collection of such copies that we may use to produce an unavoidable t -colouring. If C_2 is large enough, then it follows from Proposition 2.1 that there is a set $Y \subset V(G')$ of size at least $\mathcal{R}(t)$ such that every $X \subset Y$ of order t satisfies $|N_{R'}(X)| \geq \mathcal{R}(t)$.

Claim 3.4. *There are no blue copies of K_t in $G'[Y]$.*

Proof. If $X \subset Y$ induces a blue clique on t vertices in G' , then since $|N_{R'}(X)| \geq \mathcal{R}(t)$, the previous claim combined with Ramsey's theorem allows us to find a blue clique of order t within $N_{R'}(X)$, and consequently, an unavoidable t -colouring in G ; again, we have a contradiction. \square

We may now finish as follows. Applying Ramsey's theorem to Y , we know there must exist either a red or a blue copy of K_t within Y since $|Y| \geq \mathcal{R}(t)$. By Claim 3.4, it must be a red copy which contradicts Claim 3.3; this completes the proof of Theorem 1.1. \square

4. TOURNAMENTS

In this section, we deal with unavoidable tournaments. In what follows, to save space, we write \mathcal{U}_t to denote the unavoidable t -tournament on $3t$ vertices.

4.1. Extremal construction. As before, we start by presenting an extremal construction complementing Theorem 1.2.

Proposition 4.1. *For each integer $t \geq 3$ and all large enough $n \in \mathbb{N}$, there is a tournament on n vertices not containing \mathcal{U}_t that is δ_t -far from being transitive, where $\delta_t(n) = 10^{-6} \text{ex}(n, K_{\lceil t/2 \rceil, t})/n^2$.*

Again, if the conjectural bound (1) holds, then this tells us that we actually have $\delta_t(n) \gg n^{-1/\lceil t/2 \rceil}$ in the above construction for each $t \geq 3$. While the construction demonstrating the above proposition is analogous to the construction for colourings presented earlier in Section 3, the argument justifying this construction is somewhat more involved.

Proof of Proposition 4.1. Given $t \geq 3$, we first set $r = r(t) = \lceil t/2 \rceil \geq 2$. To prove the result, we shall now construct, for all large enough $n \in \mathbb{N}$, a tournament T on n vertices not containing any copies of \mathcal{U}_t with at least $\text{ex}(n, K_{r,t})/10^6$ backward edges in any ordering of its vertex set. As usual, we assume that n is large enough to support the estimates that follow. Note that since $t \geq r \geq 2$, we have $\text{ex}(n, K_{r,t}) \gg n^{3/2}$; see [10], for example.

Let G be an n -vertex graph with $\text{ex}(n, K_{r,t}) - 1$ edges which does not contain a copy of $K_{r,t}$ and let $H \subset G$ be a spanning bipartite subgraph of G with at least $e(G)/2$ edges with vertex classes A and B . We construct a tournament T on the same vertex set as H as follows: fix an ordering σ of the vertices of H where all vertices of A precede all the vertices of B , and for every edge $xy \in E(H)$ with $x \in A$ and $y \in B$, we direct the corresponding edge in T backwards in σ from y to x in T , and every other edge forwards. In what follows, we speak about the edges in H and the edges in T directed from B to A interchangeably, since these are in one-to-one correspondence with each other.

It is not hard to see that T does not contain any copies of \mathcal{U}_t , a fact that we record below.

Claim 4.2. *T does not contain any copies of \mathcal{U}_t .*

Proof. Suppose to the contrary that there is a copy of \mathcal{U}_t in T , and let $X, Y, Z \subset V(T)$ be the three transitive vertex classes of this copy of \mathcal{U}_t in T , with edges oriented from X to Y , from Y to Z , and from Z to X .

Observe that it cannot happen that each of X, Y and Z meet A , since this would yield a cyclic triangle in A , while $T[A]$ is transitive by construction; hence, suppose without loss of generality that $Z \subset B$. The same argument applied to B shows that one of X or Y must necessarily be contained in A ; since there are no copies of $K_{r,t}$ in H (which specifies the set of edges directed from B to A), it must be the case that $Y \subset A$.

We now know that $Y \subset A$ and $Z \subset B$. Of course, either $|X \cap A| \geq \lceil t/2 \rceil = r$ or $|X \cap B| \geq r$. If the former happens, then we find a copy of $K_{r,t}$ between $X \cap A$ and Z in H , and if the latter happens, then we find a copy of $K_{r,t}$ between $X \cap B$ and Y in H , a contradiction regardless. \square

The bulk of the work, which we accomplish in the next lemma, lies in demonstrating that T is not too close to being transitive.

Lemma 4.3. *In any ordering of $V(T)$, there are at least $e(H)/10^5$ backward edges.*

Proof. Suppose this does not hold, and let $\tau = (v_1, v_2, \dots, v_n)$ be an ordering of the vertices of T that minimises the number of backward edges, so that the number of backward edges in τ is less than $e(H)/10^5$. The basic idea now is simple. Since many of

the backward edges in σ are forward edges in τ , we expect to be able to find large sets $A' \subset A$ and $B' \subset B$ such that B' precedes A' in τ with $H[A', B']$ containing a positive fraction of the edges in H . However, we may then use the fact that H has no copies of $K_{r,t}$ to conclude that at least half of the edges, say, between A' and B' in T must be directed backward from A' to B' in τ . While this sketch is conceptually straightforward, filling in the details however necessitates dealing with some technicalities.

Let us start by recording the following fact that we shall make use of repeatedly.

Claim 4.4. *If $A' \subset A$ and $B' \subset B$ are such that every vertex of B' precedes every vertex of A' in τ , then $e(H[A', B']) \leq e(H)/10^5$.*

Proof. Suppose the claim fails for some $A' \subset A$ and $B' \subset B$. Since $e(H) \gg n^{3/2}$, we must have $|A'|, |B'| \gg \sqrt{n}$. Now, the graph $H[A', B']$ contains no copies of $K_{r,t}$, and both $|A'|$ and $|B'|$ are sufficiently large, so it follows from Proposition 2.2 that $e(H[A', B']) \leq |A'||B'|/2$. Therefore,

$$e(H)/10^5 \leq e(H[A', B']) \leq |A'||B'|/2 \leq e(\overline{H}[A', B']),$$

where \overline{H} is the complement of H . Since each non-edge of $H[A', B']$ is directed from A' to B' in T , this yields at least $e(H)/10^5$ backward edges in τ , which is a contradiction; this concludes the proof of Claim 4.4. \square

Write $|A| = a$ and $|B| = b$ so that $a + b = n$, and define X and Y to be the first a vertices and the last b vertices in τ respectively. Let $X_B = B \cap X$ be those vertices of B appearing in the first a vertices in τ , and let $Y_A = A \cap Y$. Of course, we have $|X_B| = |Y_A|$; we write m for their common size.

Observe that since there are at most $e(H)/10^5$ backward edges in τ , at least $e(H)/2$ edges of H are forward edges in τ ; each such forward edge must necessarily be directed out of some vertex in X_B or into some vertex in Y_A (or both). Hence, we assume by pigeonholing that $F \subset E(H)$ is some set of $e(H)/4$ edges of H that are all forward edges in τ directed into some vertex in Y_A (the other case being symmetric). Note that it must be the case that $m \gg n^{1/2} \geq n^{1/10}$, say, since we know that $mn \gg e(H) \gg n^{3/2}$.

We may partition F as $F = F' \cup F''$, where F' consists of all the forward edges in τ directed from $B \cap Y$ to Y_A , and F'' consists of all those forward edges in τ directed from X_B to Y_A . We know from Claim 4.4 that $|F''| \leq |F|/2$, so we must have $|F'| \geq |F|/2 \geq e(H)/8$.

We now need the notion of a ‘balanced interval’. We fix a sufficiently large constant $C_1 = C_1(t) > 0$ to support what follows, and say that a sub-interval $W \subset Y$ is *balanced* if

- A1** the number of edges of F' within W is at least $|F'|/2$,
- A2** $1/50 < |W \cap A|/|W \cap B| < 50$, and

A3 either the initial segment $W_i^+ \subset W$ of the first i vertices in W satisfies $|W_i^+ \cap A| < 50|W_i^+ \cap B|$ or the terminal segment $W_i^- \subset W$ of all but the first i vertices in W satisfies $|W_i^- \cap B| < 50|W_i^- \cap A|$ for each $C_1 \leq i \leq |W| - C_1$.

We first show that a balanced interval may always be found.

Claim 4.5. *There exists a balanced sub-interval $W \subset Y$.*

Proof. Starting with the interval Y , we shall successively refine the interval under consideration into a ‘more structured’ sub-interval, repeating this iteratively until we reach our goal. We start with $W_0 = Y$ and in each step $0 \leq j \leq 30 \log n$, we do the following. If the interval W_j is balanced, then we stop. If not, then we shall find a sub-interval $W_{j+1} \subset W_j$ with $|W_{j+1}| \leq 9|W_j|/10$ that contains all but $C_2 n$ edges of F' within W_j , for some $C_2 = C_2(t) > 0$. We claim that such an iterative process must terminate in a balanced interval. Indeed, if the process does not terminate within the first $30 \log n$ steps, then as we have lost at most $30C_2 n \log n$ edges of F' , the number of surviving edges from F' is at least $|F'| - 30C_2 n \log n \geq 3|F'|/4 \gg n^{3/2}$ since $|F'| \gg n^{3/2}$, while on the other hand, the number of surviving vertices is $O(1)$, which is clearly impossible.

We now describe how to construct $W' = W_{j+1}$ from an unbalanced $W = W_j$ at some stage $0 \leq j < 30 \log n$. We may assume inductively, as we saw earlier, that the number of edges of F' with both endpoints in W is at least $3|F'|/4 \gg n^{3/2}$, so in particular, we have $|W| \gg n^{3/4}$. Since W is not balanced, it must violate one of **A2** or **A3**. We now describe how to construct W' in the case where W violates **A2** on account of $|W \cap B| > 50|W \cap A|$, and then indicate the minor modifications needed to handle the other cases.

Consider the set Z of the last C_2 vertices from A in W and suppose that v_{k+1} is the first vertex in Z . We claim that we may take $W' = W_k^+$ to be the initial segment of those vertices preceding v_{k+1} in W .

To show that this choice of W' works, we first claim that $|W' \cap B| < 4/5|W \cap B| \leq 9|W|/10$, where the second inequality holds on account of $|W \cap B| > 50|W \cap A|$. Indeed, if this is not the case, we may find a copy of $K_{r,t}$ in H by arguing as follows. Given a vertex $v = v_{i+1} \in Z$, since τ is an ordering that minimises the number of backward edges, we know from Proposition 2.4 that at least $1/2$ of the edges between W_i^+ and v are directed into v . Since

$$|W' \cap B| \geq 4|W \cap B|/5 \geq (4/5)(50/51)|W| \geq 3|W|/4,$$

the number of edges directed from $W' \cap B$ to v is at least

$$|W' \cap B| - |W_i^+|/2 \geq 3|W|/4 - |W|/2 = |W|/4.$$

It then follows that the number of edges directed from $W' \cap B$ to Z , all necessarily edges in H , is at least $|W' \cap B||Z|/4$. Therefore, as $|W' \cap B| \geq 3|W|/4 \gg n^{3/4}$ and

$|Z| = C_2$, then provided C_2 is suitably large, we conclude from Proposition 2.2 that there is a copy of $K_{r,t}$ in H between $W' \cap B$ and Z , a contradiction.

If W violates A2 on account of $|W \cap A| > 50|W \cap B|$, then we analogously construct W' by considering the first C_2 vertices from B in W . Finally, if W violates A3 for some $C_1 \leq i \leq |W| - C_1$, then we apply the above argument to both W_i^+ and W_i^- , looking at the first $C_2/2$ vertices from B in the former interval, and the last $C_2/2$ vertices from A in the latter interval. \square

To finish the proof of Lemma 4.3, we shall show that the existence of a balanced interval $J \subset Y$ yields too many backward edges in τ . We need a little notation: for a partition of $J = J_1 \cup J_2$ into an initial segment J_1 and a terminal segment J_2 , we decompose the subset of at least $|F'|/2$ edges of F' within J into three parts as $F_1 \cup F_2 \cup F_{12}$, where $F_i \subset F'$ is the set of such edges entirely within J_i for $i = 1, 2$ and $F_{12} \subset F'$ is the set of such edges directed from J_1 to J_2 .

Now, fix J_1 to be the smallest initial segment of J for which $|F_1| \geq |F'|/4$ and set $J_2 = J \setminus J_1$. Since $|F'| \gg n^{3/2} \gg n$, it follows that $|F'|/4 \leq |F_1| \leq |F'|/3$. We cannot have $|F_{12}| \geq |F'|/10$, since this would imply that there are too many edges directed from $J_1 \cap B$ to $J_2 \cap A$, contradicting Claim 4.4. Thus, we may assume that $|F_1|, |F_2| \geq |F'|/4$. It must be the case that $|J_1 \cap B| \leq |J \cap B|/1000$, for if not, then we would have

$$|J_1 \cap B||J_2 \cap A| \geq |J_2 \cap B||J_2 \cap A|/1000 \geq |F_2|/1000 \geq |F'|/4000 \geq 2e(H)/10^5,$$

which when combined with Claim 4.4 promising us that the number of edges from $J_1 \cap B$ to $J_2 \cap A$ is at most $e(H)/10^5$, yields at least $e(H)/10^5$ backward edges in τ , a contradiction. The same reasoning also leads us to conclude that $|J_2 \cap A| \leq |J \cap A|/1000$. These two assertions taken together contradict the fact that J , being balanced, satisfies A3; this completes the proof of Lemma 4.3. \square

We have shown that the tournament T we constructed has both the properties we require: T is $\delta_t(n)$ -far from being transitive, and T does not contain a copy \mathcal{U}_t ; this establishes Proposition 4.1. \square

4.2. Proof of Theorem 1.2. Let us introduce some conventions that we adopt in the sequel. In what follows, given a tournament T , we shall work exclusively with an ordering σ of its vertex set minimising the number of backward edges, so all subsequent references to forward or backward edges, intervals of vertices, lengths of edges, etc., will be with respect to this ordering. Given two disjoint sets of vertices $A < B$ of a tournament T , we define $d(A, B)$ to be the distance between the largest vertex of A and the smallest vertex of B , and we abuse notation slightly and define $d(A)$ to be the distance between the smallest vertex and the largest vertex of A . With this language in place, we are now ready to prove our second main result.

Proof of Theorem 1.2. We start by fixing $t \geq 3$ and setting $r = r(t) = \lceil t/2 \rceil \geq 2$, and we take $C = C(t) > 1$ to be large enough to support the argument that follows.

Our argument will be by contradiction. Starting with a tournament on n_0 vertices that is at least $Cn_0^{-1/r}$ -far from being transitive with no copy of \mathcal{U}_t , we repeatedly apply Lemma 2.5 until we reach a subtournament T on n vertices which has an ordering σ of its vertex set minimising the number of backward edges in which at least $Cn^{2-1/r}/1000$ backward edges have length least $n/50$. Furthermore, we may of course suppose that n is large enough to support the arguments that follow.

We justify the above claim as follows. After k unsuccessful applications of Lemma 2.5, we are left with a tournament on $n = n_0/20^k$ vertices, whose distance from being transitive is at least $6^k Cn_0^{-1/r}$, so the number of backward edges in any ordering of such a tournament is at least

$$6^k \cdot Cn_0^{-1/r} \cdot n^2 = (6/20^{1/r})^k \cdot Cn^{2-1/r}$$

Now, we know $r \geq 2$, so $6/20^{1/r} \geq 6/\sqrt{20} > 1$, so it follows that we must have a successful application of Lemma 2.5 before n becomes too small, since the number of backward edges is both at least $Cn^{2-1/r}$ and at most $n^2/2$, and indeed, if we start with C large enough, we may assume that we succeed at a stage where n is sufficiently large.

In what follows, we shall work with T , which is a tournament on n vertices with no copy of \mathcal{U}_t . Furthermore, we also fix σ , an ordering of $V(T)$ minimising the number of backward edges with respect to which we know that there are at least $Cn^{2-1/r}/1000$ backward edges of length at least $n/50$.

First, we shall prove a claim (somewhat similar in spirit to Claim 3.2) that allows us to deal with vertices of atypically large degree with respect to the backward edges.

Claim 4.6. *For any $\varepsilon > 0$, there exist positive integers $C_1 = C_1(t, \varepsilon) > 0$ and $C_2 = C_2(t, \varepsilon) > 0$ such that the following holds. For any interval I in σ of at least $m \geq C_1$ vertices, the induced tournament $T[I]$ contains at most C_2 vertices that are incident with more than εm backward edges in σ .*

Proof. We argue by contradiction, always ensuring that the numbered constants C_1, C_2, C_3, \dots in our argument are sufficiently large as a function of t and ε . Suppose there exists a set $S \subset I$ consisting of C_2 vertices each sending out at least εm backward edges in σ ; the other case where these vertices receive many backward edges may be handled analogously.

First, by pigeonholing, we pass to a large subset $S' \subset S$ of order at least $C_3 = C_3(t, \varepsilon) > 0$ with the property that $d(S') \leq \varepsilon m/100$. Now, consider the subsegment I' of I consisting of exactly $\varepsilon m/10$ vertices, immediately to the left of the first vertex of S' in σ . Each vertex of S' sends out at least εm backward edges in σ , so each such

vertex sends at least $\varepsilon m/2$ backward edges to the left of I' to the interval I'' , where I'' is the subinterval of I preceding I' in σ .

Next, we may find a large set of at least $C_4 = C_4(t, \varepsilon) > 0$ vertices in S' with linear-sized common out-neighbourhood in I'' ; more precisely, applying Proposition 2.3 to the graph consisting of the backward edges between S' and I'' , we may find a set $A \subset S'$ of size C_4 with at least $(\varepsilon/2e)^{C_4+2}m/2$ out-neighbours in I'' ; call this set of common out-neighbours B^+ . Note that since $d(S') \leq \varepsilon m/100$ and I' is an interval of order $\varepsilon m/10$ immediately to the left of the first vertex of S' , Proposition 2.4 tells us that each vertex in A receives at least $\varepsilon m/30$ edges from I' . By exactly the same argument as above, we may pass to a large subset $A' \subset A$ of size at least $C_5 = \mathcal{T}(t)$ having at least $(\varepsilon/100)^{C_5+1}m$ common in-neighbours in I' ; call this set of common in-neighbours B^- .

We may now finish as follows. If there is a $K_{\mathcal{T}(t), \mathcal{T}(t)}$ directed from B^+ to B^- , then by passing to transitive tournaments within the partition classes of this copy and within A' , we may find a copy of \mathcal{U}_t in T , which is a contradiction. Therefore, by evoking Proposition 2.2, we conclude that there are at least $c(t, \varepsilon)m^2$ backward edges between $I'' \cup I'$ for some $c(t, \varepsilon) > 0$. Since $I'' \cup I'$ is an interval in σ of order at most m , by Proposition 2.4, we conclude that $T[I'' \cup I']$ has distance at least $c(t, \varepsilon)$ from being transitive, and therefore by the result of Fox and Sudakov [8] mentioned earlier, we know that there is a copy of \mathcal{U}_t within $I' \cup I''$, provided m is sufficiently large. This a contradiction and completes the proof of the claim. \square

We apply the previous claim with $\varepsilon = 1/(100t^2)$ to the entire tournament T , concluding that we may remove $O(1)$ vertices from T and guarantee that in the resulting tournament T' on m vertices, no vertex is incident to more than $2\varepsilon m$ backward edges in the ordering induced by σ on $V(T')$, and that T' has at least $C'm^{2-1/r}$ backward edges of length at least $m/60$ with respect to σ . Recall that since we have removed at most $O(1)$ vertices from T , every remaining backward edge in T whose endpoints are at distance at least $n/50$ in σ are at distance at most $m/60$ in T' .

In what follows, we work with T' and the ordering induced by σ on $V(T')$, though we abuse notation slightly and refer to this induced ordering as σ as well. We call a backward edge of T' *good* if its length is at least $m/60$; of course, we know that T' has at least $C'm^{2-1/r}$ good backward edges.

Claim 4.7. *We may assume that T' does not contain a copy of $K_{t,t}$ formed from good backward edges, where one partition class of this copy precedes the other in σ , and each of the partition classes of this copy forms a transitive subtournament.*

Proof. Suppose that such a copy exists, say with partition classes A and B with $A < B$ where both $T[A]$ and $T[B]$ are transitive. We know there are at least $m/60$ vertices between the last element of A and the first of B ; call this intervening interval P . Since

we removed all vertices incident to many backward edges in passing to T' , we know that A has at least $|P| - 2tm/(100t^2) \geq 2|P|/3$ common out-neighbours in P , and that B similarly has at least $2|P|/3$ common in-neighbours in P . Provided m is large enough, we can then find a set $S \subset P$ of $\mathcal{T}(t)$ vertices in P where all the edges are directed from A to S and from S to B . Passing to a transitive subtournament inside S , we find a copy of \mathcal{U}_t in T' , which is a contradiction. \square

Now, we partition $V(T')$ into 130 intervals of size $m/130$, and observe that at least a $(1/15)^4$ -fraction of the good backward edges of T' lie between two of these intervals; call these intervals I and J with $I < J$ and note that we necessarily have $d(I, J) \geq m/130$, because I, J can not be consecutive intervals of σ . To summarise, we now have two intervals $I < J$ of order $m/150$ for which there exists at least $C'm^{2-1/r}$ good backward edges directed from J and I , for some large $C' = C'(t) > 0$ and $d(I, J) \geq m/150$.

Next, we shall find two disjoint sets of edges from the good backward edges between I and J in such a way that every pair of edges across these two sets interlace nicely. To do so, we need to prepare I and J appropriately. We know that the number of good backward edges between I and J is at least $C'(|I| + |J|)^{2-1/r}$, so we pass to subintervals $I' \subset I$ and $J' \subset J$ chosen such that the number of good backward edges from J' to I' is of the form $\lambda(|I'| + |J'|)^{2-1/r}$ with $\lambda \geq C'$ as large as possible. We now show that the sizes of I' and J' must be comparable.

Claim 4.8. *We have $|J'|/3 < |I'| < 3|J'|$.*

Proof. Suppose this does not hold, and assume $|J'| \geq |I'| = q$. Writing $|J'| = p \cdot q + s$, we may partition J' into p consecutive intervals of size q and one interval of size $s \leq q$. By the maximality of λ , the number of good backward edges between I' and any interval in this decomposition of J' is at most $\lambda(2q)^{2-1/r}$, so the total number of good backward edges from J' to I' is at most

$$(p+1) \cdot \lambda(2q)^{2-1/r} < \lambda((p+1)q)^{2-1/r} < \lambda(|I'| + |J'|)^{2-1/r},$$

provided $p \geq 3$ and $r \geq 2$; this is a contradiction that proves Claim 4.8. \square

We now find an appropriate collection of interlacing edges between I' and J' using the following density increment lemma, which may be of independent interest.

Lemma 4.9. *For every $\alpha \in [0, 1/2]$ and $\varepsilon > 0$, there exists $C_7 = C_7(\alpha, \varepsilon) > 0$ such that the following holds. Let T be a tournament and τ an ordering of its vertices. For any two intervals $I < J$ of vertices with $|I| + |J| = n$ for which there are $L \geq C_7 n^{2-\alpha}$ backward edges from J to I the following holds:*

- (1) *either there exists a partition of I into two intervals $I = I_1 \cup I_2$ and a partition of $J = J_1 \cup J_2$ into two intervals with $I_1 < I_2$ and $J_1 < J_2$ such that the number of backward edges from J_1 and I_1 and from J_2 and I_2 is at least εL , or*

- (2) *there exist two intervals $I' \subset I$ and $J' \subset J$ with $|I'| + |J'| \leq n/2$ where the number of backward edges from J' and I' is at least $(1/2 - 3\varepsilon)L$.*

Proof. We argue as follows. Consider the smallest initial segment of I , say I_1 , such that the number of backward edges with an endpoint in I_1 is at least $L/2$. Since $L = \omega(n)$, the set $I \setminus I_1 = I_2$ must also be incident with at least $L/2 + o(L)$ backward edges.

Now, let $J = \{1, 2, \dots, |J|\}$ and for each $j \in J$, let us define $b(j)$ to be the difference between the number of backward edges from I_1 to $\{1, \dots, j-1\}$ and the number of backward edges between I_2 and $\{j, \dots, |J|\}$. We know that $b(1) = -L/2 + o(L)$ and $b(|J|) = L/2 + o(L)$, and since $b(t+1) = b(t) + o(L)$, there exists a vertex $p \in J$ such that $b(p) = o(L)$; accordingly, let $J_1 = \{1, \dots, p-1\}$ and $J_2 = \{p, \dots, |J|\}$.

Suppose that the number of backward edges between I_1 and J_1 is at least $2\varepsilon L$. Then, since $b(p) = o(L)$, there must also exist at least $2\varepsilon L + o(L) \geq \varepsilon L$ backward edges between I_2 and J_2 , in which case, we are done. If the above assumption does not hold, then the number of backward edges between I_1 and J_2 and the number of backward edges between I_2 and J_1 are both at least $(1/2 - 3\varepsilon)L$; we find I' and J' by now taking the pair with the smaller total size, thus proving Lemma 4.9. \square

We now apply Lemma 4.9 to I' and J' — recall these intervals were defined before Claim 4.8 — to find many interlacing good backward edges between them. Indeed, we apply Lemma 4.9 with $\alpha = 1/r$ and $\varepsilon = 1/100$ to the backward edges between I' and J' . If the latter conclusion of the lemma holds, then we find two intervals $I'' \subset I'$ and $J'' \subset J'$ such that the number of backward edges between I'' and J'' is at least $(1/2 - 1/50)\lambda(|I'| + |J'|)^{2-1/r}$. However $|I''| + |J''| \leq (|I'| + |J'|)/2$, from which it follows that

$$(1/2 - 1/50)\lambda(|I'| + |J'|)^{2-1/r} \geq (3\lambda/2)(|I''| + |J''|)^{2-1/r},$$

which contradicts the maximality of λ . Therefore the former conclusion of the lemma must hold, which means that we may find four intervals $I_1 < I_2 < J_1 < J_2$ where $|I_1| + |I_2| = |I'|$ and $|J_1| + |J_2| = |J'|$ such that the number of good backward edges between I_1 and J_1 is at least $(\lambda/400)(|I_1| + |J_1|)^{2-1/r}$ and the number of good backward edges between I_2 and J_2 is at least $(\lambda/400)(|I_2| + |J_2|)^{2-2/t}$. Moreover, as we already know, the distance between I_2 and J_1 is at least $m/150$.

In the light of the above discussion, let us select a collection of four intervals $I_1 < I_2 < J_1 < J_2$ with $d(I_2, J_1) \geq m/150$ such that the numbers of good backward edges between I_1 and J_1 and between I_2 and J_2 are respectively at least $\lambda_1(|I_1| + |J_1|)^{2-1/r}$ and $\lambda_2(|I_2| + |J_2|)^{2-1/r}$ where $\lambda_1, \lambda_2 \geq \lambda/400$ and conditional on this, $\lambda_1 + \lambda_2$ is as large as possible. In what follows, we argue under the assumption that $|I_2| \geq |I_1|$ and $|J_2| \geq |J_1|$; the three other cases may be handled analogously.

Now, let $|J_2| = p$ and $|I_2| = p'$, and assume $p' \leq p$. Using the exact same reasoning that established that $|I'|$ and $|J'|$ were comparable, namely the proof of Claim 4.8, we

may suppose that $p/3 \leq p' \leq p$. Just as we have some separation between I_2 and J_1 , it will also be convenient to introduce some separation between I_1 and I_2 and between J_1 and J_2 . For this purpose, we shall subdivide I_2 and J_2 into two new intervals. Indeed, let $I_2 = I_3 \cup I_4$ where I_3 is the initial segment of I_2 of order $|I_2|/100$, and similarly let $J_2 = J_3 \cup J_4$ where J_3 is the initial segment of J_2 of order $|J_2|/100$. Finally, using the maximality of λ_2 once I_1 and J_1 (and hence λ_1) are fixed, we see that the number of good backward edges between I_4 and J_4 is at least

$$\lambda_2(p + p')^{2-1/r} - \lambda_2(p' + p/100)^{2-1/r} - \lambda_2(p + p'/100)^{2-1/r},$$

which may be verified to be at least $(\lambda_2/100)(|I_4| + |J_4|)^{2-1/r}$.

In the rest of the argument, we work exclusively with these four intervals $I_1 < I_4 < J_1 < J_4$. Below, we summarise the properties of these four intervals that we shall need for the rest of the proof.

- B1** $I_1 < I_4 < J_1 < J_4$.
- B2** $d(I_4, J_1) \geq m/150$.
- B3** $d(I_1, I_4) \geq (|I_1| + |I_4|)/200$.
- B4** $d(J_1, J_4) \geq (|J_1| + |J_4|)/200$.
- B5** The number of good backward edges between I_1 and J_1 is at least $\lambda'(|I_1| + |J_1|)^{2-1/r}$, for some suitably large constant $\lambda' > 0$.
- B6** The number of good backward edges between I_4 and J_4 is at least $\lambda'(|I_4| + |J_4|)^{2-1/r}$, for some suitably large constant $\lambda' > 0$.

We need to rule out the possibility of finding a copy of $K_{t,t}$ in the backward edges between either I_1 and I_4 or J_1 and J_4 . We may accomplish this as before. We first observe that $|I_i|, |J_i|$ with $i \in \{1, 4\}$ must all be sufficiently large in order to satisfy these properties since λ' may be assumed to be sufficiently large. Moreover, by mimicking the argument used to prove Claim 4.6 and using the fact that we have only deleted $O(1)$ vertices so far, we may delete $O(1)$ further vertices from $I_1 \cup I_4$ so that none of remaining vertices are incident to more than $\varepsilon|V(T^*)|$ backward edges in σ , where T^* is the induced tournament on the interval spanning I_1 to I_4 and $\varepsilon = 1/(100t^2)$. We analogously remove the $O(1)$ vertices incident to many backward edges from $J_1 \cup J_4$ as well. To avoid introducing more notation, we shall allow ourselves a slight abuse and continue calling these four new intervals $I_1 < I_4 < J_1 < J_4$.

Now, our plan is to find a $K_{r,t}$ using good backward edges, where the smaller partition class H_1 of size r lies in I_1 and the larger partition class H_2 of t vertices lies in J_1 , and similarly, to find a copy of $K_{r,t}$ with the smaller partition class F_1 in J_4 and the larger partition class F_2 in I_4 . This can obviously be done by Proposition 2.1 since we know there are many good backward edges between these sets (by properties **B5** and **B6**). What is crucial however, is to find two such copies where the edges between H_1 and F_2

and between H_2 and F_1 are forward edges directed from left to right; we will accomplish this with the help of Claim 4.7.

We argue as follows using dependant random choice. For suitably large constants $C_8 = C_8(t) > 0$, $C_9 = C_9(t) > 0$ and $C_{10} = C_{10}(t) > 0$ with C_{10} sufficiently larger than C_9 and C_9 sufficiently larger than C_8 , we may appeal to Proposition 2.1 to find two sets $A \subset I_1$ and $B \subset J_4$ such that

- C1** $|A| = C_8, |B| = C_9$,
- C2** the induced tournaments on A and B are transitive,
- C3** all the edges between A and B are directed from A to B ,
- C4** every subset of size r in A has at least C_{10} common in-neighbours in J_1 , and
- C5** every subset of size r in B has at least C_{10} common out-neighbours in I_4 .

To see this, first observe that by a straightforward application of Proposition 2.1 to $I_1 \cup J_1$ and to $I_4 \cup J_4$, we may find two suitably large sets $A'' \subset I_1$ and $B'' \subset J_4$ satisfying properties **C4** and **C5**, respectively. We may invoke Ramsey's theorem for tournaments to pass to suitably large transitive subsets $A' \subset A''$ and $B' \subset B''$ so that properties **C1** and **C2** are also satisfied. Finally, we know that there is no copy of a $K_{t,t}$ in the backward edges between A' and B' by Claim 4.7, so we may now appeal to Ramsey's theorem for bipartite graphs to find a suitably large complete bipartite graph (A, B) where $A \subset A'$ and $B \subset B'$ where all the edges directed forwards from left to right, yielding **C3**.

We repeat the above process of 'two-step cleaning' via Ramsey's theorem over all pairs of the common in-neighbours of the r -sets in A and the common out-neighbours of the r -sets in B . Concretely, for each $S_1 \subset A$ of size r , start by setting $N(S_1) = N^-(S_1)$, and similarly, for each $S_2 \subset B$ of size r , start by setting $N(S_2) = N^+(S_1)$. Now, we iterate over all such pairs (S_1, S_2) and repeatedly do the following: find, by the same argument as above, sets $N'(S_1) \subset N^-(S_1)$ and $N'(S_2) \subset N^+(S_2)$ such that $N'(S_1)$ and $N'(S_2)$ form transitive tournaments where all the edges are directed from $N'(S_2)$ to $N'(S_1)$, and then replace $N(S_1)$ by $N'(S_1)$ and $N(S_2)$ by $N'(S_2)$. At the end of this procedure, we may assume that, for a suitably large $C_{11} = C_{11}(t) > 0$, we have the following: every r -set $A' \subset A$ has a set of C_{11} common in-neighbours $N(A')$ in J_1 and every r -set $B' \subset B$ has a set of C_{11} common out-neighbours $N(B')$ in I_4 such that the induced tournaments on A'' and B'' are transitive, and all the edges between B'' and A'' are directed forwards from $N(B')$ to $N(A')$. In summary, we have

- D1** $|A| = C_8, |B| = C_9$,
- D2** the induced tournaments on A and B are transitive,
- D3** all the edges between A and B are directed from A to B ;

- D4** for every r -subset $A' \subset A$, there is a set $N(A') \subset N^-(A') \cap J_1$ of size C_{11} which forms a transitive tournament, and similarly, for every r -set $B' \subset B$, there is a set $N(B') \subset N^+(B') \cap I_4$ of size C_{11} which forms a transitive tournament, and
- D5** for every r -subset $A' \subset A$ and every r -set $B' \subset B$, all edges are directed from $N(B')$ to $N(A')$.

Next, since $|B|$ is much larger than $|A|$, we refine B as follows. For each r -set $A' \subset A$, we consider its set $N(A')$ of C_{11} common in-neighbours in J_1 . We know by Claim 4.7 and B4 that there is no copy of $K_{t,t}$ in the backward edges between $N(A')$ and B , so by Ramsey's theorem, we may find a sufficiently large complete bipartite graph (A^*, B^*) consisting of forward directed edges from $N(A')$ to B , where $A^* \subset N(A')$ and $B^* \subset B$. We iterate this cleaning procedure through all the r -sets A' in A , replacing B by B^* each time. At the end, we conclude that for a suitably large constant $C_{12} = C_{12}(t) > 0$, we now have $|B| = C_{12}$ and every r -set $A' \subset A$ has a set of t common in-neighbours A^* in J_1 such that all edges between A^* and B are directed from A^* to B .

We are now done since we may find a copy of \mathcal{U}_t as follows. We select any subset $\tilde{B} \subset B$ of size r . Consider its set of $N(\tilde{B})$ of C_{11} common out-neighbours in I_4 . Again, by Claim 4.7 and B3, we know that there is no copy of $K_{t,t}$ with the edges directed from $N(\tilde{B})$ to A . Therefore, we may find, by Ramsey's theorem, a complete bipartite graph consisting of forward edges between a set $X \subset N(\tilde{B})$ of size t and a set $\tilde{A} \subset A$ of size r . Let $Y \subset N(\tilde{A})$ be a set of t common in-neighbours of \tilde{A} in J_1 . Then it is easy to see that the transitive classes $\tilde{A} \cup \tilde{B}$, X and Y together induce a copy of \mathcal{U}_t . This completes the proof by contradiction. \square

5. EXCEPTIONAL PATTERNS

The results established earlier in the paper for unavoidable patterns of order at least three might suggest at first glance that the Ramsey numbers for patterns of order two should satisfy $\mathcal{C}(2, \delta) = \Theta(\delta^{-2})$ and $\mathcal{D}(2, \delta) = \Theta(\delta^{-1})$. However, this is not the case; patterns of order two exhibit some degenerate behaviour, as we shall now demonstrate.

First, we deal with unavoidable 2-colourings. While it is not hard to prove a much more precise result, we settle for the following.

Proposition 5.1. $\mathcal{C}(2, \delta) = \Theta(1/\delta)$.

Proof. By taking a colouring of K_n where all edges are coloured blue except the edges incident with some vertex (which are coloured red), we obtain a colouring which does not contain a K_4 inducing an unavoidable 2-colouring. Clearly, both colours appear on at least $n - 1$ edges.

Next, we shall show that there exists an absolute constant $C > 0$ such that any colouring of K_n where both colours appear on at least Cn edges contains a K_4 inducing an unavoidable 2-colouring.

Suppose $G = K_n$ has a colouring where both colours appear at least Cn times. We may assume that there are at least as many blue edges as red ones. Now, following the proof of Theorem 1.1, from Claim 3.2 (with $\varepsilon = 1/6$), we are guaranteed that there is a set S of at most C_1 vertices such that in $V(G) \setminus S$ every vertex is incident with at least $2n/3$ blue edges. Assuming $C > C_1$, we deduce $G \setminus S$ must span a red edge xy . Using the fact that the blue neighbourhoods of x and y must intersect in at least two vertices, we obtain a K_4 inducing an unavoidable 2-colouring. \square

The case of unavoidable 2-tournaments is somewhat harder, and we are unfortunately unable to determine the correct rate of growth of $\mathcal{D}(2, \delta)$. Nonetheless, we are able to show the following.

Proposition 5.2. $\log(1/\delta)/\delta \ll \mathcal{D}(2, \delta) \ll (\log(1/\delta))^2/\delta$.

Proof. First, we dispose of the lower bound using an inductive construction. Let T_n be a tournament on n vertices which does not contain a copy of \mathcal{U}_2 and which is $\log n/(5n)$ -far from being transitive; such a tournament exists when $n = 3$, as can be seen from considering a cyclic triangle. Given T_n , we shall construct a tournament T on $2n + 1$ vertices with the required properties. To do so, we take two vertex-disjoint copies of T_n , say on vertex sets A and B , and direct all the edges from A to B . Then, we add a new vertex z where all the edges are directed from B to z and from z to A . We observe that this tournament does not contain a \mathcal{U}_2 . Indeed, any such copy must contain z as \mathcal{U}_2 is strongly-connected. Furthermore, note that \mathcal{U}_2 contains two vertex-disjoint copies of a cyclic triangle. Therefore, one such copy must use z , and the other must be entirely inside A or entirely inside B , but this is impossible.

Now, we claim that T is $\log(2n + 1)/(5(2n + 1))$ -far from being transitive. To see this, observe that any ordering of A must span at least $n \log n/5$ backward edges, and the same holds for B , by the induction hypothesis. Hence, any ordering of $V(T)$ must have at least $2n \log n/5$ backward edges from $E(T[A]) \cup E(T[B])$. Finally, note that one of the following must hold. Either there are $n/2$ vertices in A which precede every vertex in B , in which case, regardless of where z is in the ordering, z must be incident to at least $n/2$ backward edges, or the first vertex of B in the ordering must be incident to at least $n/2$ backward edges from A . In either case, any ordering of $V(T)$ spans at least $n/2 + 2n \log n/5 \geq (2n + 1) \log(2n + 1)/5$ backward edges, as claimed.

Next, we deal with the upper bound, again proceeding by induction on the number of vertices. The argument closely resembles the proof of Theorem 1.2, so we restrict ourselves to sketching the main points of departure. Clearly, it suffices to handle the case where the number of vertices n is sufficiently large, say, greater than a sufficiently

large constant $C > 0$. Let T be a tournament on n vertices which is $C(\log n)^2/n$ -far from being transitive, and let σ be an ordering of $V(T)$ which minimises the number of backward edges.

Let I_1 and I_2 be the intervals corresponding to the first half and the last half of σ of sizes $n/2$ each. By the induction hypothesis, both I_1 and I_2 induce at most $C(n/2)(\log(n/2))^2$ backward edges in σ . Therefore, the number of backward edges from I_2 to I_1 is at least $Cn(\log n)^2 - Cn(\log(n/2))^2 \geq Cn \log(n)/2$.

In the same fashion as in the proof of Theorem 1.2, let $J_1 < J_2$ be two disjoint intervals for which the number of backward edges between J_1 and J_2 is $\lambda(|J_1| + |J_2|) \log(|J_1| + |J_2|)$, with $\lambda \geq C/2$ as large as possible. We proceed assuming $|J_2| \geq |J_1|$, the other case being analogous. Let $X \subset J_2$ be the initial segment of J_2 of size $|J_2|/10$. By the maximality of λ , we know that the number of backward edges between J_1 and X is at most $\lambda(|J_1| + |X|) \log(|J_1| + |X|) \leq (19\lambda/20)(|J_1| + |J_2|) \log(|J_1| + |J_2|)$. Therefore, the number of backward edges between J_1 and $J_3 = J_2 \setminus X$ is at least $(\lambda/20)(|J_1| + |J_3|) \log(|J_1| + |J_3|)$, and additionally, we also know that $d(J_1, J_3) \geq (|J_1| + |J_3|)/200$. Now, let $Y_1 < Y_2$ be two disjoint intervals satisfying both the above properties where the number of backward edges between Y_1 and Y_2 is at least $\lambda'(|Y_1| + |Y_2|) \log(|Y_1| + |Y_2|)$, with $\lambda' \geq \lambda/20$ as large as possible.

A minor modification of the proof of Lemma 4.9 (roughly, partition $Y_1 = P_1 \cup Q_1$ and $Y_2 = P_2 \cup Q_2$ as in Lemma 4.9, and then note that if we do not have a linear number of edges between P_2 and P_1 and between Q_2 and Q_1 , then one of $P_1 \cup P_2$ or $Q_1 \cup Q_2$ spans more backward edges than what it should), with an extra loss of a logarithmic factor shows that we can split $Y_1 = P_1 \cup Q_1$ and $Y_2 = P_2 \cup Q_2$, with $P_1 < Q_1 < P_2 < Q_2$ such that the number of backward edges between P_1 and P_2 and between Q_1 and Q_2 are respectively at least $\lambda_1(|P_1| + |P_2|)$ and $\lambda_2(|Q_1| + |Q_2|)$. As before, we shall suppose that we have picked $P_1 < Q_1 < P_2 < Q_2$ as above for which $\lambda_1 + \lambda_2$ is as large as possible, conditioned on both λ_1 and λ_2 being large enough.

We shall sketch how to handle the case where $|Q_2| \geq |P_2|$, the other case being analogous. Moreover, in the argument that follows, we may assume without loss of generality that $|Q_1| \leq |Q_2|$.

Delete the first $|Q_2|/20$ vertices of Q_2 and denote the remaining interval by Q'_2 . As in the proof of Theorem 1.2, this allows us to separate Q'_2 from P_2 , since we now have $d(P_2, Q'_2) \geq (|P_2| + |Q'_2|)/50$. We now need to show that there are still sufficiently many backward edges between Q_1 and Q'_2 . Indeed, Q_1 and Q_2 span $\lambda_2(|Q_1| + |Q_2|)$ backward edges between them, and the maximality of λ_2 (with λ_1 fixed) allows us to bound from above the number of backward edges between Q_1 and $Q_2 \setminus Q'_2$, from which we may conclude that the number of backward edges between Q_1 and Q'_2 is at least $(\lambda_2/200)(|Q_1| + |Q'_2|)$. Replacing Q_2 by Q'_2 , we may now assume that $P_1 < Q_1 < P_2 < Q_2$

are four intervals as above, again with the number of backwards edges between these intervals assumed to be maximal in the same sense as before.

Now that we have separation between Q_1 and P_2 and between P_2 and Q_2 , all that is left to do is to enforce some separation between P_1 and Q_1 . We are led to handle two different cases, depending on whether or not Q_1 and Q_2 have comparable sizes.

Suppose first that $|Q_1| \leq |Q_2| \leq 4|Q_1|$. Then, we may proceed as we did before, deleting the first $|Q_1|/20$ vertices from Q_1 to create separation between P_1 and Q_1 while still ensuring that a positive fraction of the backward edges between Q_1 and Q_2 still survive; the rest of the argument is identical to the proof of Theorem 1.2.

Next, suppose that $|Q_1| < |Q_2|/4$. Then, let Q'_2 be the initial segment of Q_2 of the same length as Q_1 , and let $Q''_2 = Q_2 \setminus Q'_2$. If there are fewer than $(K_2/10)(|Q'_2| + |Q_1|)$ backward edges between Q_1 and Q'_2 , then the intervals Q_1 and Q''_2 must have at least $K_2(|Q_1| + |Q_2|) - (K_2/10)(|Q'_2| + |Q_1|) > K_2(|Q''_2| + |Q_1|)$ backward edges between them, which is a contradiction. This implies that, replacing Q_2 by Q'_2 if necessary, we may assume Q_1 and Q_2 have comparable sizes. Now, we take four such intervals $P_1 < Q_1 < P_2 < Q_2$ as above (with the appropriate separation between intervals), with the additional constraint that Q_1 and Q_2 have comparable sizes, and again with the number of backwards edges between intervals assumed to be appropriately maximal; the rest of the argument is identical to the proof of Theorem 1.2. \square

6. CONCLUSION

Our main contribution in this paper was to pin down the order of magnitude of the Ramsey numbers $\mathcal{C}(t, \delta)$ and $\mathcal{D}(t, \delta)$ for fixed $t \in \mathbb{N}$ as $\delta \rightarrow 0$. If one is however willing to settle for just the correct exponents governing the growth rates of these Ramsey numbers, then more can be said.

A careful rendering of our argument yields dependencies governed by iterated logarithms, allowing us to establish weaker forms of the bounds (2) and (3) that are valid as long as t is much smaller than $1/\delta$. Concretely, we have

$$\mathcal{C}(t, \delta) = (1/\delta)^{t(1+o(1))}$$

as $\delta \rightarrow 0$ with $3 \leq t \leq \log^{(3)}(1/\delta)$, and

$$\mathcal{D}(t, \delta) = (1/\delta)^{\lceil t/2 \rceil (1+o(1))}$$

as $\delta \rightarrow 0$ with $3 \leq t \leq \log^{(4)}(1/\delta)$. It would be of interest to work out, even roughly, at what point the above bounds cease to be valid.

It would also be interesting to understand the growth rate of the Ramsey numbers $\mathcal{C}(t, \delta)$ and $\mathcal{D}(t, \delta)$ in the other off-diagonal regime where t is much bigger than $1/\delta$, as well as in the diagonal regime where $t \approx 1/\delta$. Both these questions pose interesting challenges of their own, somewhat orthogonal to the problems under consideration here.

It is somewhat embarrassing that we are unable to pin down the rate of growth of $\mathcal{D}(2, \delta)$. While we have managed to estimate this degenerate case here up to a logarithmic multiplicative factor, we suspect that our lower bound gives the correct rate of growth. It would be of interest to improve on our upper bound and demonstrate that $\mathcal{D}(2, \delta) \ll \log(1/\delta)/\delta$, thereby closing a small but annoying gap in the existing bounds.

It would also be of interest to pursue similar lines of questioning in other settings that arise in Ramsey theory. Concretely, one could ask about the multi-colour setting, asymmetric Ramsey properties, as well as the infinitary setting, to name but a few; see [1] for some preliminary results on such questions.

Finally, we conclude with an application of our main result that might be of some independent interest. The following is an easy corollary of our main theorem, and resolves a problem of Gishboliner [11] rooted in generalised Turán theory.

Corollary 6.1. *For each $t \geq 3$, there is a $C = C(t) > 0$ such that any tournament T on n vertices with at least $Cn^{3-1/\lceil t/2 \rceil}$ directed triangles contains an unavoidable t -tournament.*

The implication is trivial: any tournament on n vertices which contains $Cn^{3-1/\lceil t/2 \rceil}$ directed triangles must be at least $((C/10)n^{-1/\lceil t/2 \rceil})$ -far from being transitive. Perhaps more interesting is the fact that this bound is tight up to the constant $C(t)$ in its statement (again assuming the truth of the Kovári–Sós–Turán conjecture); this may be seen by considering the same construction used to prove Proposition 4.1, but we omit the details.

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