Abstract. A family of vectors in $[k]^n$ is said to be intersecting if any two of its elements agree on at least one coordinate. We prove, for fixed $k \geq 3$, that the size of a symmetric intersecting subfamily of $[k]^n$ is $o(k^n)$, which is in stark contrast to the case of the Boolean hypercube (where $k = 2$). Our main contribution addresses limitations of existing technology: while there are now methods, first appearing in work of Ellis and the third author, for using spectral machinery to tackle problems in extremal set theory involving symmetry, this machinery relies crucially on the interplay between up-sets, biased product measures, and threshold behaviour in the Boolean hypercube, features that are notably absent in the problem considered here. To circumvent these barriers, introducing ideas that seem of independent interest, we develop a variant of the sharp threshold machinery (underlying earlier work) that applies at the level of products of posets.

1. Introduction

We pursue a line of investigation initiated by Babai [2] and Frankl [9] about forty years ago concerning the role of symmetry in extremal set theory. Our starting point is the Erdős–Ko–Rado theorem [8], which asserts that for $n, k \in \mathbb{N}$ with $k < n/2$, the largest families of $k$-subsets of $[n]$ are precisely the trivial ones, namely those that consist of all $k$-sets containing some fixed element of $[n] = \{1, 2, \ldots, n\}$. Many variants and generalisations of this theorem (involving different intersection conditions and different discrete structures such as permutations, vectors and graphs) have since been established. A common theme in this line of enquiry is that the extremal constructions are often highly asymmetric, depending only on a small number of ‘coordinates’; see [15, 10, 1, 19], for example. It is therefore natural to ask what happens when one further imposes a symmetry requirement on the family under consideration, the most natural such requirement being that the family be invariant under some transitive subgroup of the symmetric group $S_n$. Indeed, this direction was proposed by Babai [2] a few decades back, and has since been rather fruitful; see [9, 5] for some classical results, and [7, 6, 13] for more recent developments.

Here, again, as in our discussion of families of sets, the above extremal examples are highly asymmetric (membership being determined by a single coordinate), though now with a small caveat: in the Boolean hypercube $[2]^n$ with $n$ odd, the family of vectors with more 1’s than 2’s is intersecting, of the maximum possible size $2^{n-1}$, and invariant under all of $S_n$. However, this has no counterpart for $k \geq 3$, where, even without symmetry, it is known that the only extremal examples are the trivial ones (see [16]). In fact, a little
thought suggests a more interesting possibility: might it be true that (for \( k \geq 3 \)) symmetric, intersecting families must be \textit{much} smaller? Our main purpose here is to show that this is indeed the case.

Before stating a precise result, we repeat, a little more formally the definition of symmetry. As above, we use \([n]\) for \(\{1,2,\ldots,n\}\), and \(S_n\) for the symmetric group on \([n]\), which acts on \([k]^n\) in the natural way, namely \((\sigma(x))_i = x_{\sigma(i)}\) for \(\sigma \in S_n\) and \(x \in [k]^n\). The \textit{automorphism group} of \(A \subset [k]^n\) is, as usual, \(\text{Aut}(A) = \{\sigma \in S_n: \sigma(A) = A\}\), and we say \(A \subset [k]^n\) is \textit{symmetric} if \(\text{Aut}(A)\) is a transitive subgroup of \(S_n\). Our main result is then as follows.

\textbf{Theorem 1.1.} \textit{For fixed} \(k \geq 3\), \textit{if} \(A \subset [k]^n\) \textit{is symmetric and intersecting}, \textit{then} \(|A| = o(k^n)\).

Perhaps surprisingly, this simple and natural statement seems resistant to elementary proof, and it may be that the more important point of this work is its contribution to methodology. Giving us a starting point, Ellis and the third author [7], in resolving a conjecture of Frankl [9] on symmetric 3-wise intersecting families, introduced the use of spectral machinery for tackling problems in extremal set theory involving symmetry; this framework has since been successfully adapted — see [6, 13] — to resolve other old extremal problems in the Boolean hypercube involving symmetry constraints. Note, though, that this approach depends crucially on the interplay between up-sets, biased product measures, and ‘sharp threshold’ behaviour, all features absent from the problem under consideration here; for example, all of [7, 6, 13] start with the elementary observation that the \(p\)-biased measure of an up-set in \([2]^n\) is monotone increasing in \(p\), but even this fact has no useful analogue in \([k]^n\) for \(k \geq 3\). This situation is reminiscent of difficulties occasioned by a lack of useful notions of monotonicity in some probabilistic contexts; see the ‘all blue’ problem of [17] for one particularly egregious example.

Here, one could, for example, try working in \([k]^n\) with the natural product order, but one is then confronted with the following: compressing an intersecting family ‘upwards’ preserves the intersection condition but not the automorphism group, while replacing a family by its ‘up-closure’ preserves symmetries but not the intersection condition; furthermore, there appears to be no natural analogue in \([k]^n\) of the biased product measures on \([2]^n\) that are at the heart of the arguments of [7, 6, 13].

Our (at first unpromising-looking) way around these obstacles is to embed \([k]^n\) in a larger ‘covering space’, a suitable product of posets, in which up-closure avoids the above difficulties and appropriate analogues of biased product measures still provide the leverage we need. Having transferred our problem to this larger space, we deduce Theorem 1.1 using a suitable variant of the sharp threshold theorem of Friedgut and Kalai [14], based, like the original, on results of Bourgain, Kahn, Kalai, Katznelson, and Linial [4].

The paper is organised as follows. In Section 2, we prove our variant of the Friedgut–Kalai sharp threshold theorem for products of posets; the proof of Theorem 1.1 follows in Section 3. Finally, we conclude in Section 4 with a brief discussion of what might come next.

\section{Biased measures on products of posets}

We now present a general construction that is at the heart of our approach. In what follows, the reader may find it helpful to keep \(p\)-biased product measures on the Boolean hypercube in mind.

Let \((W, \preceq)\) be a finite poset. We say that \(A \subset W\) is an \textit{up-set} if \(x \in A\) and \(x \preceq y\) imply \(y \in A\). Recall, for probability measures \(\mu_0\) and \(\mu_1\) on \(W\), that \(\mu_1\) \textit{(stochastically) dominates} \(\mu_0\) if \(\mu_1(A) \geq \mu_0(A)\) for every up-set \(A \subset W\). We extend this, saying that \(\mu_1\) \textit{dominates} \(\mu_0\) \textit{with strength} \(\kappa\) if

\[\mu_1(A) - \mu_0(A) \geq \kappa\]

(\dagger)
for every up-set $A \subset W$ other than $\emptyset$ and $W$. Given probability measures $\mu_0$ and $\mu_1$ on $W$, we consider the interpolation from $\mu_0$ to $\mu_1$ — our analogue of biased product measures — obtained by taking $\mu_t = (1 - t)\mu_0 + t\mu_1$ to be the measure at ‘time’ $t \in [0, 1]$. We need the following variant of the Friedgut–Kalai [14] theorem; in what follows, as usual, if $\mu$ is a probability measure on $W$, then $\mu_n$ is the corresponding product measure on $W^n$.

**Theorem 2.1.** Assume that $A \subset W^n$ is a symmetric up-set, $\mu_0$ and $\mu_1$ are probability measures on $W$, and $\mu_1$ dominates $\mu_0$ with strength $\kappa > 0$. If $0 \leq p < q \leq 1$ and $\mu_n^p(A), \mu_n^q(A) \in [\varepsilon, 1 - \varepsilon]$, then

$$q - p \leq C\kappa^{-1} \log(1/2\varepsilon)/\log n,$$

where $C > 0$ is a universal constant.

**Proof.** We begin with a variant of the Margulis–Russo formula [18, 21], namely

$$\frac{d}{dp} \mu_n^p(A) = \sum_{i=1}^{n} (\mu_p^{i-1} \times (\mu_1 - \mu_0) \times \mu_p^{n-i})(A).$$

Next, recall that the influence $I_{A,p}(i)$ of a coordinate $i$ is the probability that, for $x \sim \mu_n^p$, changing the value of $x_i$ can affect whether $x \in A$, i.e., the probability that the ‘slice’

$$A_i(x) = \{w \in W: (x_1, \ldots, x_{i-1}, w, x_{i+1}, \ldots, x_n) \in A\}$$

is neither $W$ nor $\emptyset$. By (†),

$$(\mu_p^{i-1} \times (\mu_1 - \mu_0) \times \mu_p^{n-i})(A) \geq \kappa I_{A,p}(i),$$

implying

$$\frac{d}{dp} \mu_n^p(A) \geq \kappa \sum_{i=1}^{n} I_{A,p}(i).$$

On the other hand, as in [14], symmetry and [4] give

$$\sum_{i=1}^{n} I_{A,p}(i) = \Omega(\min(\mu_p^n(A), 1 - \mu_p^n(A)) \log n);$$

so, combining, we have

$$\frac{d}{dp} \mu_n^p(A) = \Omega(\kappa \min(\mu_p^n(A), 1 - \mu_p^n(A)) \log n).$$

The stated inequality now follows by elementary calculus. $\square$

### 3. Proof of the main result

As in Theorem 1.1, we assume $A \subset [k]^n$ is symmetric and intersecting, and wish to show that $|A| = o(k^n)$. In outline, the proof of this fact goes as follows.

1. Enlarge $[k]^n$ to a space $W^n$, where $W$ is a suitably chosen ‘covering poset’, equipped with an appropriate $\mu_0$ and $\mu_1$.
2. Use the fact that $A$ is intersecting to conclude that its up-closure in $W^n$ has $\mu_t$-measure at most $1/2$ for a suitable time $t$ (in the interpolation from $\mu_0$ to $\mu_1$).
3. Deduce from Theorem 2.1 that $A$ must have been negligibly small in the original space $[k]^n$. 


Proof of Theorem 1.1. Write $[k]^{(r)}$ for the collection of $r$-subsets of $[k]$, and let $(W, \preceq)$ be the poset
\[ W = [k]^{(1)} \cup [k]^{(k-1)}, \]
with $\preceq$ defined by inclusion. We embed $[k]$ in $W$ by identifying $[k]$ with $[k]^{(1)}$ in the obvious way.

Let $\mu_0$ and $\mu_1$ be, respectively, the uniform (probability) measures on $[k]^{(1)}$ and $[k]^{(k-1)}$, and, as in Section 2, set $\mu_t = (1-t)\mu_0 + t\mu_1$, noting that $\mu_{1/2}$ is the uniform measure on $W$.

Claim 3.1. $\mu_1$ dominates $\mu_0$ with strength $1/k$.

Proof. Let $A \subset W$ be a proper, nontrivial up-set. If $A \subset [k]^{(k-1)}$ or $A \supset [k]^{(k-1)}$, then it is clear that
\[ (\mu_1 - \mu_0)(A) \geq 1/k. \]
The only other possibilities are the ‘stars’
\[ A = \{\{i\}\} \cup ([k]^{(k-1)} \setminus \{[k] \setminus \{i\}\}), \]
for which we have
\[ (\mu_1 - \mu_0)(A) = 1 - 2/k \geq 1/k. \]

We now extend the notion of an intersecting family to $W^n$ by saying that $A \subset W^n$ is intersecting if for any $x, y \in A$, there is some $i \in [n]$ such that $x_i \cap y_i \neq \emptyset$.

Claim 3.2. If $A \subset W^n$ is intersecting, then $\mu_{1/2}(A) \leq 1/2$.

Proof. Note that if $x \sim \mu_{1/2}$, then we also have $x^c \sim \mu_{1/2}$, where $x^c = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is the point-wise complement of $x$. Since at most one of $x$ and $x^c$ can belong to $A$, we have
\[ 2\mu_{1/2}(A) = E[|A \cap \{x, x^c\}|] \leq 1. \]

We may now finish as follows. Given $A \subset [k]^n$ symmetric and intersecting as in Theorem 1.1, let $B$ be its up-closure in $W^n$ and notice that $B$ is again symmetric and intersecting. Claim 3.2 thus gives $\mu_{1/2}(B) \leq 1/2$, so, applying Theorem 2.1 with $p = 0, q = 1/2, \varepsilon = \mu_0(B)$ and $\kappa = 1/k$, we have
\[ 1/2 \leq C\kappa^{-1} \log(1/2\varepsilon)/\log n, \]
or, rearranging,
\[ k^{-1}|A| = \mu_0(B) \leq \frac{1}{2}n^{-\kappa/2C} (= o(1)). \]

4. Conclusion

The most obvious question raised by the present work is that of estimating more accurately — beyond the $o(k^n)$ of Theorem 1.1 — how large a symmetric, intersecting subfamily of $[k]^n$ can really be (for $k \geq 3$). The best examples $A$ that we know are set-intersecting, in the sense that there is a symmetric, intersecting family $B$ of subsets of $[n]$ and an $\ell \in [k]$ such that $x \in [k]^n$ belongs to $A$ if and only if there is some $B \in B$ such that $x_i = \ell$ for all $i \in B$. For instance, if $n = q^2 + q + 1$ with $q$ a prime power, then we may take $B$ to be the set of lines of the classical projective plane $PG(2, q)$ (see [23], for instance), yielding an $A$ of size roughly $k^{n-\sqrt{n}}$. Note that the family consisting of all strings with 1’s in more than half the coordinates, the counterpart of the exceptional example for $[2]^n$ mentioned in the introduction, does much worse.
It seems possible (but maybe impossible to prove) that the largest symmetric intersecting families in $[k]^n$ are set-intersecting. Failing that, it would be very interesting to at least show that there are constants $c, \delta > 0$ (possibly depending on $k$) such that for any symmetric intersecting $A \subseteq [k]^n$, we have
\[
\log_k |A| \leq n - cn^\delta.
\]

Finally, we expect that the main technical contribution of this paper — dealing with intersection problems by situating them in a suitable covering space — will be applicable to further questions in extremal set theory; we hope to return to this circle of ideas in future work.

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References

2. L. Babai, Personal communication.