Subgraphs of large connectivity and chromatic number

António Girão and Bhargav Narayanan

Abstract. Resolving a problem raised by Norin in 2020, we show that for each $k \in \mathbb{N}$, the minimal $f(k) \in \mathbb{N}$ with the property that every graph G with chromatic number at least $f(k)+1$ contains a subgraph H with both connectivity and chromatic number at least k satisfies $f(k) \leq 7k$. This result is best-possible up to multiplicative constants, and sharpens earlier results of Alon–Kleitman–Thomassen–Saks–Seymour from 1987 showing that $f(k) = O(k^3)$, and of Chudnovsky–Penev–Scott–Trotignon from 2013 showing that $f(k) = O(k^2)$. Our methods are robust enough to handle list colouring as well: we additionally show that for each $k \in \mathbb{N}$, the minimal $f_\ell(k) \in \mathbb{N}$ with the property that every graph G with list chromatic number at least $f_\ell(k)+1$ contains a subgraph H with both connectivity and list chromatic number at least k is well-defined and satisfies $f_\ell(k) \leq 4k$. This result is again best-possible up to multiplicative constants; here, unlike with $f(\cdot)$, even the existence of $f_\ell(\cdot)$ appears to have been previously unknown.

1. Introduction

Many of the central open problems in graph theory concern structures that are unavoidable in graphs of large chromatic number, Hadwiger's conjecture [4] being perhaps the most notable example. Here, we shall address two related questions that arise in the study of Hadwiger's conjecture and its list colouring analogue.

Our starting point is the following well-known fact: every graph of chromatic number at least $4k+1$ contains a subgraph of connectivity at least k, as follows from a classical result of Mader [6] asserting that every graph of minimum degree at least $4k$ contains a k-connected subgraph. It is natural to then ask if a graph of large chromatic number must contain a subgraph of both large connectivity and large chromatic number; this was answered by Alon, Kleitman, Thomassen, Saks and Seymour [1] who showed for each $k \in \mathbb{N}$ that there exists a minimal $f(k) \in \mathbb{N}$ such that every graph with chromatic number at least $f(k)+1$ contains a subgraph whose connectivity and chromatic number are both at least k, and that $f(k) = O(k^3)$. This was improved by Chudnovsky, Penev, Scott and Trotignon [3] who (amongst other things) showed that $f(k) = O(k^2)$, and

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the lower order terms in this result were later improved by Penev, Thomassé and Trotignon [10].

The results described above have since found many applications in the study of graphs of large chromatic number. Motivated by applications (see [8, 9]) to the study of Hadwiger's conjecture, Norin [7] asked if the aforementioned results could be sharpened to show an essentially best-possible estimate of $f(k) = O(k)$; our first result, which has already proved crucial in improving the state of the art on Hadwiger's conjecture (see [11]), answers this question affirmatively.

Theorem 1.1. *For each $k \in \mathbb{N}$, every graph G with chromatic number at least $7k+1$ contains a subgraph H with connectivity and chromatic number at least k.*

In other words, Theorem 1.1 asserts that $f(k) \leq 7k$, and from below, Alon, Kleitman, Thomassen, Saks and Seymour [1] showed that $f(k) \geq 2k-3$. While these bounds are not too far apart, we make no particular effort to optimise the multiplicative constant in our result since it seems unlikely that this will completely bridge the gap between the upper and lower bounds.

With the analogue of Hadwiger's conjecture for list colourings [5] in mind, it is also natural to ask if, for each $k \in \mathbb{N}$, there exists a minimal $f_\ell(k) \in \mathbb{N}$ such that every graph with list chromatic number at least $f_\ell(k)+1$ contains a subgraph whose connectivity and list chromatic number are both at least k. While even the existence of $f_\ell(\cdot)$ appears to have been previously unknown, the methods we use to prove Theorem 1.1 are robust enough to both demonstrate the existence of $f_\ell(\cdot)$, and prove an essentially best-possible estimate; once again, this result has already been instrumental in improving the state of the art on the list colouring analogue of Hadwiger's conjecture (see [12]).

Theorem 1.2. *For each $k \in \mathbb{N}$, every graph G with list chromatic number at least $4k+1$ contains a subgraph H with connectivity and list chromatic number at least k.*

In other words, Theorem 1.2 asserts that $f_\ell(k) \leq 4k$, and the construction of Alon, Kleitman, Thomassen, Saks and Seymour [1] mentioned earlier also shows that $f_\ell(k) \geq 2k-3$.

It is worth mentioning that all of [1, 3, 10] treat the more general 'asymmetric' problem of finding a subgraph of connectivity at least k and chromatic number at least m. Our arguments also yield asymmetric analogues of Theorems 1.1 and 1.2 with essentially optimal bounds, but since the symmetric problems seem to be at the heart of the matter, we restrict our attention to these for the most part and confine ourselves to a few brief remarks about the modifications necessary for the relevant asymmetric variants.

This paper is organised as follows. We introduce the key notions that we need in Section 2, give the proofs of both our results in Section 3, and conclude with a discussion of the aforementioned asymmetric variants and some other related questions in Section 4.

2. Preliminaries

We start by establishing some notation. For a set X, we write 2^X for the power set of X, and given a function λ defined on X and a subset $Y \subset X$, we write $\lambda|_Y$ for the restriction of λ to Y. Given a graph G, as is usual, we write $\chi(G)$, $\chi_\ell(G)$ and $\kappa(G)$ for the chromatic number, list chromatic number and the connectivity of G respectively, and for a subset $X \subset V(G)$, we write $G[X]$ for the subgraph of G induced by X; for any graph-theoretic terminology not defined here, we refer the reader to [2].

In what follows, we fix $k \in \mathbb{N}$ and work with a fixed palette $\mathcal{C}$ of $7k$ colours. Our proof of Theorem 1.1 will hinge around two notions, those of templates and extensibility, that we define below (a related notion, that of 'colouring constraints', appears in [3]); the modifications needed for Theorem 1.2 will be indicated in place where needed, in Section 3.

Templates. A template on a graph consists of a set of properly 'pre-coloured' vertices, along with lists of 'forbidden' colours at each of the remaining vertices. Formally, a *template* $T = (S, c, F)$ on a graph G consists of

(1) a subset $S \subset V(G)$,
(2) a proper *pre-colouring* $c : S \to \mathcal{C}$ of the induced subgraph $G[S]$, and
(3) a function $F : V(G) \setminus S \to 2^{\mathcal{C}}$ specifying a list of *forbidden* colours at each remaining vertex.

Let $G = (V, E)$ be a graph and let $T = (S, c, F)$ be a template on G. We define the *degree* of T by

$$\deg(T) = k|S| + \sum_{v \in V \setminus S} |F(v)|. \tag{1}$$

For a set $X \subset V$, the *restriction* of T to the induced subgraph $G[X]$ is naturally the template

$$T_X = (S \cap X, c|_{S\cap X}, F|_{S\cap X}).$$

Let us note that the degree of a template is additive across disjoint restrictions, i.e., if $X \cup Y$ is a partition of V, then

$$\deg(T) = \deg(T_X) + \deg(T_Y).$$

Finally, we say that a proper colouring $\hat{c} : V \to \mathcal{C}$ of G *respects* T if it extends the pre-colouring specified by T while avoiding the forbidden colours at all the other vertices, i.e., if

(1) $\hat{c}(v) = c(v)$ for all $v \in S$, and

(2) $\hat{c}(v) \notin F(v)$ for all $v \in V \setminus S$.

Extensibility. We say that a graph G is *inextensible* if there exists a template $T = (S, c, F)$ on G such that

(1) $\deg(T) \leq 2k^2$,
(2) $|F(v)| \leq 2k$ for each $v \in V(G) \setminus S$, and
(3) there is no proper colouring of G using the palette $\mathcal{C}$ that respects T;

we call this template T a *witness* for the inextensibility of G, and also note that there may be multiple templates witnessing this fact. Naturally, we say that G is *extensible* if it is not inextensible.

First, we observe that graphs of sufficiently large chromatic number are inextensible.

Lemma 2.1. *If G is a graph with $\chi(G) \geq 7k+1$, then G is inextensible.*

Proof. This is obvious; the empty template, with no colours forbidden at any vertex and no pre-coloured vertices, shows that G is inextensible. $\square$

Next, there is some slack in the definition of inextensibility, as shown by the following observation; while this slack is not essential, it will nonetheless help make our argument more transparent.

Lemma 2.2. *If a graph G is inextensible, then this is witnessed by a template $T = (S, c, F)$ on G for which $|F(v)| \leq k-1$ for each $v \in V(G) \setminus S$.*

Proof. Of all the templates $T = (S, c, F)$ witnessing the inextensibility of G, choose one with $|S|$ maximal, and suppose for the sake of contradiction that there exists a vertex $v \in V(G) \setminus S$ with $|F(v)| \geq k$. We claim that we may find another template T' witnessing the inextensibility of G in which $S \cup \{x\}$ is pre-coloured, contradicting the maximality of $|S|$.

To see this, first note that since $\deg(T) \leq 2k^2$, we must have $|S| \leq 2k$. Next, as $|F(v)| \leq 2k$ and $|\mathcal{C}| = 7k$, there exists a colour in $\mathcal{C} \setminus F(v)$ not appearing anywhere in the pre-colouring of S; to obtain T' from T, we pre-colour v with this colour and remove the forbidden list of colours associated with v; any proper colouring of G respecting T' also respects T, so it suffices to check that $\deg(T') \leq \deg(T) \leq 2k^2$. This is straightforward: in passing from T to T', the first summand in the definition of the degree increases by k, but since $|F(v)| \geq k$, the second summand decreases by at least k. $\square$

3. Proofs of the main results

With the notions of templates and extensibility in hand, we are now ready to prove our first main result.

Proof of Theorem 1.1. Let G be a graph with $\chi(G) \geq 7k+1$; our goal is to find a subgraph H of G with both $\chi(H) \geq k$ and $\kappa(H) \geq k$. By Lemma 2.1, G is inextensible, so let H be a *minimal inextensible* induced subgraph of G on some vertex set U, and let $T = (S, c, F)$ be a template on H witnessing its inextensibility with $|F(v)| \leq k-1$ for each $v \in U \setminus S$, as promised by Lemma 2.2. As observed earlier, since $\deg(T) \leq 2k^2$, it must be the case that $|S| \leq 2k$. We shall show that H has large connectivity and chromatic number.

Claim 3.1. $\kappa(H) \geq k$.

Proof. Suppose for the sake of contradiction that H is not k-connected. We shall find a proper colouring $\hat{c}$ of H using $\mathcal{C}$ that respects T, contradicting the inextensibility of H as witnessed by T.

First, if H is isomorphic to a complete graph K_k, then the construction of $\hat{c}$ is straightforward. It suffices to find a proper colouring of $H[U \setminus S]$ where each vertex v receives a colour from the list $L(v)$ of colours in $\mathcal{C}$ appearing neither somewhere in S, nor in $F(v)$. Since $|F(v)| \leq k-1$, we see that for each vertex $v \in U \setminus S$, we have (with room to spare)

$$|L(v)| \geq |\mathcal{C}| - |S| - |F(v)| \geq 7k - 2k - (k-1) \geq k. \tag{2}$$

Since H has k vertices and $|L(v)| \geq k$ for each $v \in U \setminus S$, we may colour each vertex $v \in U \setminus S$ with a colour from $L(v)$ in such a way that these vertices all get distinct colours.

Next, suppose that there is a subset $X \subset U$ of size at most $k-1$ which disconnects H, and fix a partition $Y \cup Z$ of $U \setminus X$ with both Y and Z nonempty in which there are no edges between Y and Z. Since

$$\deg(T_Y) + \deg(T_Z) \leq \deg(T) \leq 2k^2,$$

we assume, without loss of generality, that $\deg(T_Z) \leq k^2$.

First, let $U' = X \cup Y$ and consider $H' = H[U']$. Starting with the template $T_{U'}$, we construct a new template $T' = (S \cap U', c\,|_{S \cap U'}, F')$ on H' by defining F' as follows:

(1) $F'(v) = F(v)$ for each $v \in Y \setminus S$, and
(2) for each $v \in X \setminus S$, we include $F(v)$ in $F'(v)$, and then for each $z \in S \cap Z$ that v is adjacent to, we add the colour $c(z)$ to $F'(v)$.

It is easy to see that $|F'(v)| \leq 2k$ for each $v \in U'$; indeed,

$$|F'(y)| = |F(y)| \leq k-1$$

for each $y \in Y \setminus S$, and since $\deg(T_Z) \leq k^2$, we must have $|S \cap Z| \leq k$, so

$$|F'(x)| \leq |S \cap Z| + |F(x)| \leq k + (k-1) = 2k-1$$

for each $x \in X \setminus S$.

It is also not hard to see that $\deg(T') \leq \deg(T)$; indeed, in passing from T to T', the removal of the pre-coloured vertices in $S \cap Z$ decreases the degree of the template by $k|S \cap Z|$, while the addition of the colours of these vertices to the lists of forbidden colours in X increases the degree of the template by at most $|X||S \cap Z| \leq (k-1)|S \cap Z|$.

From the minimality of H, we know that H' is extensible, so there exists a proper colouring c' of H' using $\mathcal{C}$ that respects T'.

Next, we take $U'' = X \cup Z$ and $H'' = H[U'']$, and construct a new template T'' on H'' as follows: we start with the restriction T_Z, and then additionally pre-colour the vertices in $X \setminus S$ according to c'. Clearly, we have

$$\deg(T'') \leq k|X| + \deg(T_Z) \leq k(k-1) + k^2 \leq 2k^2,$$

and since all the lists of forbidden colours in T'' are inherited from T, these lists all have size at most $k-1$.

From the minimality of H, we again know that H'' is extensible, so we may find a proper colouring c'' of H'' using $\mathcal{C}$ that respects T''.

Of course, we have ensured that $c'\,|_X = c''\,|_X$, so gluing these two colourings together along X gives us a proper colouring $\hat{c}$ of H as required. □

Claim 3.2. *$\chi(H) \geq k$.*

Proof. Suppose again for the sake of contradiction that $\chi(H) \leq k-1$. This certainly means that we may partition $U \setminus S$ into $k-1$ independent sets $J_1, J_2, \dots, J_{k-1}$.

Now, we order the vertices of $U \setminus S$ in such a way that each independent set J_i forms an interval in this ordering. We process the vertices of $U \setminus S$ in order and partition them into (at most) $3k$ intervals $I_1, I_2, \dots, I_{3k}$, each contained within some independent set J_i, as follows: having constructed $I_1, I_2, \dots, I_{m-1}$, we consider as yet unprocessed vertices in order and add them one by one to I_m, and when considering a vertex v, we decide to stop the construction of I_m and move on to I_{m+1} based on the following pair of rules:

(1) we stop *without adding* v to I_m if the addition of v stops I_m from being contained within a single independent set J_i, and otherwise
(2) we stop *by adding* v to I_m if this causes the sum $\sum_{u \in I_m} |F(u)|$ to exceed k.

It is not hard to see that this procedure produces at most $3k$ intervals: the number of times we stop on account of the first rule is at most $k-1$, and the number of times we stop on account of the second rule is at most $2k$ since $\deg(T) \leq 2k^2$. Notice that these intervals $I_1, I_2, \dots, I_{3k}$ have the following properties:

(1) each I_m is an independent set in H, and
(2) each I_m satisfies $\sum_{v \in I_m} |F(v)| \leq 2k$.

We now construct a proper colouring $\hat{c}$ of H using $\mathcal{C}$ that respects T, again contradicting the inextensibility of H as witnessed by T. For each interval I_m, we form the list L_m of colours appearing neither somewhere in S, nor in $\cup_{v\in I_m} F(v)$, noting that

$$|L_m| \geq |\mathcal{C}| - |S| - \sum_{v\in L_m} |F(v)| \geq 7k - 2k - 2k \geq 3k.$$

Since there are at most $3k$ intervals, it is now clear, as before, that we can colour each interval I_m with some colour from L_m in such a way that distinct intervals get distinct colours. This colouring yields a proper colouring $\hat{c}$ of H as required. $\square$

Claims 3.1 and 3.2 together show that H has the requisite properties, proving the result. $\square$

The proof of our second main result is almost identical to, and easier in parts than, the proof of Theorem 1.1. The main point here is to modify the notion of a template to deal with list colourings. Let $G = (V, E)$ be a graph equipped with a list $L(v)$ of colours at each vertex $v \in V$. A template $T = (S, c, F)$ on G now consists of a properly pre-coloured set of vertices S with each vertex $v \in S$ coloured with a colour $c(v) \in L(v)$ along with a set of forbidden colours $F(v) \subset L(v)$ at each vertex $v \in V \setminus S$. The rest of the notions from Section 2 around templates carry over verbatim: the degree of a template T is again given by (1), i.e., by

$$\deg(T) = k|S| + \sum_{v\in V\setminus S} |F(v)|,$$

a list colouring of G respects T if it obeys the constraints imposed by T, we say G is inextensible if there is a template $T = (S, c, F)$ on G such that

(1) $\deg(T) \leq 2k^2$,
(2) $|F(v)| \leq 2k$ for each $v \in V \setminus S$, and
(3) there is no list colouring of G that respects T,

and we finally note that Lemma 2.2 holds in this setting as well. With these alterations in place, the proof of Theorem 1.2 is more or less identical to that of Theorem 1.1.

Proof of Theorem 1.2. Let $G = (V, E)$ be a graph with $\chi_\ell(G) \geq 4k + 1$, and suppose that we are given a list $L(v)$ of $4k$ colours at each vertex $v \in V$ witnessing this bound. As before, the empty template shows that G is inextensible, and we claim that a minimal inextensible induced subgraph of G has the desired properties.

In a little more detail, let H be a minimal inextensible induced subgraph of G on some vertex set U, and let $T = (S, c, F)$ be a template on H witnessing its inextensibility. As before, we may assume by Lemma 2.2 that $|F(v)| \leq k - 1$ for each $v \in U \setminus S$, and since $\deg(T) \leq 2k^2$, it must be the case that $|S| \leq 2k$.

The proof of the fact that $\kappa(H) \geq k$ is identical to the proof of Claim 3.1; indeed, the only difference is that the weak inequality in (2) now becomes tight, with a $4k$ in place of the $7k$ in the proof of Claim 3.1.

The argument showing $\chi_\ell(H) \geq k$ is easier than the one needed to establish Claim 3.2. Indeed, suppose that $\chi_\ell(H) \leq k-1$; we claim that this contradicts the inextensibility of H. To see this, note that to find a list colouring of H extending T, it suffices to find a proper colouring $\hat{c}$ of $H' = H[U \setminus S]$ with $\hat{c}(v)$ belonging to the set

$$L'(v) = L(v) \setminus (F(v) \cup \{c(u) : u \in S\});$$

now noticing that

$$|L'(v)| \geq |L(v)| - |S| - |F(v)| \geq 4k - 2k - (k-1) = k+1$$

for each $v \in U \setminus S$, we conclude that such a colouring $\hat{c}$ exists by definition since $\chi_\ell(H') \leq \chi_\ell(H) \leq k-1$, completing the proof. □

4. Conclusion

We conclude with a discussion of some problems that still remain. In what follows, we restrict ourselves to proper colourings, but similar considerations apply to list colourings as well.

As mentioned earlier, we now know that $2k-3 \leq f(k) \leq 7k$. There are reasons to believe that the lower bound is more reflective of the truth, as we now explain. As mentioned before, all of [1, 3, 10] study an asymmetric analogue of the problem treated here: for $k, m \in \mathbb{N}$, let $g(k,m)$ be the least natural number such that every graph G with chromatic number at least $g(k,m)+1$ contains a subgraph H with connectivity at least k and chromatic number at least m. Of course, it is clear from our results that $g(k,m) = \Theta(m+k)$, but more precise results are available in the 'off-diagonal' case when m is much larger than k: it is shown in [1] that $g(k,m) \geq m+k-3$, and [10] shows that $g(k,m) \leq m+2k-3$ when $m \geq 2k^2$. It is not hard to modify the arguments here to establish something like this latter bound unconditionally, i.e., to show that $g(k,m) = m + O(k)$ for all $k, m \in \mathbb{N}$. Obtaining a precise description of $g(k,m)$ when m is much larger than k may be a good starting point towards pinning down the exact value of $f(k) = g(k,k)$.

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School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK

Email address: giraoa@bham.ac.uk

Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA

Email address: narayanan@math.rutgers.edu