A universal exponent for homeomorphs

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ABSTRACT. We prove a uniform bound on the topological Turán number of an arbitrary two-dimensional simplicial complex S: any two-dimensional complex on n vertices with at least $C_S n^{3-1/5}$ facets contains a homeomorph of S, where $C_S > 0$ is a constant depending on S alone. This result, a two-dimensional analogue of a classical one-dimensional result of Mader, sheds some light on an old problem of Linial from 2006.

1. INTRODUCTION

An r-uniform hypergraph G over a finite set V (the *vertices* of G) consists of a collection of r-element subsets of V (the *edges* of G), and any such r-uniform hypergraph G may be identified with an $(r-1)$-dimensional simplicial complex S_G whose $(r-1)$-skeleton is the edge set of G and whose 0-skeleton is the vertex set of G. A number of natural geometric problems arise when we view r-uniform hypergraphs (or r-graphs, for short) as $(r-1)$-dimensional simplicial complexes (or $(r-1)$-complexes, for short). For example, questions of this nature arise in the high-dimensional combinatorics programme of Linial [12, 10], and have also been raised by Gowers [5]; for a sample of some recent results in this programme, see [4, 14, 13, 11].

In this paper, we study the Turán problem for 2-complexes, or equivalently, the topological Turán problem for 3-graphs. In the Turán theory of 3-graphs (see [17, 7]), one is concerned with finding an *isomorphic copy* of a fixed 3-graph inside a larger 3-graph, whereas in the context of 2-complexes (which is our focus here), one is concerned with finding a *homeomorphic copy* of a fixed 2-complex inside a larger 2-complex.

Since we shall adhere to the language of 3-graphs in what follows, let us explain what it means for a pair of 3-graphs to be homeomorphic. We say that two 3-graphs G and H are *homeomorphic* if the associated 2-complexes S_G and S_H (as described above) are homeomorphic as topological spaces, and we say that G *contains a homeomorph of H* if there is a subgraph of G homeomorphic to H. The following example may help clarify this point of view: a 3-graph H is a homeomorph of the complete 3-graph K_4^3 on four vertices (i.e., the tetrahedron, or equivalently, the two-dimensional sphere S^2) if we can place the vertices of H on the sphere and then triangulate the sphere using
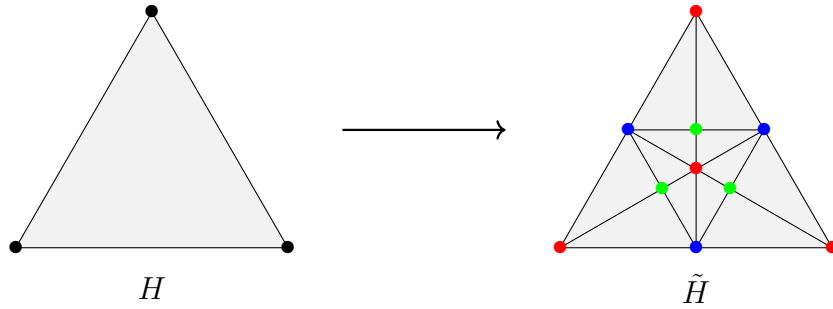
FIGURE 1. Each edge of H maps to twelve new edges in $\tilde{H}$; the colours red, blue and green describe the tripartition of $\tilde{H}$.

those vertices in such a way that the resulting triangles are precisely the edges of H. With this language in place, our main contribution is the following theorem.

Theorem 1.1. *For each 3-graph H, there exists $C_H > 0$ such that any 3-graph G on n vertices with at least $C_H n^{3-1/5}$ edges contains a homeomorph of H.*

Here, it is worth mentioning that the topological Turán problem for 2-graphs (i.e., graphs) is understood reasonably well: indeed, the homeomorphs of a graph H are essentially just the subdivisions of H, and a classical result of Mader [15] asserts that for any graph H, there exists $C_H > 0$ such that every n-vertex graph with at least $C_H n$ edges contains a subdivision of H (and this is tight in general up to the multiplicative constant). Linial [8, 9] has raised the question of an analogous result for 3-graphs, and while answers are available for a few *specific* 3-graphs, no general results for 3-graphs in the spirit of Mader's theorem appear to have been previously known; our main result, Theorem 1.1, fills in this gap.

We shall in fact prove Theorem 1.1 with $C_H = 2000v(H)^6$ for all sufficiently large $n \in \mathbb{N}$. However, we make no attempt to optimise this constant since we do not believe the exponent of $3 - 1/5$ in our result to be tight; instead, we expect the right exponent to be $3 - 1/2 = 5/2$, and make the following conjecture.

Conjecture 1.2. *For each 3-graph H, there exists $C_H > 0$ such that any 3-graph G on n vertices with at least $C_H n^{5/2}$ edges contains a homeomorph of H.*

This conjectural exponent of $5/2$ requires explanation, and this brings us to the starting point of the line of investigation we pursue in this paper. In the specific case of the tetrahedron K_4^3 (or equivalently, the sphere), a classical result of Brown, Erdős and Sós [1] says that $5/2$ is indeed the correct exponent: the minimum number of edges guaranteeing a homeomorph of the sphere in an n-vertex 3-graph is $\Theta(n^{5/2})$. Conjecture 1.2 is then motivated by the following line of reasoning: it turns out that we may find homeomorphs in 2-graphs roughly once we are able to find *cycles*, i.e.,

homeomorphs of S^1, and our investigations suggest that we ought to be able to find homeomorphs in 3-graphs roughly once we are able to find *spheres*, i.e., homeomorphs of S^2. While Theorem 1.1 presents some progress towards Conjecture 1.2, we emphasise that even the specialisation of this conjecture to specific 3-graphs leads to interesting open problems; indeed, the special case of H being (a triangulation of) the torus remains open, and has been reiterated by Linial [8, 9] on multiple occasions as a natural starting point.

While the arguments of Brown, Erdős and Sós [1] are rather specific to the sphere and, slightly more generally, to 'double-pyramidal' complexes, their arguments may nonetheless be generalised to show that for any 3-graph H, there is an $\varepsilon_H > 0$ such that any n-vertex 3-graph with at least $n^{3-\varepsilon_H}$ edges contains a homeomorph of H. An alternative (and somewhat easier) way of seeing this is as follows: for any 3-graph H, it is not difficult to construct a 3-*partite* 3-graph $\tilde{H}$ that is homeomorphic to H (as shown in Figure 1), and since finding a copy of $\tilde{H}$ as a *subgraph* is a degenerate Turán problem, it follows from a classical result of Erdős [3] that there is an $\varepsilon_H > 0$ such that any n-vertex 3-graph with at least $n^{3-\varepsilon_H}$ edges contains a copy of $\tilde{H}$ as a subgraph, and hence a homeomorphic copy of H. In contrast, Theorem 1.1 says that this H-specific exponent ε_H may actually be replaced by the universal exponent $1/5$.

The level of generality at which Theorem 1.1 applies comes at a price, however: for a few specific 3-graphs of interest, such as the sphere and the torus for example, the aforementioned arguments (i.e., that of Brown–Erdős–Sós, and the one based on the degenerate 3-graph Turán problem) yield better estimates than what is promised by Theorem 1.1.

This paper is organised as follows. We begin with some definitions and establish some of the basic notions we need for the proof of our main result in Section 2. The proof of Theorem 1.1 then follows in Section 3. We conclude with a discussion about the limits of our approach, as well as some open problems, in Section 4.

2. Preliminaries

Our notation is for the most part standard. Given a 2-graph or a 3-graph G, we write $v(G)$ and $e(G)$ for the number of vertices and edges of G respectively. For a set S of vertices in a 2-graph G, we write $\Gamma(S)$ for the set of common neighbours of S in G, and following a common abuse, we write $\Gamma(x)$ for $\Gamma(\{x\})$, $\Gamma(x, y)$ for $\Gamma(\{x, y\})$, and so on; in the sequel, whenever we refer to $\Gamma(\cdot)$, the underlying graph will always be clear from the context, so there should be no cause for confusion. In those arguments that will involve working with both 2-graphs and 3-graphs in close proximity, we shall refer to the edges of 3-graphs as *faces* to avoid confusion. Finally, in what follows, pairs and triples refer respectively to unordered two-element and three-element sets; again, we
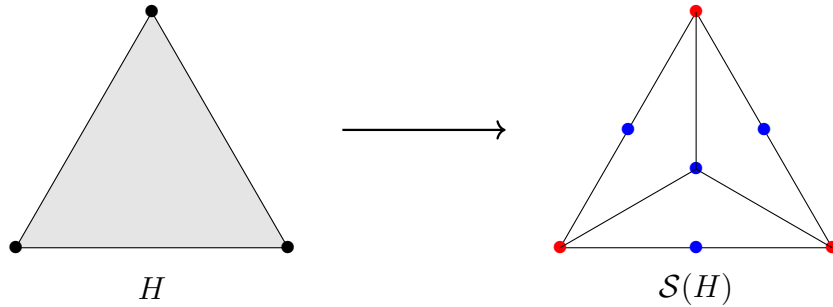
FIGURE 2. The construction of $\mathcal{S}(H)$ from H; the colours red and blue describe the bipartition of $\mathcal{S}(H)$.

abuse notation slightly and abbreviate a pair $\{x, y\}$ as xy, a triple $\{x, y, z\}$ as xyz, and so on.

It will be convenient to work with 3-partite 3-graphs; the following fact facilitates this, and follows from an easy averaging argument.

Proposition 2.1. *Any 3-graph on $3n$ vertices with m edges contains a 3-partite subgraph with vertex classes of size n and at least $2m/9$ edges.* □

Now, let H be a fixed 3-graph and let G be a 3-partite 3-graph whose three vertex classes X, Y and Z are each of size n. Our strategy to construct a homeomorph of H in G will involve gluing various building blocks together appropriately; below, we introduce the notions we require to execute this strategy.

First, we shall construct an *auxiliary 2-graph* $\mathcal{S}(H)$ from H that will be helpful in finding a homeomorphic copy of H in G. The construction of $\mathcal{S}(H)$ from H, illustrated in Figure 2, is as follows: first, for each pair xy that is contained in some face of H, we introduce a new vertex $u = u_{xy}$ in $\mathcal{S}(H)$ and add the edges xu and yu to $\mathcal{S}(H)$; then, for each face xyz of H, we introduce a new vertex $u = u_{xyz}$ in $\mathcal{S}(H)$ and add the edges xu, yu and zu to $\mathcal{S}(H)$.

We record a few facts about $\mathcal{S}(H)$ below.

(1) Each face xyz of H gives rise to three specific 4-cycles in $\mathcal{S}(H)$, namely the 4-cycles $\{x, u_{xy}, y, u_{xyz}\}$, $\{y, u_{yz}, z, u_{xyz}\}$ and $\{z, u_{zx}, x, u_{xyz}\}$; we call the 4-cycles of this form the *special* 4-cycles of $\mathcal{S}(H)$.
(2) $\mathcal{S}(H)$ is bipartite, with the set V_1 of the original vertices of H and the set V_2 of the new vertices added in the construction of $\mathcal{S}(H)$ forming a bipartition.
(3) The degree of any vertex of $\mathcal{S}(H)$ in V_2 is at most 3.

Next, we describe the structures within G that will serve as building blocks in constructing a homeomorph of H. The *link graph* $\mathcal{L}_z$ of a vertex $z \in Z$ is the bipartite graph between X and Y whose edges are those pairs xy for which xyz is a face of G.

Notice that a 4-cycle in the link graph $\mathcal{L}_z$ corresponds to four faces of G (all sharing the vertex z) that, taken together, are homeomorphic to a disk; we call such a collection of four faces of G a 4-*disk* with *centre z*, and call the associated 4-cycle in the link graph $\mathcal{L}_z$ the *boundary* of the 4-disk.

Observe that a fixed 4-cycle in the complete bipartite graph between X and Y may be the boundary of anywhere between 0 and n different 4-disks in G. We set $K_H = 3v(H)^3$, and call a 4-cycle between X and Y

(1) H-*admissible* if the cycle is the boundary of more than K_H different 4-disks in G, and
(2) H-*forbidden* if this cycle is the boundary of between 0 and K_H different 4-disks in G.

The definitions of admissible and forbidden 4-cycles are motivated by the following observation. As noted earlier, each face of H corresponds to three special 4-cycles in $\mathcal{S}(H)$, and if we manage to find a copy of $\mathcal{S}(H)$ in $X \times Y$ with the property that all its 4-cycles form boundaries of 4-disks in G with distinct centres, then we may glue the corresponding 4-disks together to obtain a homeomorph of H in G.

We shall rely on H-admissible 4-cycles between X and Y to build a homeomorph of H in G. First, assuming G has sufficiently many edges, we shall show that we may pass to a subgraph G' of G in which most of the 4-cycles in $X \times Y$ are H-admissible. Next, we shall show, using G', that we may find a copy of $\mathcal{S}(H)$ between X and Y with the property that each of the 4-cycles in this copy is H-admissible. Finally, since an H-admissible 4-cycle is contained in at least $3v(H)^3 \geq 3e(H)$ different link graphs, we will be able to ensure that we never re-use central vertices when gluing the appropriate 4-disks in G' together to construct a homeomorph of H.

3. Proof of the main result

As before, let H be a fixed 3-graph, take $K_H = 3v(H)^3$, and let G be a 3-partite 3-graph whose three vertex classes X, Y and Z are each of size n.

Our first goal is to find a vertex $z \in Z$ whose link graph $\mathcal{L}_z$ is sufficiently dense so as to contain many copies of the auxiliary 2-graph $\mathcal{S}(H)$ defined in Section 2, and which has a small number of H-forbidden 4-cycles. In order to achieve this, we use a straightforward application of dependent random choice; with the set-up as above, we have the following claim.

Lemma 3.1. *If $e(G) \geq Cn^{3-\delta}$, then there exists a vertex $z \in Z$ such that*

(1) $e(\mathcal{L}_z) \geq (C/2)n^{2-\delta}$, *and*
(2) *the number of H-forbidden 4-cycles in $\mathcal{L}_z$ is at most $(2K_H/C)n^{1+\delta}e(\mathcal{L}_z)$.*

Proof. Select a vertex $z \in Z$ uniformly at random. It is clear that $\mathbb{E}[e(\mathcal{L}_z)] = e(G)/n$. Note that the probability that any given H-forbidden 4-cycle is contained in $\mathcal{L}_z$ is at most K_H/n. Therefore, writing B_z for the number of H-forbidden 4-cycles in $\mathcal{L}_z$, we have $\mathbb{E}[B_z] \leq K_H n^3$.

Putting the two estimates above together, we have

$$\mathbb{E}\big[e(\mathcal{L}_z) - (C/2)n^{2-\delta} - (C/2K_H)n^{-1-\delta}B_z)\big] \geq 0,$$

so there must exist a vertex $z \in Z$ for which we both have

$$e(\mathcal{L}_z) - (C/2)n^{2-\delta} \geq 0$$

and

$$e(\mathcal{L}_z) - (C/2K_H)n^{-1-\delta}B_z \geq 0,$$

proving the claim. □

Our proof of Theorem 1.1 will hinge around finding a copy of the auxiliary 2-graph $\mathcal{S}(H)$ within the link graph $\mathcal{L}_z$ promised by Lemma 3.1 while avoiding H-forbidden 4-cycles. To find this copy, we first show that we can pass to a large subset $Y' \subset Y$ within which almost all pairs and triples are well-behaved. To quantify what it means to be well-behaved, we make the following definitions.

(1) We call a pair y_1y_2 of vertices in Y *good* if

$$|\Gamma(y_1, y_2)| \geq n^{1-2\varepsilon}$$

and there are at most

$$(K_H/C)n^{1-3\varepsilon}|\Gamma(y_1, y_2)|$$

H-forbidden 4-cycles containing both y_1 and y_2, and *bad* otherwise.
(2) We call a triple $y_1y_2y_3$ of vertices in Y *good* if

$$|\Gamma(y_1, y_2, y_3)| \geq n^{1-3\varepsilon},$$

and *bad* otherwise.

With this set-up, we next show the following.

Lemma 3.2. *Let $\varepsilon \leq 1/5$, $C \geq 1$ and let $\mathcal{L}_z$ be a bipartite graph between X and Y with $(C/2)n^{2-\varepsilon}$ edges in which the number of H-forbidden 4-cycles is at most $K_H n^{3+1/5-\varepsilon}$. Then there is a subset Y' of Y of size at least $n^{1-\varepsilon}/4$ within which*

(1) *at most $(400/C)\binom{|Y'|}{2}$ pairs are bad, and*
(2) *at most $(600/C)\binom{|Y'|}{3}$ triples are bad.*

Proof. To prove the lemma, we start by selecting a vertex $x \in X$ uniformly at random. Clearly, we have

$$\mathbb{E}[|\Gamma(x)|] = (C/2)n^{1-\varepsilon}.$$

Notice that if a pair y_1y_2 in Y is bad, then either

(a) $|\Gamma(y_1, y_2)| < n^{1-2\varepsilon}$, or
(b) the number of H-forbidden 4-cycles through y_1 and y_2 is at least

$$(K_H/C)n^{1-3\varepsilon}|\Gamma(y_1, y_2)|,$$

or possibly both.

First, given a bad pair y_1y_2 in Y for which (a) holds, since $|\Gamma(y_1, y_2)| < n^{1-2\varepsilon}$, the probability that both y_1 and y_2 belong to $\Gamma(x)$ is at most $n^{-2\varepsilon}$; hence, the number P_1 of such pairs surviving in $\Gamma(x)$ satisfies

$$\mathbb{E}[P_1] \leq n^{2-2\varepsilon}.$$

Next, let $\mathcal{Q}$ denote the set of bad pairs y_1y_2 for which (b) holds, so each pair $y_1y_2 \in \mathcal{Q}$ lies in at least $(K_H/C)n^{1-3\varepsilon}|\Gamma(y_1, y_2)|$ H-forbidden 4-cycles. Since the total number of H-forbidden 4-cycles in $\mathcal{L}_z$ is at most $K_H n^{3+1/5-\varepsilon}$, we get

$$\sum_{y_1y_2 \in \mathcal{Q}} (K_H/C)n^{1-3\varepsilon}|\Gamma(y_1, y_2)| \leq K_H n^{3+1/5-\varepsilon},$$

which implies that

$$\sum_{y_1y_2 \in \mathcal{Q}} |\Gamma(y_1, y_2)| \leq Cn^{2+1/5+2\varepsilon}.$$

It follows that the number P_2 of pairs in $\mathcal{Q}$ surviving in $\Gamma(x)$ satisfies

$$\mathbb{E}[P_2] = \sum_{y_1y_2 \in \mathcal{Q}} \frac{|\Gamma(y_1, y_2)|}{n} \leq Cn^{1+1/5+2\varepsilon}.$$

Thus, the total number P_x of bad pairs surviving in $\Gamma(x)$, which is clearly at most the sum $P_1 + P_2$, satisfies

$$\mathbb{E}[P_x] \leq \mathbb{E}[P_1] + \mathbb{E}[P_2] \leq Cn^{1+1/5+2\varepsilon} + n^{2-2\varepsilon} \leq (1+C)n^{2-2\varepsilon};$$

here, the last inequality relies on the fact that $\varepsilon \leq 1/5$.

Finally, given a bad triple $y_1y_2y_3$ in Y, since $|\Gamma(y_1, y_2, y_3)| < n^{1-3\varepsilon}$, the probability that this triple survives in $\Gamma(x)$ is at most $n^{-3\varepsilon}$. Writing T_x for the number of bad triples surviving in $\Gamma(x)$, we again have

$$\mathbb{E}[T_x] \leq n^{3-3\varepsilon}.$$

Putting the above estimates together, we get

$$\mathbb{E}\left[|\Gamma(x)| - \frac{Cn^{1-\varepsilon}}{4} - \frac{CP_x}{12(1+C)n^{1-\varepsilon}} - \frac{CT_x}{6n^{2-2\varepsilon}}\right] \geq 0,$$

which in particular implies that there is some $x \in X$ for which we have

(A) $|\Gamma(x)| \geq Cn^{1-\varepsilon}/4$,
(B) $|\Gamma(x)| \geq CP_x/12(1+C)n^{1-\varepsilon}$, and
(C) $|\Gamma(x)| \geq CT_x/6n^{2-2\varepsilon}$.

Multiplying the inequality in (B) by the one in (A), the inequality in (C) by the square of the one in (A), we see that for this choice of $x \in X$, we have

$$P_x \leq \big(48(1+C)/C^2\big)|\Gamma(x)|^2 \leq (400/C)\binom{|\Gamma(x)|}{2},$$

and

$$T_x \leq \big(96/C^3\big)|\Gamma(x)|^3 < \big(600/C^3\big)\binom{|\Gamma(x)|}{3},$$

provided n is sufficiently large. Taking $Y' = \Gamma(x)$ for this choice of x proves the claim. $\qquad\square$

We are now ready to put these two lemmas together to prove our main result.

Proof of Theorem 1.1. Our goal given is to find a homeomorph of a given 3-graph H in any large 3-graph G with sufficiently many faces. Appealing to Proposition 2.1, we start by assuming that G is a 3-partite 3-graph whose three vertex classes X, Y and Z are each of size n, and which has at least $Cn^{3-1/5}$ faces for some suitably large constant C depending on H alone. As described in Section 2, we shall work with the auxiliary 2-graph $\mathcal{S}(H)$ to find a homeomorph of H in G.

First, we apply Lemma 3.1 to G with $\delta = 1/5$. This gives us a vertex $z \in Z$ whose link graph $\mathcal{L}_z$ contains $(C/2)n^{2-\varepsilon}$ edges for some $\varepsilon \leq \delta = 1/5$ in which the number of H-forbidden 4-cycles is at most

$$(2K_H/C)n^{1+1/5}e(\mathcal{L}_z) = K_H n^{3+1/5-\varepsilon}.$$

This link graph satisfies the requirements of Lemma 3.2, so we apply the lemma to pass to a subset $Y' \subset Y$ within which most pairs and most triples are good.

Recall that $\mathcal{S}(H)$ is bipartite and admits a bipartition (V_1, V_2) where each vertex in V_2 has degree at most 3, and where V_1 is in fact the original set of vertices of H.

We shall first embed the vertices of V_1 into Y' in such a way that no embedded pair is bad and no embedded triple is bad. In order to show that this is possible, we note that the proportion of bad pairs in Y' is at most $400/C$ and the proportion of bad triples in Y' is at most $600/C^3$. We define a 3-graph $\mathcal{D}(Y')$ on the vertex set Y' whose edges are those that are potentially problematic for our embedding, i.e., those triples $y_1 y_2 y_3$ which are either bad, or for which one of the pairs $y_1 y_2$, $y_2 y_3$ or $y_1 y_3$ is bad. The density of this 3-graph $\mathcal{D}(Y')$ is at most $1200/C + 600/C^3 \leq 2000/C$.

Our goal now is to find a complete 3-graph on $|V_1|$ vertices in the complement of $\mathcal{D}(Y')$, since the existence of such a subgraph enables us to inject V_1 into Y' whilst avoiding all bad pairs and bad triples. A bound of de Caen [2] shows that a copy of the complete 3-graph K_t^3 on t vertices can be found in any 3-graph on n vertices of density at least

$$1 - \binom{t-1}{2}^{-1},$$

provided n is sufficiently large. Therefore, we may find our embedding (again, assuming that n is sufficiently large) provided that

$$2000/C \le \binom{|V_1|}{2}^{-1},$$

which we may ensure by taking $C \ge 1000v(H)^2$.

It remains to find an embedding of the vertices of V_2 into X. For each vertex u of degree 3 in V_2 that we need to embed into X, we have a choice of $n^{1-3\varepsilon}$ vertices in the common neighbourhood of its three already-embedded neighbours from V_1; we choose its image from these candidates uniformly at random. Similarly, for each vertex v of degree 2 in V_2, we choose its image uniformly at random from the $n^{1-2\varepsilon}$ vertices in the common neighbourhood of its two already-embedded neighbours from V_1.

The probability that this embedding is not proper, i.e., that some two vertices in V_2 get mapped to the same vertex in X, is at most $|V_2|^2 n^{3\varepsilon-1} < 1/2$, provided n is large (since $\varepsilon \le 1/5$ and $|V_2| \le 10v(H)^3$).

We shall next show that for this embedding, the probability of some special 4-cycle in $\mathcal{S}(H)$ mapping to an H-forbidden 4-cycle is also at most $1/2$. Since the number of special 4-cycles in $\mathcal{S}(H)$ is $3e(H)$, it suffices to show for each special 4-cycle $\mathcal{C}$ in $\mathcal{S}(H)$ that the probability of its image being H-forbidden is at most $1/(6e(H))$. Let y_1 and y_2 be the images of vertices of $\mathcal{C}$ in V_1 (which have been fixed earlier deterministically), and consider u' and v', the (random) images of the two vertices u and v of $\mathcal{C}$ from V_2 whose degrees in $\mathcal{S}(H)$ are respectively 3 and 2. Let y_3 be the vertex in Y' so that u' is chosen uniformly at random from $\Gamma(y_1, y_2, y_3)$. Suppose for a contradiction that the probability of the image of $\mathcal{C}$ being H-forbidden is at least $1/(6e(H))$. Then this implies that at least a $1/(6e(H))$ proportion of the 4-cycles formed by taking y_1 and y_2, together with a vertex $x_1 \in \Gamma(y_1, y_2, y_3)$ and a vertex $x_2 \in \Gamma(y_1, y_2)$ are H-forbidden. This leads us to conclude that the number of H-forbidden 4-cycles through y_1 and y_2 is at least

$$(1/6e(H))n^{1-3\varepsilon}|\Gamma(y_1, y_2)|.$$

However, if $1/(6e(H)) \ge K_H/C$, then this would imply that $y_1 y_2$ is a bad pair, a contradiction; this final inequality may be ensured by taking $C \ge 18v(H)^6$, since $e(H) \le v(H)^3$ and $K_H = 3v(H)^3$.

We have shown that it is possible to embedded $\mathcal{S}(H)$ into $\mathcal{L}_z$ in such a way that all of the special 4-cycles in this embedding are H-admissible. This embedding extends to a homeomorph of H inside G as follows. For each 4-cycle $\mathcal{C}$ in $\mathcal{L}_z$ that is the image of some special 4-cycle of $\mathcal{S}(H)$, we claim that we may choose a unique vertex $z(\mathcal{C}) \in Z$ such that $\mathcal{C}$ is also contained in the link graph $\mathcal{L}_{z(\mathcal{C})}$: indeed, $\mathcal{C}$ is H-admissible, so there are at least $K_H = 3v(H)^3 \geq 3e(H)$ choices for $z(\mathcal{C})$. We then use $z(\mathcal{C})$ to turn each of the embedded special 4-cycles $\mathcal{C}$ in $\mathcal{L}_z$ into a 4-disk in G, noting that these 4-disks all have distinct centres; the result is a homeomorph of H in G. $\qquad\square$

4. Conclusion

Below, we address some of the limitations of our approach to finding homeomorphs of a fixed target 3-graph H, as well as some potential avenues for improvement.

It seems plausible that the exponent of $3 - 1/5$ that we obtain may be improved somewhat by a more judicious application of the methods developed here. However, the ideas developed in this paper reach a bottleneck, conjecturally, at the exponent of $3 - 1/4$. This is because it is believed [16] that there exist n-vertex 3-graphs with $\Omega(n^{3-1/4})$ edges that do not contain any octahedra, though the best constructions presently known, see [6], only manage $\Omega(n^{3-1/3})$ edges. If a 3-graph does not contain any octahedra, then our approach based on H-admissible 4-cycles falls apart, since if all the 4-cycles in the link graphs are H-forbidden, then our method for extending $\mathcal{S}(H)$ to a homeomorph of H fails due to degeneracy concerns.

Another important fact to bear in mind is that while $\Omega(n^{5/2})$ edges guarantee a homeomorph of S^2 in any n-vertex 3-graph, the number of edges needed to guarantee a homeomorph of S^2 *of bounded size* comes with an exponent strictly greater than $5/2$, as can be verified by a standard deletion argument applied to a (binomial) random 3-graph of the appropriate density. Our methods here end up finding bounded-size homeomorphs: indeed, we find a homeomorphic copy of H that has $12e(H)$ edges. Any strategy that does not plan for the possibility of finding large homeomorphs of H, i.e., of size unbounded in terms of H, cannot prove Conjecture 1.2.

We leave the reader with a reminder of the specialisation of Conjecture 1.2 to the torus as reiterated by Linial [8, 9].

Conjecture 4.1. *There is a $C > 0$ such that any 3-graph G on n vertices with at least $Cn^{5/2}$ edges contains a homeomorph of the torus.*

By adapting the arguments of Brown, Erdős, and Sós [1] to triple-pyramidal complexes (from double-pyramidal complexes), it may be shown that an exponent of $3-1/3$ suffices for the torus, but improving on this bound remains an attractive starting point to Conjecture 1.2 in its full generality.

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