Slowdown for the geodesic-biased random walk

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Abstract. Given a connected graph G with some subset of its vertices excited and a fixed target vertex, in the *geodesic-biased random walk* on G, a random walker moves as follows: from an unexcited vertex, she moves to a uniformly random neighbour, whereas from an excited vertex, she takes one step along some fixed shortest path towards the target vertex. We show, perhaps counterintuitively, that the geodesic-bias can slow the random walker down exponentially: there exist connected, bounded-degree n-vertex graphs with excitations where the expected hitting time of a fixed target is at least $\exp(\sqrt[4]{n}/100)$.

1. Introduction

In this paper, we investigate a model of excited random walk on a connected graph, namely *geodesic-biased random walk*, where the excitations are designed to decrease the hitting time of a fixed target vertex. The model originates in the theoretical computer science and computational biology communities [8, 7, 5], and was brought to our attention by Sousi [17]. By way of context, let us mention that various matters relating to hitting times — recurrence and return times [4, 18, 2, 3], speed [15, 9, 16] and slowdown [13, 14] — have been investigated in a number of different models of excited random walk; for a broad overview, see [12, 10].

Geodesic-biased random walk is defined on a connected n-vertex graph G. Having fixed a starting vertex $a \in V(G)$, a target vertex $b \in V(G)$ and a subset $\mathcal{X} \subset V(G)$ of excited vertices, a random walker walks from a until she hits b as follows: from an unexcited vertex of G, she moves to a uniformly random neighbour, whereas from an excited vertex, she takes one step along some predetermined shortest path to the target vertex b. Our focus here is the hitting time $\tau_a(b, \mathcal{X})$ i.e., the first time at which the walker hits b starting from a when the set of excited vertices is $\mathcal{X}$.

When every vertex is excited, i.e., $\mathcal{X} = V(G)$, the geodesic-biased walk reduces to a deterministic walk along a shortest path to the target vertex, in which case we have $\mathbb{E}[\tau_a(b, V(G))] = O(n)$. On the other hand, when no vertices are excited, i.e., $\mathcal{X} = \varnothing$, the geodesic-biased walk reduces to the simple random walk on G, and an old result of Lawler [11] gives a uniform polynomial bound (see also [1, 6]) for the expected hitting

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time of $\mathbb{E}[\tau_a(b, \varnothing)] = O(n^3)$. Many of the existing results in the literature [8, 7, 5] show that the expected hitting time of a fixed target in the geodesic-biased walk, for various graphs G and random choices of the set $\mathcal{X}$ of excited vertices, is significantly smaller than Lawler's uniform bound. Motivated by this, we shall investigate how much the geodesic-bias can decrease the hitting time of a fixed target.

While the geodesic-bias ostensibly aims to decrease hitting times, it is actually not hard to construct examples where the expected hitting time of a fixed target in the geodesic-biased walk is *slightly larger* than the expected hitting time in the analogous simple random walk. To wit, consider a graph where two vertices a and b are connected by two paths of lengths 2 and 3, with the middle vertex of the shorter path being attached to a 'trap', say a large clique; here, it is not hard to see that exciting a increases the expected hitting time of b, since the random walker ends up spending more time in the 'trap'. However, the digraph formed by taking a shortest path from each vertex to a fixed target is acyclic, so one cannot string together multiple such 'traps' in a cyclic fashion; in particular, such constructions cannot hope to slow the geodesic-biased walk down by more than a constant factor in comparison to the simple random walk.

In the light of the above discussion, it is natural to ask if the results in [8, 7, 5] are indicative of a broader phenomenon, and if there is a uniform polynomial bound for the expected hitting time of a fixed target in the geodesic-biased walk, much like Lawler's bound [11] for the simple random walk. Our first result shows, perhaps surprisingly, that this is not the case: even a single excitation can cause an exponential slowdown.

Theorem 1.1. *For infinitely many $n \in \mathbb{N}$, there exists a connected graph G on n vertices with $a, b \in V(G)$ such that*

$$\mathbb{E}[\tau_a(b, \{a\})] = \Omega\left(\exp\left(\frac{\sqrt[4]{n}\log n}{100}\right)\right).$$

The construction proving Theorem 1.1 produces graphs of unbounded degree. In the context of the simple random walk, bounded-degree graphs are known to behave somewhat differently from those of unbounded degree; for example, as shown by Lawler [11], expected hitting times in a bounded-degree n-vertex graph are $O(n^2)$. Our second result, also in the spirit of Theorem 1.1, shows that exponential slowdown is unavoidable on graphs of bounded degree as well, though more excitations are required in this case.

Theorem 1.2. *For infinitely many $n \in \mathbb{N}$, there exists a connected graph G on n vertices of maximum degree 3 with $a, b \in V(G)$ and a set $\mathcal{X} \subset V(G)$ of $O(\sqrt{n})$ excited vertices such that*

$$\mathbb{E}[\tau_a(b, \mathcal{X})] = \Omega\left(\exp\left(\frac{\sqrt[4]{n}}{100}\right)\right).$$

This paper is organised as follows. We give the proofs of Theorems 1.1 and 1.2 in Section 2. We conclude with a discussion of some open problems in Section 3.

2. Proofs of the main results

In this section, we prove our two main results. It will be helpful to have some notation. As is usual, we write $[n]$ for the set $\{1, 2, \ldots, n\}$. In the geodesic-biased random walk on a graph G, when the target vertex b and set $\mathcal{X}$ of excited vertices are clear from the context, we abbreviate the expected hitting time $\tau_x(y, \mathcal{X})$ of y from x by $T(x, y)$.

We shall make use of a well-known Chernoff-type bound.

Proposition 2.1. *Let $X = X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \ldots, X_n$ are independent Bernoulli random variables. Writing $\mu = \mathbb{E}[X]$, we have*

$$\mathbb{P}(X \geq (1+\delta)\mu) \leq \exp\left(\frac{-\delta^2 \mu}{2+\delta}\right)$$

for all $\delta > 0$. □

We also require the following well-known gambler's ruin estimate.

Proposition 2.2. *The probability that the simple random walk on $\{0, 1, \ldots, n\}$ started at 1 visits n before it visits 0 is $1/n$.* □

We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. We build an infinite family of graphs as follows. We fix $k \in \mathbb{N}$, set $m = \lfloor \sqrt{k} \rfloor$, and consider a graph G as follows: we start with a path of length $m+1$ between a and b, say $a, v_1, v_2 \ldots, v_m, b$, and then connect each v_i to a by k disjoint paths of length $i+1$ as shown in Figure 1. Formally, we take

$$V(G) = \{a, b\} \cup \{v_1, v_2, \ldots, v_m\} \cup \bigcup_{j=1}^{m} \bigcup_{i=1}^{j} R_{i,j},$$

where $R_{i,j} = \{r_{i,j,l} : l \in [k]\}$, and specify $E(G)$ as follows:

- $\forall i \in [m-1] : \{v_i, v_{i+1}\} \in E(G)$,
- $\forall j \in [m], \forall i \in [j-1], \forall l \in [k] : \{r_{i,j,l}, r_{i,j+1,l}\} \in E(G) \wedge \{r_{i,1,l}, a\} \in E(G) \wedge \{r_{i,k,l}, v_i\} \in E(G)$,
- $\{a, v_1\} \in E(G)$ and $\{v_m, b\} \in E(G)$.

We consider the geodesic-biased random walk on this graph with target b and $\mathcal{X} = \{a\}$. The unique shortest path to b from a is the path $a, v_1, v_2 \ldots, v_m, b$, so the random walker always moves to v_1 from a.

Lemma 2.3. *For $1 \leq j \leq m+1$, we have $T(a, v_j) \geq \frac{k^{j-1}}{4^{j-1} \cdot (j-1)!}$.*

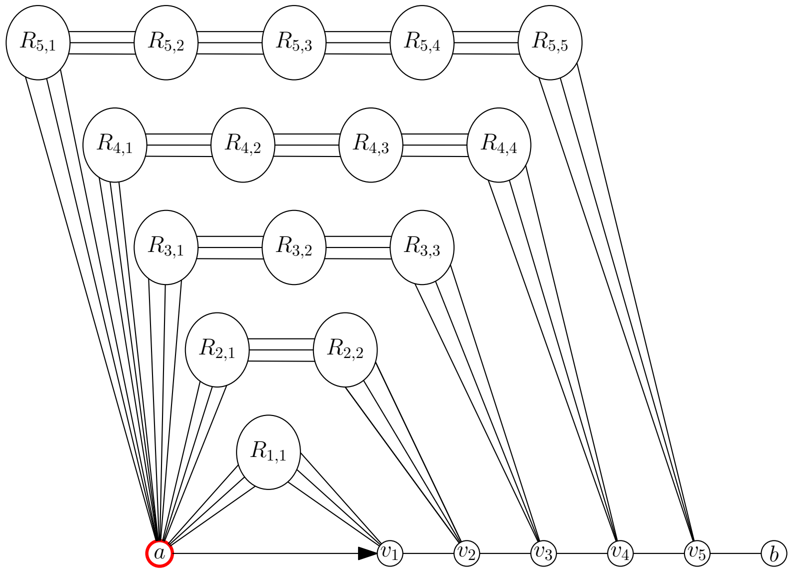

FIGURE 1. The construction with $m = 5$.

Proof. We will prove this lemma by induction. For $j = 1$, we have $T(a, v_1) = 1$ and the bound clearly holds. Now, assume the lemma holds for j and note that $T(a, v_{j+1}) = T(a, v_j) + T(v_j, v_{j+1})$, as we can only reach v_{j+1} from v_j. We may then bound $T(v_j, v_{j+1})$ by

$$\begin{aligned} T(v_j, v_{j+1}) &= 1 + \frac{1}{k+2}T(v_{j-1}, v_{j+1}) + \frac{1}{k+2}T(v_{j+1}, v_{j+1}) + \frac{k}{k+2}T(R_{j,j}, v_{j+1}) \\ &\geq \frac{k}{k+2}T(R_{j,j}, v_{j+1}) \end{aligned}$$

From Proposition 2.2, it follows that the probability of walking from $R_{j,j}$ to v_j before a is $j/(j+1)$, and the complementary event has the probability $1/(j+1)$. We then see that

$$T(R_{j,j}, v_{j+1}) \geq \frac{1}{j+1}T(a, v_{j+1}) + \frac{j}{j+1}T(v_j, v_{j+1}).$$

Using this bound, we obtain

$$\begin{aligned} T(v_j, v_{j+1}) &\geq \frac{k}{k+2}\frac{1}{j+1}T(a, v_{j+1}) + \frac{k}{k+2}\frac{j}{j+1}T(v_j, v_{j+1}), \text{ so} \\ \frac{k+2j+2}{(k+2)(j+1)}T(v_j, v_{j+1}) &\geq \frac{k}{(k+2)(j+1)}T(a, v_{j+1}), \text{ whence} \\ T(v_j, v_{j+1}) &\geq \frac{k}{k+2j+2}T(a, v_{j+1}) \end{aligned}$$

Combining the above bound with the bound on $T(a, v_{j+1})$, we get

$$\begin{aligned}
T(a, v_{j+1}) &\geq T(a, v_j) + \frac{k}{k+2j+2} T(a, v_{j+1}), \text{ so} \\
\frac{2j+2}{k+2j+2} T(a, v_{j+1}) &\geq T(a, v_j), \text{ whence} \\
T(a, v_{j+1}) &\geq \frac{k+2j+2}{2j+2} T(a, v_j) \geq \frac{k}{4j} T(a, v_j)
\end{aligned}$$

By the induction hypothesis, we now conclude that

$$T(a, v_{j+1}) \geq \frac{k}{4j} \frac{k^{j-1}}{4^{j-1} \cdot (j-1)!} = \frac{k^j}{4^j \cdot j!};$$

the result follows. □

From Lemma 2.3, we conclude that $T(a, b) \geq k^m / 4^m m!$; since $m = \lfloor \sqrt{k} \rfloor$, standard bounds for the factorial show that

$$T(a, b) \geq \frac{1}{4} \left(\frac{\sqrt{k}}{4} \right)^{\sqrt{k}-1}$$

and since $n = |V(G)| = \Theta(m^2 k) = \Theta(k^2)$, we deduce that

$$T(a, b) = \Omega\left(\exp\left(\frac{\sqrt[4]{n} \log n}{100} \right) \right),$$

proving the result. □

Next, we present the (slightly more involved) proof of Theorem 1.2.

Proof of Theorem 1.2. To prove the result, we build an infinite family of graphs as follows. We fix $m \in \mathbb{N}$, and consider a graph G constructed as follows: as before, we start with a path of length $m+1$ between a and b, say $a, v_1, v_2 \ldots, v_m, b$, and then attach a path of length $2m+2$ to each v_i, and finally chain the ends of these paths to a by another path as shown in Figure 2. Formally, we set

$$V(G) = \{a, b\} \cup \{v_1, v_2, \ldots, v_m\} \cup \{s_1, s_2, \ldots, s_m\} \cup \bigcup_{j=1}^{2m+1} \bigcup_{i=1}^{m} \{r_{i,j}\}$$

and specify $E(G)$ as follows:

- $\forall i \in [m-1] : \{v_i, v_{i+1}\} \in E(G) \wedge \{s_i, s_{i+1}\} \in E(G)$,
- $\forall j \in [2m], \forall i \in [m] : \{r_{i,j}, r_{i,j+1}\} \in E(G) \wedge \{r_{i,1}, s_i\} \in E(G) \wedge \{r_{i,2m+1}, v_i\} \in E(G)$,
- $\{a, v_1\} \in E(G)$, $\{v_m, b\} \in E(G)$, and $\{a, s_1\} \in E(G)$.

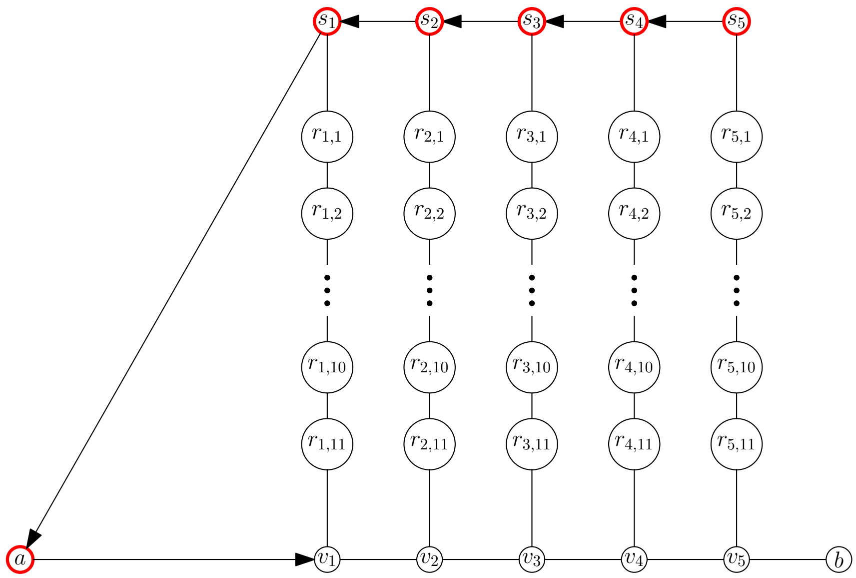

FIGURE 2. The bounded-degree construction with $m = 5$.

We consider the geodesic-biased random walk on this graph with target b and $\mathcal{X} = \{a, s_1, s_2, \ldots, s_m\}$. Notice that our choice of path lengths ensures that the random walker moves deterministically from s_i to s_{i-1} (or to a in the case of s_1), and from a to v_1.

Lemma 2.4. *We have* $T(v_1, b) \geq \exp(\sqrt{m}/10)/(m^{3/2} + 1)$.

Proof. We proceed via a renewal argument. Observe that $T(v_1, b) \geq 1 + q \cdot T(v_1, b)$, where q is the probability of the event that the random walker visits a before b after leaving v_1. It will be more convenient to work with the complementary event, namely, that the random walker visits b before a after leaving v_1; we write $p = 1 - q$ for the probability of this event. From the previous inequality, we then have $T(v_1, b) \geq 1/(1 - q) = 1/p$.

Now, we shall estimate p, the probability that the geodesic-biased walk starting at v_1 hits b before a. To do so, we consider the Markov chain $(x_t)_{t \geq 0}$ induced by the geodesic-biased walk on the states $a, v_1, \ldots, v_m, b$ with a and b being absorbing; of course, p is exactly the probability that this induced chain started at v_1 reaches the absorbing state b before it hits the absorbing state a.

For each non-absorbing state v_i, there are three possibilities for the next state of the induced chain hit by the random-walker: v_{i-1}, v_{i+1} or a. The probabilities of these transitions are as follows: we write ε for the probability of returning to a via s_i, and note that the other two transitions have the same probability, i.e.,

$$\mathbb{P}[x_{t+1} = v_{i+1} \,|\, x_t = v_i] = \mathbb{P}[x_{t+1} = v_{i-1} \,|\, x_t = v_i] = \frac{1 - \varepsilon}{2}.$$

We may calculate ε, the probability of *retracing*, i.e., returning to a via s_i, as follows. The probability of reaching s_i before v_i starting from $r_{i,2k+1}$ is, by Proposition 2.2, exactly $1/(2m+2)$. It then follows that $\varepsilon = \frac{1}{3}(\frac{2m+1}{2m+2}\varepsilon + \frac{1}{2m+2})$, from which we get $\varepsilon = 1/(4m+5)$.

We shall estimate $p = p_s + p_l$ by separately estimating p_s, the probability of the chain hitting b before a starting from v_1 in at most $m^{3/2}$ steps, and p_l, the probability of the chain hitting b before a starting from v_1 and taking more than $m^{3/2}$ steps to do so.

First, we dispose of 'long' excursions. We claim that $p_l \le (1-\varepsilon)^{m^{3/2}}$; indeed, if the chain does not hit either of a or b in the first $m^{3/2}$ steps, then the chain does not, in particular, retrace on any of the first $m^{3/2}$ steps. Thus

$$p_l \le (1-\varepsilon)^{m^{3/2}} \le \left(1 - \frac{1}{4m+5}\right)^{m^{3/2}} \le \exp\left(\frac{-\sqrt{m}}{10}\right).$$

Next, we focus on the 'short' excursions. Note that we may write $p_s = \sum_{t=0}^{m^{3/2}} p(t)$, where

$$p(t) = \mathbb{P}[\{x_t = b\} \wedge \{\forall\, 1 \le i < t : (x_i \ne a \wedge x_i \ne b)\}].$$

We may then bound $p(t)$ by conditioning on the chain never retracing to get

$$p(t) \le \mathbb{P}[\{x_t = b\} \wedge \{\forall\, 1 \le i < t : (x_i \ne a \wedge x_i \ne b)\} \,|\, \text{No Retrace}].$$

This upper bound may be interpreted in terms of the simple random walk on the integers; indeed, conditional on never retracing, the chain is isomorphic to the simple random walk on the integer line. Concretely, consider the simple random walk $\{y_t\}_{t \ge = 0}$ on the integers and note that

$$\begin{aligned}
&\mathbb{P}[\{x_t = b\} \wedge \{\forall\, 1 \le i < t : (x_i \ne a \wedge x_i \ne b)\} \,|\, \text{No Retrace}] \\
&\qquad\qquad = \mathbb{P}[\{y_0 = 1 \wedge y_t = m+1\} \wedge \{\forall 1 \le i < t : (y_i \ne 0 \wedge y_i \ne m+1)\}] \\
&\qquad\qquad \le \mathbb{P}[\{y_0 = 1 \wedge y_t = m+1\}] \le \mathbb{P}[\{y_0 = 1 \wedge y_t \ge m+1\}].
\end{aligned}$$

The last probability above is easy to estimate since the simple random walk on the integers may be viewed as a sum of independent Bernoulli random variables, so by applying Proposition 2.1 (with $\delta = m/t$) to such a representation of the random walk on the integers, we obtain

$$\mathbb{P}[\{y_0 = 1 \wedge y_t \ge m+1\}] \le \exp\left(\frac{-m^2}{4t+2m}\right) \le \exp\left(\frac{-\sqrt{m}}{10}\right),$$

where the second inequality holds for all $t \le m^{3/2}$. Consequently, we have

$$p_s \le m^{3/2} \exp\left(\frac{-\sqrt{m}}{10}\right).$$

Combining the above estimates for p_s and p_l and the fact that $T(v_1, b) \ge 1/(p_s + p_l)$ now yields the required bound. $\square$

The theorem immediately follows from the above lemma. Indeed, $T(a,b) = 1+T(v_1,b)$, and writing the above bound for $T(v_1,b)$ in terms of $n = |V(G)| = 2 + m(2m+3)$ proves the result. □

3. Conclusion

Our results raise a few different natural questions; we discuss two such problems below.

There remains the question of determining the right order of uniform bound for the expected hitting time of a fixed target in the geodesic-biased walk: we have shown that on a connected n-vertex graph, this may be as large as $\exp(n^{1/4}\log n/100)$, while it is more or less trivial to show a uniform upper bound of $\exp(n \log n)$; it would be interesting to close this gap and pin down the truth.

Another problem that we have been unable to resolve concerns bounded-degree graphs. While we have exhibited exponential slowdown for the geodesic-biased walk on bounded-degree graphs, our constructions nonetheless require an unbounded number of excitations, which leads to the following: in the geodesic-biased walk on a bounded-degree graph with a bounded number of excitations, is there a uniform polynomial bound on the expected hitting time of the fixed target?

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