An improved lower bound for Folkman's theorem

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Abstract. Folkman's theorem asserts that for each $k \in \mathbb{N}$, there exists a natural number $n = F(k)$ such that whenever the elements of $[n]$ are two-coloured, there exists a set $A \subset [n]$ of size k with the property that all the sums of the form $\sum_{x \in B} x$, where B is a nonempty subset of A, are contained in $[n]$ and have the same colour. In 1989, Erdős and Spencer showed that $F(k) \geq 2^{ck^2/\log k}$, where $c > 0$ is an absolute constant; here, we improve this bound significantly by showing that $F(k) \geq 2^{2^{k-1}/k}$ for all $k \in \mathbb{N}$.

1. Introduction

Schur's theorem, proved in 1916, is one of the central results of Ramsey theory and asserts that whenever the elements of $\mathbb{N}$ are finitely coloured, there exists a monochromatic set of the form $\{x, y, x + y\}$ for some $x, y \in \mathbb{N}$. About fifty years ago, a wide generalisation of Schur's theorem was obtained independently by Folkman, Rado and Sanders, and this generalisation is now commonly referred to as Folkman's theorem (see [2], for example). To state Folkman's theorem, it will be convenient to have some notation. For $n \in \mathbb{N}$, we write $[n]$ for the set $\{1, 2, \ldots, n\}$, and for a finite set $A \subset \mathbb{N}$, let

$$S(A) = \left\{ \sum_{x \in B} x : B \subset A \text{ and } B \neq \varnothing \right\}$$

denote the set of all *finite sums* of A. In this language, Folkman's theorem states that for all $k, r \in \mathbb{N}$, there exists a natural number $n = F(k, r)$ such that whenever the elements of $[n]$ are r-coloured, there exists a set $A \subset [n]$ of size k such that $S(A)$ is a monochromatic subset of $[n]$; of course, it is easy to see that Folkman's theorem, in the case where $k = 2$, implies Schur's theorem.

In this note, we shall be concerned with lower bounds for the two-colour Folkman numbers, i.e., for the quantity $F(k) = F(k, 2)$. In 1989, Erdős and Spencer [1] proved that

$$F(k) \geq 2^{ck^2/\log k} \tag{1}$$

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for all $k \in \mathbb{N}$, where $c > 0$ is an absolute constant; here, and in what follows, all logarithms are to the base 2. Our primary aim in this note is to improve (1).

Before we state and prove our main result, let us say a few words about the proof of (1). Erdős and Spencer establish (1) by considering uniformly random two-colourings. In particular, they show that if $[n]$ is two-coloured uniformly at random and additionally $n \le 2^{ck^2/\log k}$ for some suitably small absolute constant $c > 0$, then with high probability, there is no k-set $A \subset [n]$ for which $S(A)$ is monochromatic. On the other hand, it is not hard to check that if $n \ge 2^{Ck^2}$ for some suitably large absolute constant $C > 0$, then a two-colouring of $[n]$ chosen uniformly at random is such that, with high probability, there exists a set $A \subset [n]$ of size k for which $S(A)$ is monochromatic; indeed, to see this, it is sufficient to restrict our attention to sets of the form $\{p, 2p, \dots, kp\}$, where p is a prime in the interval $[n/\log^2 n, 2n/\log^2 n]$, and notice that the sets of finite sums of such sets all have size $k(k+1)/2$ and are pairwise disjoint. With perhaps this fact in mind, in their paper, Erdős and Spencer also describe some of their attempts at removing the factor of $\log k$ in the exponent in (1); nevertheless, their bound has not been improved upon since.

Our main contribution is a new, doubly exponential, lower bound for $F(k)$, significantly strengthening the bound due to Erdős and Spencer.

Theorem 1.1. *For all $k \in \mathbb{N}$, we have*

$$F(k) \ge 2^{2^{k-1}/k}. \tag{2}$$

This short note is organised as follows. We give the proof of Theorem 1.1 in Section 2 and conclude with some remarks in Section 3.

2. Proof of the main result

Proof of Theorem 1.1. The result is easily verified when $k \le 3$, so suppose that $k \ge 4$ and let $n = \lfloor 2^{2^{k-1}/k} \rfloor$. In the light of our earlier remarks, a uniformly random colouring of $[n]$ is a poor candidate for establishing (2). Instead, we generate a (random) red-blue colouring of $[n]$ as follows: we first red-blue colour the odd elements of $[n]$ uniformly at random, and then extend this colouring uniquely to all of $[n]$ by insisting that the colour of $2x$ be different from the colour of x for each $x \in [n]$; hence, for example, if 5 is initially coloured blue, then 10 gets coloured red, 20 gets coloured blue, and so on.

Fix a set $A \subset [n]$ of size k with $S(A) \subset [n]$. We have the following estimate for the probability that $S(A)$ is monochromatic in our colouring.

Claim 2.1. $\mathbb{P}(S(A)$ *is monochromatic*$) \le 2^{1-2^{k-1}}$.

Proof. First, if $|S(A)| \le 2^k - 2$, then it is easy to see from the pigeonhole principle that there exist two subsets $B_1, B_2 \subset A$ such that $\sum_{x \in B_1} x = \sum_{x \in B_2} x$, and by removing

$B_1 \cap B_2$ from both B_1 and B_2 if necessary, these sets may further be assumed to be disjoint; in particular, this implies that $S(A)$ contains two elements one of which is twice the other. It therefore follows from the definition of our colouring that $S(A)$ cannot be monochromatic unless $|S(A)| = 2^k - 1$. Next, suppose that $|S(A)| = 2^k - 1$. For each odd integer $m \in \mathbb{N}$, we define $G_m = \{m, 2m, 4m, \dots\} \cap [n]$, and note that these geometric progressions partition $[n]$. Observe that $S(A)$ intersects at least 2^{k-1} of these progressions. Indeed, if there is an odd integer $r \in A$ for example, then $S(A)$ contains exactly 2^{k-1} distinct odd elements and these elements must lie in different progressions. More generally, if each element of A is divisible by 2^s and s is maximal, then there exists an element r of A with $r = 2^s t$, where t is odd; it is then clear that precisely 2^{k-1} elements of $S(A)$ are divisible by 2^s but not by 2^{s+1} and these elements must necessarily lie in different progressions. With this in mind, we define B_A to be a maximal subset of $S(A)$ with the property $|B_A \cap G_m| \leq 1$ for each m; for example, we may take B_A to consist of the least elements (where they exist) of the sets $S(A) \cap G_m$. Clearly, our red-blue colouring restricted to B_A is a uniformly random colouring, so the probability that B_A is monochromatic is $2^{1-|B_A|}$; it follows that the probability that $S(A)$ is monochromatic is at most $2^{1-|B_A|} \leq 2^{1-2^{k-1}}$. □

It is now easy to see that if X is the number of sets $A \subset [n]$ of size k for which $S(A)$ is a monochromatic subset of $[n]$ in our colouring, then

$$\mathbb{E}[X] \leq \binom{n}{k} 2^{1-2^{k-1}} \leq \left(\frac{en}{k}\right)^k 2^{1-2^{k-1}} \leq \left(\frac{e 2^{2^{k-1}/k}}{k}\right)^k \left(2^{1-2^{k-1}}\right) = 2\left(\frac{e}{k}\right)^k < 1,$$

where the last inequality holds for all $k \geq 4$. Hence, there exists a red-blue colouring of $[n]$ without any set A of size k for which $S(A)$ is a monochromatic subset of $[n]$, proving the result. □

3. Conclusion

We conclude this note with two remarks. First, using the original arguments of Erdős and Spencer [1] in conjunction with an inverse Littlewood–Offord theorem of Nguyen and Vu [3], it is possible to improve (1) (up to removing the factor of $\log k$ in the exponent) by just considering uniformly random two-colourings. Second, we note that while (2) improves significantly on (1), this lower bound is still considerably far from the best upper bound for $F(k)$, which is of tower type; see [4], for instance.

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References

1. P. Erdős and J. Spencer, *Monochromatic sumsets*, J. Combin. Theory Ser. A **50** (1989), 162–163. 1, 3
2. R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey theory*, 2nd ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., New York, 1990. 1
3. H. Nguyen and V. Vu, *Optimal inverse Littlewood-Offord theorems*, Adv. Math. **226** (2011), 5298–5319. 3
4. A. D. Taylor, *Bounds for the disjoint unions theorem*, J. Combin. Theory Ser. A **30** (1981), 339–344. 3

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