A counterexample to directed-KKL

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Abstract. We show that the natural directed analogues of the Kahn–Kalai–Linial theorem and the Eldan–Gross inequality from the analysis of Boolean functions fail to hold. This is in contrast to several other isoperimetric inequalities on the Boolean hypercube (such as the Poincaré inequality, Margulis's inequality and Talagrand's inequality) for which directed strengthenings have recently been established.

In this note, we consider isoperimetric inequalities over the Boolean hypercube $\{0,1\}^n$. Our notation and terminology follow O'Donnell [11]; in particular, we refer the reader to introductory chapters of [11] for further background.

Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$ and an input $x \in \{0,1\}^n$, we define the *sensitivity of f at x* as

$$\mathrm{sens}_f(x) = \left|\left\{i : f(x) \neq f(x^{\oplus i})\right\}\right|,$$

where $x^{\oplus i} = (x_1, \dots, 1 - x_i, \dots, x_n)$. Two closely related isoperimetric quantities are the *influence of a variable $i \in [n]$ on f*, given by

$$\mathrm{Inf}_i[f] = \mathop{\mathbb{P}}_{x\sim\{0,1\}^n}\left[f(x) \neq f(x^{\oplus i})\right],$$

and the *total influence of f*, given by

$$\mathbf{I}[f] = \sum_{i=1}^{n} \mathrm{Inf}_i[f].$$

It is easy to check that $\mathbf{I}[f] = \mathbb{E}[\mathrm{sens}_f(x)]$, and so the total influence of a function is sometimes also referred to as its *average sensitivity*.

To set the stage, we recall perhaps the simplest isoperimetric inequality on the Boolean hypercube, the Poincaré inequality, which says that

$$\mathbf{I}[f] \geq \mathrm{Var}[f].$$

The follow strengthening of the Poincaré inequality was obtained by Talagrand [14], which is known to imply yet another isoperimetric inequality due to Margulis [9].

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Theorem 1. *Given a Boolean function* $f : \{0,1\}^n \to \{0,1\}$, *we have*

$$\mathop{\mathbb{E}}_{x\sim\{0,1\}^n}\left[\sqrt{\mathrm{sens}_f(x)}\right] = \Omega(\mathrm{Var}[f]).$$

An alternative (and incomparable) strengthening of the Poincaré inequality is given by the celebrated Kahn–Kalai–Linial theorem [6].

Theorem 2. *Given a Boolean function* $f : \{0,1\}^n \to \{0,1\}$, *there exists* $i \in [n]$ *such that*

$$\mathrm{Inf}_i[f] = \Omega\left(\mathrm{Var}[f] \cdot \frac{\log n}{n}\right).$$

Talagrand [15] conjectured the following common generalization of Theorems 1 and 2, which was proved by Eldan and Gross [4].

Theorem 3. *Given a Boolean function* $f : \{0,1\}^n \to \{0,1\}$, *we have*

$$\mathop{\mathbb{E}}_{x\sim\{0,1\}^n}\left[\sqrt{\mathrm{sens}_f(x)}\right] = \Omega\left(\mathrm{Var}[f]\sqrt{\log\left(2 + \frac{e}{\sum_{i=1}^n \mathrm{Inf}_i[f]^2}\right)}\right).$$

Here, we will be concerned with directed versions of such results in the Boolean hypercube. Recall that a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is said to be *monotone* (respectively *anti-monotone*) if for all $x, y \in \{0,1\}^n$, $x \preccurlyeq y$ implies $f(x) \le f(y)$ (respectively $f(x) \ge f(y)$), where we write $x \preccurlyeq y$ to mean $x_i \le y_i$ for all $i \in [n]$. In connection with the problem of monotonicity testing, Khot, Minzer, and Safra [7] obtained a 'directed' analogue of Theorem 1. Writing

$$\mathrm{sens}_f^-(x) = \left|\left\{i : f(x) > f(x^{\oplus i}) \text{ and } x \preccurlyeq x^{\oplus i}\right\}\right|$$

for the *negative sensitivity of* f *at* x, and

$$\varepsilon(f) = \min_{g \text{ monotone}} \mathrm{dist}(f, g)$$

for the *distance to monotonicity of* f, where $\mathrm{dist}(f,g) = \mathbb{P}_{x\sim\{0,1\}^n}[f(x) \ne g(x)]$, the following result, a slight sharpening of the result of Khot, Minzer, and Safra [7], was established by Pallavoor, Raskhodnikova, and Waingarten [12].

Theorem 4. *Given a Boolean function* $f : \{0,1\}^n \to \{0,1\}$, *we have*

$$\mathop{\mathbb{E}}_{x\sim\{0,1\}^n}\left[\sqrt{\mathrm{sens}_f^-(x)}\right] = \Omega(\varepsilon(f)).$$

Indeed, prior results on monotonicity testing due to Goldreich, Goldwasser, Lehman, Ron and Samordinsky [5] and Chakrabarty and Seshahdri [3] can be viewed as directed analogues of the Poincaré inequality and Margulis's inequality [9] respectively. Finally, a directed analogue of an inequality due to Pisier [13] was obtained by Canonne, Chen, Kamath, Levi and Waingarten [2].

Although the directed analogues are known to imply their undirected counterparts, see [7], their proofs bear little resemblance to the proofs in the undirected setting (with the exception of the directed Pisier inequality) and are usually much more involved.

These results suggest an informal analogy between the undirected and the directed cube, with isoperimetric quantities being replaced with their directed counterparts and $\mathrm{Var}[f]$ being replaced with $\varepsilon(f)$ in the latter. Writing

$$\mathrm{Inf}_i^-[f] = \left|\left\{x : f(x) > f(x^{\oplus i}) \text{ and } x \preccurlyeq x^{\oplus i}\right\}\right| \cdot \frac{1}{2^{n-1}}$$

for the *negative influence of* i *on* f, the following directed variant of Theorem 2 would yield a natural directed analogue of the KKL inequality.

Conjecture 5. *Given a Boolean function* $f : \{0,1\}^n \to \{0,1\}$, *there exists* $i \in [n]$ *such that*

$$\mathrm{Inf}_i^-[f] \geq \Omega\left(\varepsilon(f) \cdot \frac{\log n}{n}\right).$$

Conjecture 5, as well as a Fourier analytic reformulation thereof, appears to have been first raised by Khot [8]. Our aim in this short note is to show that Conjecture 5 fails to hold.

Theorem 6. *There is a function* $f : \{0,1\}^{2n} \to \{0,1\}$ *with*

(1) $\mathrm{Inf}_i^-[f] = 0$ *for all* $i \in [n]$,
(2) $\mathrm{Inf}_i^-[f] = O(1/n)$ *for all* $i \in [2n] \setminus [n]$, *and*
(3) $\varepsilon(f) = \Omega(1)$,

We note that this further rules out a natural directed analogue of Theorem 3 (which would imply Conjecture 5).

To prove Theorem 6, We view $\{0,1\}^{2n}$ as $\{0,1\}^n \times \{0,1\}^n$ and construct a function $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ with

(1) $\mathrm{Inf}_i^-[f] = 0$ for all $i \in [n]$,
(2) $\mathrm{Inf}_i^-[f] = O(1/n)$ for all $i \in [2n] \setminus [n]$, and
(3) $\varepsilon(f) = \Omega(1)$,

thereby refuting Conjecture 5.

Proof of Theorem 6. Let $T_1, \dots, T_n \in \binom{[n]}{\log n}$ be drawn independently and uniformly at random. Set

$$f(x,y) := \bigvee_{i=1}^{n} \left(\left(\bigwedge_{j \in T_i} x_j\right) \wedge (1 - y_i)\right).$$

We note that this function is closely related to the well-known 'tribes' function due to Ben-Or and Linial [1].

It is clear that f is monotone in the first n coordinates and anti-monotone in the last n coordinates; consequently $\mathrm{Inf}_i^-[f] = 0$ for all $i \in [n]$. A coordinate $i \in [2n] \setminus [n]$ is relevant only on $x \in \{0,1\}^n$ for which $\bigwedge_{j \in T_i} x_j = 1$; as $|T_i| = \log n$, this set has measure at most

$$\frac{2^{n-\log n}}{2^n} = \frac{1}{n}.$$

It follows that $\mathrm{Inf}_i^-[f] = O(1/n)$ for all $i \in [2n]$.

Before turning to the third item above, we recall the following fact from [7] without proof.

Lemma 7. *For $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ such that f is monotone in the first n coordinates and anti-monotone in the last n coordinates, we have*

$$\varepsilon(f) = \Theta\left(\mathop{\mathbb{E}}_{x \sim \{0,1\}^n} \left[\mathop{\mathrm{Var}}_{y \sim \{0,1\}^n} [f(x,y)] \right] \right). \qquad \square$$

Suppose, for convenience, that $x \in \{0,1\}^n$ is such that $\bigwedge_{j \in T_i} x_j = 1$ for exactly one $i \in [n]$. Then the restricted function $f(x, \cdot) : \{0,1\}^n \to \{0,1\}$ is simply the anti-dictatorship $(1 - y_i)$, and has $\mathrm{Var}[f(x, \cdot)] = \Omega(1)$. We will be done if we can show that this happens for $\Omega(1)$ fraction of $x \in \{0,1\}^n$. As before, for fixed $i \in [n]$ we have

$$\mathop{\mathbb{P}}_{x \sim \{0,1\}^n} \left[\bigwedge_{j \in T_i} x_j \right] = \frac{1}{n},$$

which tells us that

$$\mathop{\mathbb{E}}_{x \sim \{0,1\}^n} \left[\left| \left\{ i : \bigwedge_{j \in T_i} x_j = 1 \right\} \right| \right] = 1.$$

By Markov's inequality, we thus have

$$\mathop{\mathbb{P}}_{x \sim \{0,1\}^n} \left[\left| \left\{ i : \bigwedge_{j \in T_i} x_j = 1 \right\} \right| \geq 2 \right] \leq \frac{1}{2}.$$

We also have

$$\mathop{\mathbb{P}}_{x \sim \{0,1\}^n} \left[\left| \left\{ i : \bigwedge_{j \in T_i} x_j = 1 \right\} \right| = 0 \right] \approx \left(1 - \frac{1}{n}\right)^n \approx \frac{1}{e},$$

and so the desired event happens with constant probability, and we are done. $\square$

Note added in proof

After a draft of this paper was circulated, it was brought to our attention that Minzer and Khot [10] have independently discovered a similar construction to the one establishing our main result.

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