Connections in randomly oriented graphs

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ABSTRACT. Given an undirected graph G, let us randomly orient G by tossing independent (possibly biased) coins, one for each edge of G. Writing $a \to b$ for the event that there exists a directed path from a vertex a to a vertex b in such a random orientation, we prove that for any three vertices s, a and b of G, we have

$$\mathbb{P}(s \to a \cap s \to b) \geq \mathbb{P}(s \to a)\mathbb{P}(s \to b).$$

1. INTRODUCTION

A very natural notion of a random directed graph is that of a random orientation of a fixed undirected graph. Random orientations of graphs often exhibit counter-intuitive properties. For example, Alm and Linusson [3] showed that in a random orientation of any sufficiently large complete graph, the event that there is a directed path from a to s and the event that there is a directed path from s to b are positively correlated for any three distinct vertices s, a and b; this is surprising since conditioning on the existence of a path from a to s would intuitively suggest that edges are typically 'oriented towards s', and that it should consequently be harder to walk from s to b. Random orientations in general, and the correlations between connection events in particular, have been studied by a number of different authors; see, for instance, [9, 2, 7] and the references therein.

Given a finite undirected graph $G = (V, E)$ and a collection of probabilities $\mathbf{p} = (p_e)_{e \in E}$, we orient the edges of G independently by tossing a p_e-biased coin to decide the orientation of an edge $e \in E$. More formally, given $G = (V, E)$ and $\mathbf{p}$ as above, suppose that $V \subset \mathbb{N}$ and define $\vec{G}(\mathbf{p})$ to be a random orientation of G where an edge $e = \{a, b\} \in E$ with $a < b$ is oriented from a to b with probability p_e and from b to a otherwise, independently of the other edges. We call $\vec{G}(\mathbf{p})$ a $\mathbf{p}$-*biased orientation* of G and write $\mathbb{P}_{G,\mathbf{p}}$ for the corresponding probability measure. Note that $\vec{G}(\mathbf{p})$ is an unbiased, uniformly random orientation of G when $p_e = 1/2$ for every $e \in E$.

For a pair of vertices a and b of G, let $a \to b$ denote the *connection event* that there is a directed path from a to b in a random orientation of G. Our aim in this short paper is to establish the following correlation inequality.

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Theorem 1.1. *Let $G = (V, E)$ be an undirected graph. For any three vertices $s, a, b \in V$ and any collection of probabilities $\mathbf{p} = (p_e)_{e \in E}$, we have*

$$\mathbb{P}_{G,\mathbf{p}}(s \to a \cap s \to b) \geq \mathbb{P}_{G,\mathbf{p}}(s \to a)\mathbb{P}_{G,\mathbf{p}}(s \to b).$$

The motivation for considering biased orientations in Theorem 1.1 comes from our lack of understanding of biased orientations of a number of natural graphs, most important of which is perhaps the square lattice. For $0 \leq p \leq 1$, let $\vec{\mathbb{Z}}^2(p)$ denote a random orientation of the square lattice obtained as follows: orient a horizontal edge, independently of the other edges, rightwards with probability p and otherwise leftwards, and similarly, orient a vertical edge, independently of the other edges, upwards with probability p and otherwise downwards. The following conjecture is due to Grimmett [5] and remains wide open.

Conjecture 1.2. *For each $p \neq 1/2$, $\vec{\mathbb{Z}}^2(p)$ almost surely contains an infinite directed path.*

Let us mention that while we state and prove Theorem 1.1 for finite graphs, the result also holds for any graph on a countably infinite vertex set (such as the square lattice); indeed, this follows from a standard limiting argument. We also remark that the challenge in establishing Theorem 1.1 arises entirely from having to deal with genuinely biased orientations.

Indeed, the main difficulty in working with random orientations is that a connection event $a \to b$ is not 'up-closed' in general. In other words, it is not necessarily true that one can find a 'good' orientation for each edge with the property that the event $a \to b$ is closed under the operation of changing the orientation of an edge from 'bad' to 'good'. For example, it is clear from Figure 1 that connection events in a (large) finite grid are neither closed under the operation of changing the orientation of a horizontal edge to the left, nor closed under the operation of changing the orientation of a horizontal edge to the right.

This 'up-closedness' issue however disappears when we restrict ourselves to unbiased orientations. Indeed, in this case, as was observed by McDiarmid [9], the distribution of the set of vertices reachable from a vertex s in an unbiased orientation of G is identical to the distribution of the connected component of s in the standard percolation model (at density $1/2$) on G. Therefore, as noted by Linusson [8], our result follows instantly from Harris's lemma [6] in this case. However, we see no simple way of deducing Theorem 1.1 from Harris's lemma in general; instead, our proof relies on the powerful four-functions theorem of Ahlswede and Daykin [1].

The proof of Theorem 1.1 is given in Section 2. We make a few remarks and conclude this note in Section 3.

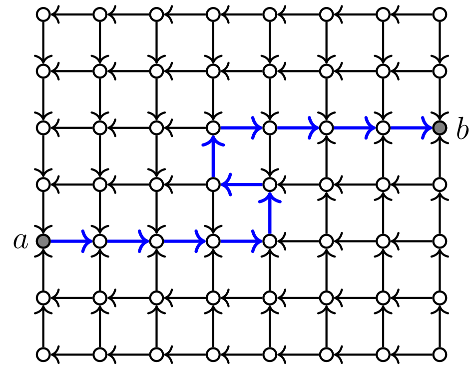

FIGURE 1. Left-to-right connection events in the grid are not 'up-closed'.

2. PROOF OF THE MAIN RESULT

To prove Theorem 1.1, we shall require the four-functions theorem of Ahlswede and Daykin [1]; see [4] for a proof and several related results.

Theorem 2.1. *Let S be a finite set and let α, β, γ and δ be functions from the set of all subsets of S to the non-negative reals. If we have*

$$\alpha(X_1)\beta(X_2) \leq \gamma(X_1 \cup X_2)\delta(X_1 \cap X_2)$$

for any two subsets $X_1, X_2 \subset S$, then

$$\sum_{X \subset S} \alpha(X) \sum_{X \subset S} \beta(X) \leq \sum_{X \subset S} \gamma(X) \sum_{X \subset S} \delta(X).$$

Before we proceed further, let us introduce some additional notation. For a set of vertices A and a vertex b, we write $A \to b$ for the union of all the events $a \to b$ with $a \in A$. Theorem 1.1 is a special case of the following result.

Theorem 2.2. *Let $G = (V, E)$ be an undirected graph. For any nonempty set $S \subset V$, any pair of vertices $a, b \in V$ and any collection of probabilities $\mathbf{p} = (p_e)_{e \in E}$, we have*

$$\mathbb{P}_{G,\mathbf{p}}(S \to a \cap S \to b) \geq \mathbb{P}_{G,\mathbf{p}}(S \to a)\mathbb{P}_{G,\mathbf{p}}(S \to b).$$

Proof. We prove the theorem by induction on the number of vertices. Clearly, the result holds trivially when G has only one vertex. Therefore, suppose that G has more than one vertex and that we have proved the result for all graphs with fewer vertices than G. The inequality is also trivial if either $a \in S$ or $b \in S$, so suppose that neither a nor b belongs to S.

Let H denote the graph obtained by deleting S from G. Let T denote the set of those vertices of H that are adjacent to some vertex of S in G. We write $O_S \subset T$ for the (random) set of those vertices $v \in T$ for which there exists an edge oriented from S to v in $\vec{G}(\mathbf{p})$.

In what follows, to reduce clutter, we write $\mathbb{P}$ for the measure $\mathbb{P}_{G,\mathbf{p}}$ and $\widehat{\mathbb{P}}$ for the measure induced by $\mathbb{P}$ on the graph H. For a subset $X \subset T$, let us define

$$\begin{aligned}
\alpha(X) &= \mathbb{P}(O_S = X)\widehat{\mathbb{P}}(X \to a),\\
\beta(X) &= \mathbb{P}(O_S = X)\widehat{\mathbb{P}}(X \to b),\\
\gamma(X) &= \mathbb{P}(O_S = X)\widehat{\mathbb{P}}(X \to a \cap X \to b), \text{ and}\\
\delta(X) &= \mathbb{P}(O_S = X).
\end{aligned}$$

Note that

$$\mathbb{P}(S \to a) = \sum_{X \subset T} \mathbb{P}(O_S = X)\mathbb{P}(S \to a \,|\, O_S = X) = \sum_{X \subset T} \mathbb{P}(O_S = X)\widehat{\mathbb{P}}(X \to a),$$

so we have

$$\begin{aligned}
\sum_{X \subset T} \alpha(X) &= \mathbb{P}(S \to a),\\
\sum_{X \subset T} \beta(X) &= \mathbb{P}(S \to b),\\
\sum_{X \subset T} \gamma(X) &= \mathbb{P}(S \to a \cap S \to b), \text{ and}\\
\sum_{X \subset T} \delta(X) &= 1.
\end{aligned}$$

Therefore, by Theorem 2.1, to prove our result, it suffices to show that

$$\alpha(X_1)\beta(X_2) \leq \gamma(X_1 \cup X_2)\delta(X_1 \cap X_2)$$

for any two subsets $X_1, X_2 \subset T$. We may inductively assume that we have established Theorem 2.2 for H. Hence, it follows that

$$\begin{aligned}
\widehat{\mathbb{P}}(((X_1 \cup X_2) \to a) \cap ((X_1 \cup X_2) \to b)) &\geq \widehat{\mathbb{P}}((X_1 \cup X_2) \to a)\widehat{\mathbb{P}}((X_1 \cup X_2) \to b)\\
&\geq \widehat{\mathbb{P}}(X_1 \to a)\widehat{\mathbb{P}}(X_2 \to b).
\end{aligned}$$

Therefore, it suffices to show that

$$\mathbb{P}(O_S = X_1)\mathbb{P}(O_S = X_2) \leq \mathbb{P}(O_S = X_1 \cup X_2)\mathbb{P}(O_S = X_1 \cap X_2).$$

This is easy to check. Indeed, each $v \in T$ belongs to O_S with some probability p_v, independently of the other vertices of T. Hence, we have

$$\begin{aligned}
\mathbb{P}(O_S = X_1)\mathbb{P}(O_S = X_2) &= \prod_{v \in X_1 \cap X_2} p_v^2 \prod_{v \in X_1 \triangle X_2} p_v(1 - p_v) \prod_{v \notin X_1 \cup X_2} (1 - p_v)^2\\
&= \mathbb{P}(O_S = X_1 \cup X_2)\mathbb{P}(O_S = X_1 \cap X_2).
\end{aligned}$$

The conditions of Theorem 2.1 have been verified; Theorem 2.2 now follows by induction. $\square$

3. Conclusion

The correlation inequality proved in this paper is 'intuitively obvious', and it therefore feels somewhat unsatisfactory that our proof must rely on the four-functions theorem. Finding a more elementary proof of our main result remains an interesting problem.

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