Selected Publications

Bhargav Narayanan

Discrete Probability and Statistical Physics

1. *Thresholds versus fractional expectation-thresholds*, **Annals of Mathematics**, 194 (2021). With K. Frankston, J. Kahn and J. Park.

   Down-set thresholds, **Random Structures & Algorithms**, to appear. With B. Gunby and X. He.

   These papers settle various questions about thresholds raised by Kahn, Kalai and Talagrand in the 2000s. In the first paper, we prove a conjecture of Talagrand from 2010 about thresholds of up-sets in product probability spaces. Concretely, for any increasing family F, we show that $p_c(F) = O(q_f(F) \log \ell(F))$, where $p_c(F)$ and $q_f(F)$ are the threshold and fractional expectation-threshold of F, and $\ell(F)$ is the maximum size of a minimal member of F. This easily implies several difficult results and conjectures in probabilistic combinatorics: thresholds for perfect hypergraph matchings (Johansson–Kahn–Vu), bounded-degree spanning trees (Montgomery), and bounded-degree spanning graphs (new), amongst others. We also prove a version of this result without a 'logarithmic correction' in product spaces where the Bernoulli measure is replaced by a smooth measure, and this allows us resolve some conjectures of Martin, Mézard and Rivoire in statistical physics from the early 2000s. In the second paper, we show that there are no analogues for down-sets of the conjectures of Kahn–Kalai and Talagrand about thresholds of up-sets, and to do so, we prove a new large-deviations principle for the number of triangles in a random graph. For more about these results and their ramifications, see this blog post by **Gil Kalai**.

2. *Friendly bisections of random graphs*, **Communications of the American Mathematical Society**, 2 (2022). With A. Ferber, M. Kwan, A. Sah and M. Sawhney.

   We prove an old conjecture of Füredi from 1988 around which there has been renewed interest owing to its inclusion in Green's list of '100 open problems' (as Problem 91). We show that the random graph $G(n, 1/2)$ asymptotically almost surely has a bisection in which a $(1 - o(1))$-fraction of the vertices have more neighbours on their own side than across. The main contribution of this paper, and likely of greater interest than the result itself, is the abstract framework we develop for establishing concentration of the trajectories of random processes.

3. *The threshold for the square of a Hamilton cycle*, **Proceedings of the American Mathematical Society**, 149 (2021). With J. Kahn and J. Park.

   Resolving a conjecture of Kühn and Osthus from 2012, we show that $p = 1/\sqrt{n}$ is the threshold for the random graph $G(n, p)$ to contain the square of a Hamilton cycle. This was a stubborn problem in probabilistic combinatorics, particularly from the point of view of technique: while higher powers of the Hamilton cycle are easily handled by a second-moment calculation, this does not work for the square, but one nonetheless expects the first-moment estimate to predict the truth; this is what we establish.

4. *Coalescence on the real line*, **Transactions of the American Mathematical Society**, 371 (2019). With P. Balister, B. Bollobás and J. Lee.

   We study a non-monotone coalescence process on the real line and resolve several questions raised by Holroyd in the late 2000s. Given probability distributions $\mathbb{P}_R$ and $\mathbb{P}_B$ on the positive reals with finite means, colour the real line alternately with red and blue intervals so that the lengths of the red intervals have distribution $\mathbb{P}_R$, the lengths of the blue intervals have distribution $\mathbb{P}_B$, and distinct intervals have independent lengths. Now iteratively update this colouring of the line by coalescing intervals: change the colour of any interval that is surrounded by longer intervals so that these three consecutive intervals subsequently form a single monochromatic interval. We say that a colour wins if every point of the line is eventually of that colour. Proving that a colour wins, even if it has a significant advantage over the other colour, does not seem to be easily accomplished. We give some general sufficient conditions to guarantee victory. We can use this to, for example, analyse the outcome of the process when all the red intervals are deterministically of length 1 initially, while the blue intervals have lengths distributed uniformly in the interval $[0, 1 + x]$; we prove that red wins when x is small, and that blue wins when x is large. We also demonstrate two results (with high confidence) which we find very surprising. First, we show that red can stochastically dominate blue and still end up loosing. Second, we show that the relationship of a colour 'beating' another colour is not transitive! The main difficulty in studying this coalescence process is the non-monotone behaviour illustrated by the previous claims. Many natural approaches to track the process fail; for example, we can set up a pair of partial differential equations to try and track the process analytically, but solving these seems to be essentially hopeless; all the results in this paper are proved using combinatorial strategies to approximately follow the process.

Extremal Combinatorics

5. *Antichain codes*, **Bulletin of the London Mathematical Society**, to appear. With B. Gunby, X. He and S. Spiro.

   Building on the classical solution of the Littlewood–Offord problem by Erdős (using Sperner's theorem), Sárközy–Szemerédi and Halász proved increasingly precise results about the anticoncentration of the randomly signed sum of a given set of nonzero reals in terms of the amount of 'arithmetic structure' within this set. Here, using a combination of coding-theoretic, combinatorial and probabilistic arguments, we prove an abstract, purely combinatorial generalisation of these results: if $A \subset 2^{[n]}$ is both an antichain and a distance-$(2r+1)$ code, then $|A| = O_r(2^n n^{-r-1/2})$, and this is best-possible up to implied constants. While all previously known proofs of Halász's theorem are Fourier-analytic, our proof is very much in the spirit of Erdős's combinatorial approach to the Littlewood–Offord problem.

6. *Subgraphs of large connectivity and chromatic number*, **Bulletin of the London Mathematical Society**, 54 (2022). With A. Girão.

   Alon, Kleitman, Thomassen, Saks and Seymour asked in 1987 if a graph of large chromatic number must contain a subgraph of both large connectivity and large chromatic number, and gave an affirmative answer with rather weak bounds. Motivated by more recent developments around Hadwiger's conjecture, Norin raised this problem again in the late 2010s, now asking for best-possible bounds. We establish essentially best-possible bounds for this problem: for each $k \in \mathbb{N}$, there exists an $f(k) \leq 7k$ such that every graph G with chromatic number at least $f(k) + 1$ contains a subgraph H with both connectivity and chromatic number at least k; also, $f(k)$ cannot be smaller than $2k - 3$.

This was the crucial missing ingredient needed to get past the so-called 'logarithmic barrier' for Hadwiger's conjecture, as was accomplished recently by **Delcourt and Postle** using our result.

7. *On symmetric 3-wise intersecting families*, **Proceedings of the American Mathematical Society**, 145 (2017). With D. Ellis.

   On regular 3-wise intersecting families, **Proceedings of the American Mathematical Society**, 146 (2018). With K. Frankston and J. Kahn.

   These papers resolve various problems raised by Babai, Cameron, Frankl and Kantor in the late 1970s and early 1980s about the dichotomy between symmetry and structure in extremal set theory. In the first paper, we prove that a 3-wise intersecting family in $\{0,1\}^n$ invariant under a transitive group of symmetries must have cardinality $o(2^n)$, and do this by establishing a new connection between extremal set theory and the 'sharp threshold' machinery of Bourgain, Friedgut and Kalai. In the second paper, we prove a new sharp threshold result that is applicable under very mild hypotheses, and use this in conjunction with the aforementioned connection to resolve other problems in the same circle.

8. *The multiplication table problem for bipartite graphs*, **Combinatorica**, 37 (2017). With J. Sahasrabudhe and I. Tomon.

   As the title suggests, this paper considers a generalisation of the Erdős multiplication table problem over the integers to the setting of bipartite graphs. Given a bipartite graph, its multiplication table is the set of sizes of its induced subgraphs. We prove that the multiplication table of a bipartite graph on m edges contains at least $m/(\log m)^{12}$ distinct elements, which is essentially tight in the sense that $(\log m)^{12}$ cannot be replaced by $(\log m)^{0.01}$. To see the connection to the multiplication table problem, note that the multiplication table of a complete bipartite graph is exactly the Minkowski product $[n] \cdot [n] = \{ab : 1 \leq a, b \leq n\}$, and Erdős's problem from 1955 asks for the size of this product set. While it is easy to show that $|[n] \cdot [n]| \gg n^2/(\log n)^2$ by counting the number of primes less than n, it does not seem to be easy to prove this in a purely combinatorial fashion without at least a weak form of the prime number theorem. Hence, to prove our result for general bipartite graphs, we develop a mixture of graph-theoretic, additive-combinatorial and number-theoretic tools.

Topological Combinatorics

9. *A universal exponent for homeomorphs*, **Israel Journal of Mathematics**, 243 (2021). With P. Keevash, J. Long and A. Scott.

   Simplicial homeomorphs and trace-bounded hypergraphs, **Discrete Analysis**, 2022:6 (2022). With J. Long and C. Yap.

   How many facets must an n-vertex k-complex have to necessarily contain a topological copy of a fixed k-complex S? Linial popularised this problem in the early 2000s and suggested a very intriguing possibility, namely that the answer depends only on the dimension k and not on S: there is a $\lambda_k > 0$ depending only on the dimension k such that $n^{k+1-\lambda_k}$ facets suffice for any fixed k-complex S as $n \to \infty$. This was previously known only for 1-complexes, being a result of Mader from the 1967. In the first paper, we dispose of the two-dimensional conjecture, and in the second paper, we settle this 'universal geometric exponent' conjecture in its full generality (i.e., in every dimension). The key new ingredient is a result about the Turán numbers of 'trace-bounded' hypergraphs, which strengthens previous results of Conlon, Fox and Sudakov from 2009; to prove this new result, we introduce a new

variant of the dependent random choice technique. For more about these results and the one below, see this blog post by **Gil Kalai**.

10. *Spanning surfaces in 3-graphs*, **Journal of the European Mathematical Society**, 24 (2022). With A. Georgakopoulos, J. Haslegrave and R. Montgomery.

    We prove a topological extension of Dirac's theorem suggested by Gowers in 2005: for any closed surface $\mathcal{S}$, we show that any two-dimensional simplicial complex on n vertices in which each pair of vertices belongs to at least $n/3+o(n)$ facets contains a homeomorph of $\mathcal{S}$ spanning all n vertices. This result is asymptotically sharp, and implies in particular that any 3-graph on n vertices with minimum codegree exceeding $n/3+o(n)$ contains a spanning triangulation of the sphere. To prove this result, we import some of the major techniques from extremal graph theory to simplicial topology: as a concrete example, we show how tools such as regularity, absorption and dependent random choice can be used to perform topological surgery on two-dimensional simplicial complexes.

11. *Disproportionate division*, **Bulletin of the London Mathematical Society**, 52 (2019). With L. Crew and S. Spirkl.

    How efficiently can we divide a cake, here $[0,1]$, among n agents each with a utility, here Borel probability measures, with different demands $\alpha_1, \alpha_2, \ldots, \alpha_n$ summing to 1? When all the agents have equal demands of $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1/n$, it is well-known that there exists a fair division with $n-1$ cuts, and this is optimal. For arbitrary demands on the other hand, folklore arguments from algebraic topology going back to the 1980s show that $O(n \log n)$ cuts suffice, and this had been the state of the art for decades. Here, we improve the state of affairs in two ways: we prove that disproportionate division may always be achieved with $3n-4$ cuts, and give an effective algorithm to construct such a division (and also show that $2n-2$ cuts may be necessary in general).