Symbolic Evaluation of Determinants and Rhombus Tilings of Holey Hexagons

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Plane Partitions (III): The Weak Macdonald Conjecture

George E. Andrews*

The Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.

Dedicated to the memory of Alfred Young and F.J.W. Whipple

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$$D_{0,0}(n).$$

Andrews's Result (1979)

Theorem. We have

$$D_{0,0}(n) = 2 \prod_{i=1}^{n-1} R_{0,0}(i),$$

in other words $R_{0,0}(n) = D_{0,0}(n+1)/D_{0,0}(n)$,

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in other words $R_{0,0}(n) = D_{0,0}(n+1)/D_{0,0}(n)$, where

$$R_{0,0}(2n) = \frac{\left(\mu+2n\right)_n \left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}}{\left(n\right)_n \left(\frac{\mu}{2}+n+\frac{1}{2}\right)_{n-1}},$$

$$R_{0,0}(2n-1) = \frac{\left(\mu+2n-2\right)_{n-1} \left(\frac{\mu}{2}+2n-\frac{1}{2}\right)_{n}}{\left(n\right)_n \left(\frac{\mu}{2}+n-\frac{1}{2}\right)_{n-1}},$$

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$$R_{0,0}(2n-1) = \frac{\left(\mu + 2n - 2\right)_{n-1} \left(\frac{\mu}{2} + 2n - \frac{1}{2}\right)_n}{\left(n\right)_n \left(\frac{\mu}{2} + n - \frac{1}{2}\right)_{n-1}},$$

and where $(a)_n$ denotes the Pochhammer symbol

$$(a)_n := a \cdot (a+1) \cdots (a+n-1).$$

Another question is the possibility of other general determinants of this nature. At first glance

$$E_m(\mu) = \det\left(\delta_{ij} + \binom{\mu+i+j}{i+1}\right)_{0\le i,j\le m-1}$$

looks interesting. Indeed it turns out that

$$E_1(\mu) = \mu + 1,$$

$$E_2(\mu) = (\mu+2)(\mu+1),$$

$$E_3(\mu) = \frac{(\mu+14)(\mu+3)(\mu+2)(\mu+1)}{12},$$

$$E_4(\mu) = \frac{(\mu+14)(\mu+9)(\mu+4)(\mu+3)(\mu+2)(\mu+1)}{72},$$

$$E_5(\mu) = \frac{(\mu+9)(\mu+5)(\mu+4)(\mu+3)(\mu+2)(\mu+1)(\mu^3+45\mu^2+722\mu+3432)}{8640}.$$

Emprically it seems reasonable to guess that

$$\frac{E_{2m}(\mu)}{E_{2m-1}(\mu)} = f_{2m,2m}(\mu-2),$$

George Andrews (1980): Macdonald's conjecture and descending plane partitions

Andrews's Conjecture (1980)

We let $D_{1,1}(n)$ denote Andrews's 1980 determinant

$$\det_{1\leqslant i,j\leqslant n}\left(\delta_{i,j}+\binom{\lambda+i+j}{j}\right),$$

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with $\lambda := \mu - 2$.

Conjecture. The following holds:

$$\frac{D_{1,1}(2n)}{D_{1,1}(2n-1)} = (-1)^{\frac{(n-1)(n-2)}{2}}\,2^n\,\frac{\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}\left(\frac{\mu}{2}+n\right)_{\lfloor (n+1)/2\rfloor}}{\left(n\right)_n\left(-\frac{\mu}{2}-2n+\frac{3}{2}\right)_{\lfloor (n-1)/2\rfloor}}$$

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Theorem. The following holds:

$$\begin{aligned}\frac{D_{1,1}(2n)}{D_{1,1}(2n-1)} &= (-1)^{\frac{(n-1)(n-2)}{2}}\,2^n\,\frac{\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}\left(\frac{\mu}{2}+n\right)_{\lfloor (n+1)/2\rfloor}}{\left(n\right)_n\left(-\frac{\mu}{2}-2n+\frac{3}{2}\right)_{\lfloor (n-1)/2\rfloor}}\\ &= 2^n\,\frac{\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}\left(\frac{\mu}{2}+n\right)_{\lfloor (n+1)/2\rfloor}}{\left(n\right)_n\left(\frac{\mu}{2}+\left\lfloor\frac{3n}{2}\right\rfloor+\frac{1}{2}\right)_{\lfloor (n-1)/2\rfloor}}\\ &= \frac{\left(\mu+2n\right)_n\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}}{\left(n\right)_n\left(\frac{\mu}{2}+n+\frac{1}{2}\right)_{n-1}}.\end{aligned}$$

$\longrightarrow$ Proven in 2013 using computer algebra.

$$
\begin{aligned}
D_{1,1}(1) &= \mu + 1 \\
D_{1,1}(2) &= (\mu+1)(\mu+2) \\
D_{1,1}(3) &= \tfrac{1}{12}(\mu+1)(\mu+2)(\mu+3)(\mu+14) \\
D_{1,1}(4) &= \tfrac{1}{72}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+9)(\mu+14) \\
D_{1,1}(5) &= \tfrac{1}{8640}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+9) \\
&\quad \times(\mu^3+45\mu^2+722\mu+3432) \\
D_{1,1}(6) &= \tfrac{1}{518400}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+6) \\
&\quad \times(\mu+8)(\mu+13)(\mu+15)(\mu^3+45\mu^2+722\mu+3432) \\
D_{1,1}(7) &= \tfrac{1}{870912000}(\mu+1)\circ\circ\circ(\mu+34)(\mu^3+47\mu^2+954\mu+5928) \\
D_{1,1}(8) &= \tfrac{1}{731566080000}(\mu+1)\circ\circ\circ(\mu+34)(\mu^3+47\mu^2+954\mu+5928) \\
D_{1,1}(9) &= \tfrac{1}{22122558259200000}(\mu+1)(\mu+2)\circ\circ\circ(\mu+21)^2 \\
&\quad \times(\mu^6+142\mu^5+8505\mu^4+277100\mu^3+5253404\mu^2+52937808\mu \\
D_{1,1}(10) &= \tfrac{1}{334493080879104000000}(\mu+1)(\mu+2)\circ\circ\circ(\mu+25)(\mu+27) \\
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Generalization of Andrews's Determinant

Definition: For $n, s, t \in \mathbb{Z}$, $n \geqslant 1$, and $\lambda := \mu - 2$ with μ being an indeterminate, we define $D_{s,t}(n)$ to be the following $(n \times n)$-determinant:

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- closed form for $D_{0,0}(n)$ (Andrews 1979)
- closed form for $D_{1,1}(2n)/D_{1,1}(2n-1)$ (Andrews 1980)
- monstrous conjecture for $D_{1,1}(n)$ (Christoph and Aek 2013)

Determinant with Kronecker-Delta

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Determinant with Kronecker-Delta

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where M_j^i denotes the (i, j)-minor of the corresponding matrix.

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where B^I_J denotes the matrix that is obtained by deleting all rows with indices in I and all columns with indices in J from the matrix

$$\left(\underbrace{\binom{\lambda + i + j + s + t - 2}{j + t - 1}}_{b_{i,j,s,t}} \right)_{1 \leqslant i,j \leqslant n} .$$

Lindström-Gessel-Viennot Lemma

Consider 'base' and 'destination' vertices of a directed acyclic graph denoted by $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, respectively.

Lindström-Gessel-Viennot Lemma

Consider 'base' and 'destination' vertices of a directed acyclic graph denoted by $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, respectively. For each path P, let $\omega(P)$ be the product of its edge weights. Let

$$e(a,b) = \sum_{P:a\to b} \omega(P) \qquad \text{and}$$

$$M = \begin{pmatrix} e(a_1,b_1) & e(a_1,b_2) & \cdots & e(a_1,b_n) \\ e(a_2,b_1) & e(a_2,b_2) & \cdots & e(a_2,b_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(a_n,b_1) & e(a_n,b_2) & \cdots & e(a_n,b_n) \end{pmatrix}.$$

Lindström-Gessel-Viennot Lemma

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Then the determinant of M is the signed sum over all n-tuples $P = (P_1, \ldots, P_n)$ of non-intersecting paths from A to B:

$$\det(M) = \sum_{(P_1, \ldots, P_n):\ A \to B} \operatorname{sign}(\sigma(P)) \prod_{i=1}^{n} \omega(P_i).$$

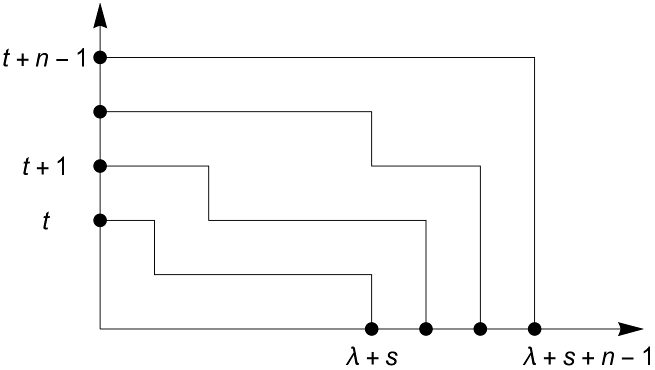
where σ denotes a permutation that is applied to B.

Lindström-Gessel-Viennot Lemma

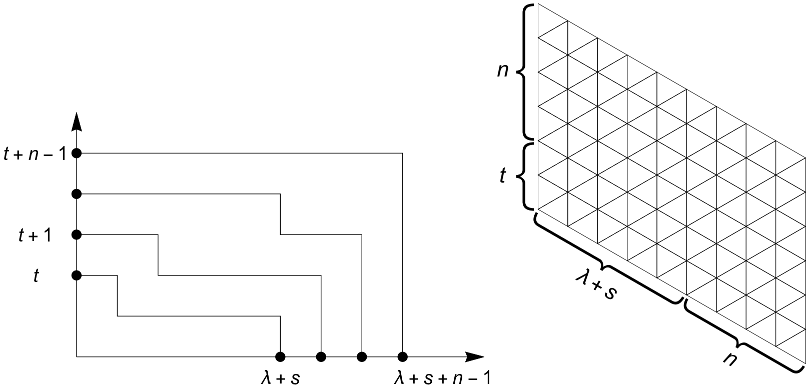
In our context, it implies that the determinant without the Kronecker-Delta

$$\det_{1\leqslant i,j\leqslant n}\left(\binom{\lambda+i+j+s+t-2}{j+t-1}\right)$$

counts n-tuples of non-intersecting paths in the lattice $\mathbb{N}^2$:



Lattice Paths ⟶ Rhombus Tilings



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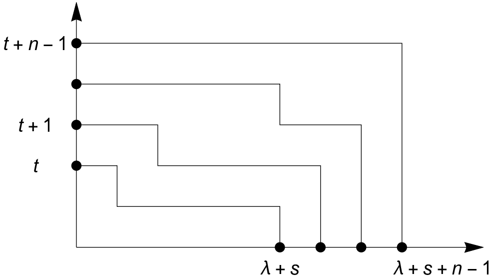

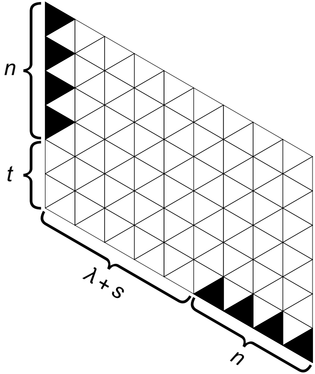

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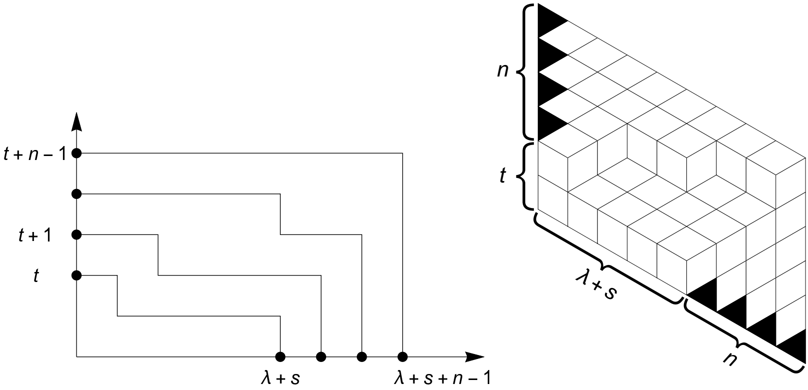
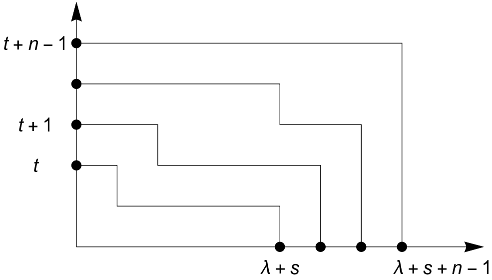
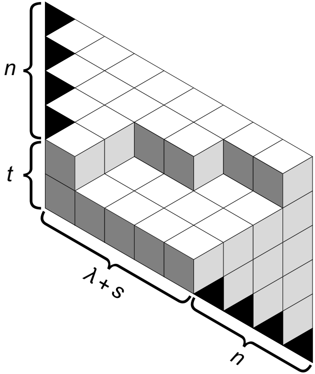


Lattice Paths $\longrightarrow$ Rhombus Tilings





Kronecker-Deltas on the Main Diagonal

If $s = t$, the previous formula for $D_{s,t}$ simplifies to

$$D_{s,s}(n) = \sum_{I \subseteq \{1,\dots,n\}} \det\bigl(B_I^I\bigr),$$

i.e., $D_{s,s}(n)$ is the sum of principal minors of the binomial matrix.

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Hence:

$D_{s,s}(n)$ counts all k-tuples of non-intersecting lattice paths, $k = 0, \dots, n$, and where the start and end points are given by the same k-subset.

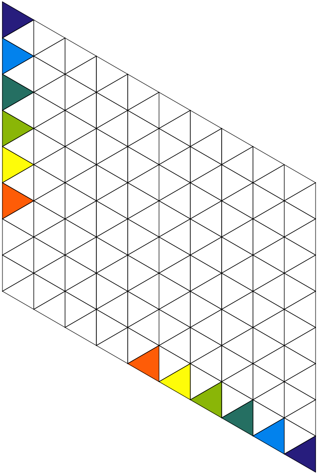
Kronecker-Deltas on the Main Diagonal

$s = 2$

$t = 2$

$n = 6$

$\lambda = 2$



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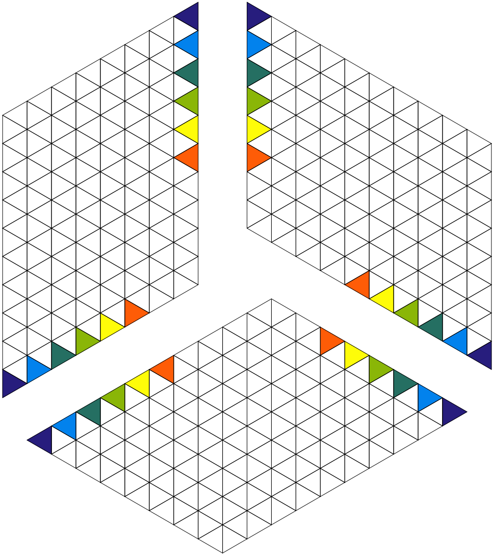

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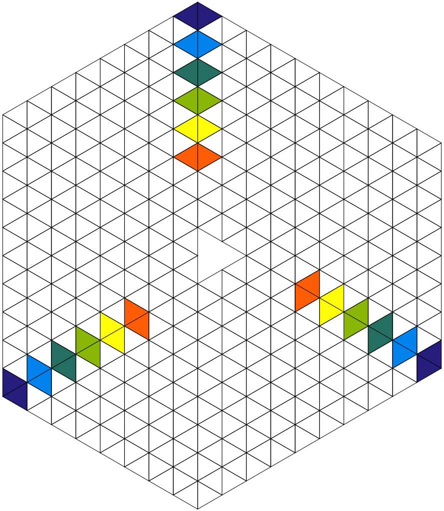

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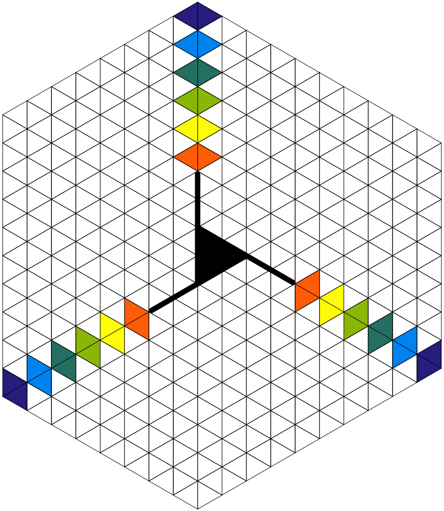

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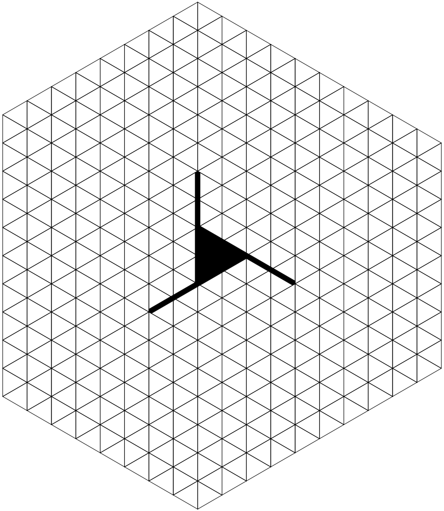

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Remark: This combinatorial interpretation is due to Krattenthaler and Ciucu (at least for $s = 0$).

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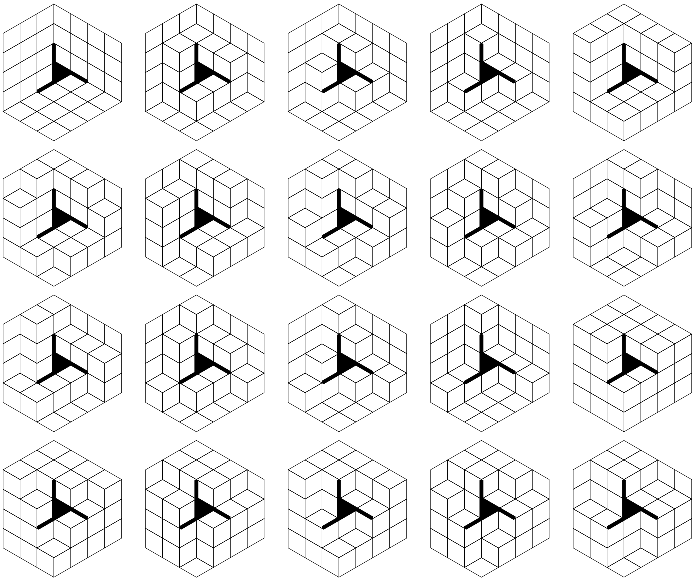
Example: For $s = t = 1$, $n = 2$, and $\lambda = 1$ we obtain

$$D_{1,1}(2)\big|_{\lambda \to 1} = \begin{vmatrix} 4 & 6 \\ 4 & 11 \end{vmatrix} = 20.$$

Cyclically Symmetric Rhombus Tilings of a Holey Hexagon



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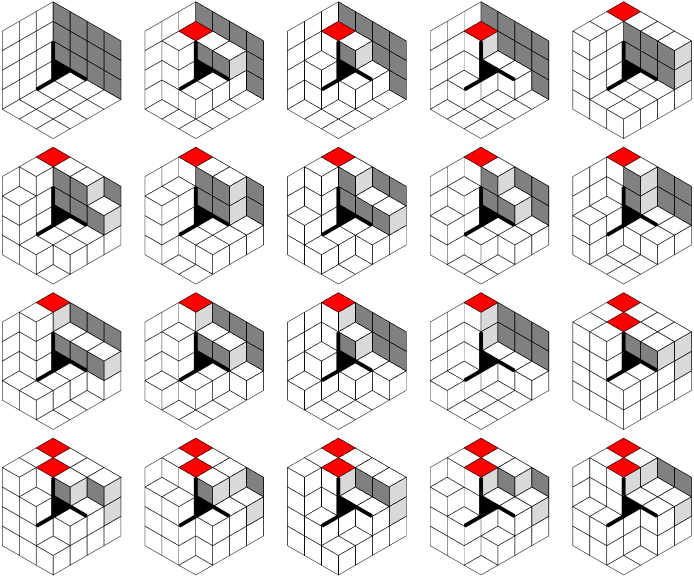

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$$\times \left(\prod_{i=1}^{\left\lfloor \frac{3}{2} \left\lfloor \frac{n}{2} \right\rfloor - 2 \right\rfloor} \Big(\mu + 2i + 2\left\lfloor \frac{3}{2}\left\lfloor \frac{n}{2} + 1 \right\rfloor \right\rfloor - 1\Big)^{2\left\lfloor \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{3}(i-1) \right\rfloor - 1} \right),$$

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$$F_m(n) = \left(\prod_{i=1}^{\left\lfloor \frac{1}{4}(n-1) \right\rfloor} (\mu + 2i + n + m)^{1-2i-m} \right)$$

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$$\begin{aligned} T(k) = {} & 55296k^6 + 41472(\mu - 1)k^5 + 384(30\mu^2 - 66\mu + 53)k^4 \\ & + 96(\mu - 1)(15\mu^2 - 42\mu + 61)k^3 \\ & + 4(19\mu^4 - 122\mu^3 + 419\mu^2 - 544\mu + 72)k^2 \\ & + (\mu - 1)(\mu^4 - 14\mu^3 + 101\mu^2 - 160\mu - 84)k \\ & + 2(\mu - 3)(\mu - 2)(\mu - 1)(\mu + 1), \end{aligned}$$

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Then for every positive integer n we have

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Desnanot-Jacobi-Dodgson Identity (DJD)

Theorem. Let $\left(m_{i,j}\right)_{i,j\in\mathbb{Z}}$ be an infinite sequence and denote by $M_{s,t}(n)$ the determinant of the $(n\times n)$-matrix whose upper left entry is $m_{s,t}$, more precisely the matrix $\left(m_{i,j}\right)_{s\leqslant i<s+n,t\leqslant j<t+n}$. Then:

$$M_{s,t}(n)M_{s+1,t+1}(n-2) = \\ M_{s,t}(n-1)M_{s+1,t+1}(n-1) - M_{s+1,t}(n-1)M_{s,t+1}(n-1).$$

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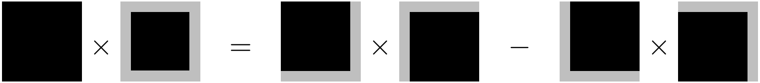
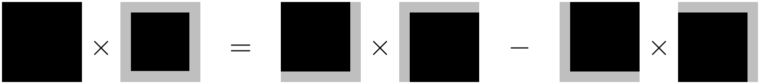
Schematically:



(DJD) for $D_{1,1}(n)$



By (DJD) we obtain a recurrence equation for $D_{1,1}(n)$:

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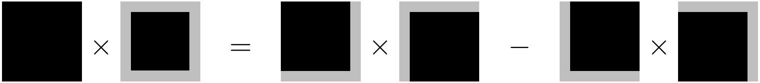

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We rewrite it slightly:

$$D_{1,1}(n) = \underbrace{\frac{D_{0,0}(n+1)}{D_{0,0}(n)}}_{= R_{0,0}(n)} D_{1,1}(n-1) + \frac{D_{1,0}(n)D_{0,1}(n)}{D_{0,0}(n)}.$$

$\longrightarrow$ Hence we need to know $D_{1,0}(n)$ and $D_{0,1}(n)$.

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- It has always dim. 1: $\ker(M^{(2n)}) = \langle c_n \rangle$ for $c_n \in \mathbb{Q}(\mu)^{2n}$.
- For each n, normalize the generator c_n (last component $= 1$).

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Task: We want to evaluate $D_{1,0}(n)$ and $D_{0,1}(n)$.

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- Use the holonomic systems approach to prove

$$M^{(2n)} \cdot c_n = 0, \text{ i.e., } \quad \sum_{j=1}^{2n} M_{i,j}^{(2n)} c_{n,j} = 0 \quad \text{for all } i \text{ and } n.$$

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We obtain product formulas for $D_{1,0}(2n-1)$ and $D_{0,1}(2n-1)$, by using a variant of Zeilberger's "HOLONOMIC ANSATZ".

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Method: "Pull out of the hat" a holonomic function $c_{n,j}$ and prove

$$
\begin{aligned}
c_{n,n} &= 1 && (n \geqslant 1),\\
\sum_{j=1}^{n} c_{n,j} a_{i,j} &= 0 && (1 \leqslant i < n),\\
\sum_{j=1}^{n} c_{n,j} a_{n,j} &= \frac{d_n}{d_{n-1}} && (n \geqslant 1).
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Then $\det\left(a_{i,j}\right)_{1 \leqslant i,j \leqslant n} = d_n$ holds.

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Example: For $D_{0,0}(2n)$ we obtain the following holonomic system of recurrence relations for $c_{n,j}$.

Recurrence Equations for $c_{n,j}$ for $D_{0,0}(2n)$

$$
\begin{aligned}
&\{(j+\mu+2n-3)(2\mu j^6+8nj^6-2j^6+3\mu^2 j^5-48n^2 j^5-12\mu j^5-24nj^5+9j^5+\\
&\mu^3 j^4+48n^3 j^4-11\mu^2 j^4-84\mu n^2 j^4+204n^2 j^4+21\mu j^4-20\mu^2 n j^4+38\mu n j^4-\\
&10nj^4-11j^4+216n^4 j^3-2\mu^3 j^3+312\mu n^3 j^3-408n^3 j^3+7\mu^2 j^3+28\mu^2 n^2 j^3+\\
&122\mu n^2 j^3-198n^2 j^3-2\mu j^3-9\mu^3 n j^3+68\mu^2 n j^3-113\mu n j^3+78nj^3-3j^3-\\
&864n^5 j^2-756\mu n^4 j^2+432n^4 j^2-\mu^3 j^2-112\mu^2 n^3 j^2-308\mu n^3 j^2+600n^3 j^2+\\
&11\mu^2 j^2-3\mu^3 n^2 j^2-66\mu^2 n^2 j^2+189\mu n^2 j^2-168n^2 j^2-23\mu j^2-2\mu^4 n j^2+\\
&15\mu^3 n j^2-28\mu^2 n j^2+33\mu n j^2-34nj^2+13j^2+864n^6 j+432\mu n^5 j+432n^5 j-\\
&144\mu^2 n^4 j+1116\mu n^4 j-1104n^4 j+2\mu^3 j-88\mu^3 n^3 j+384\mu^2 n^3 j-392\mu n^3 j-\\
&36n^3 j-10\mu^2 j-14\mu^4 n^2 j+45\mu^3 n^2 j+40\mu^2 n^2 j-317\mu n^2 j+270n^2 j+14\mu j-\\
&\mu^5 nj+3\mu^4 nj+17\mu^3 nj-89\mu^2 nj+112\mu nj-42nj-6j+432\mu n^6-864n^6+\\
&432\mu^2 n^5-1080\mu n^5+432n^5+144\mu^3 n^4-324\mu^2 n^4-156\mu n^4+456n^4+20\mu^4 n^3-\\
&18\mu^3 n^3-220\mu^2 n^3+470\mu n^3-204n^3+\mu^5 n^2+3\mu^4 n^2-37\mu^3 n^2+57\mu^2 n^2+\\
&36\mu n^2-60n^2+2\mu^4 n-18\mu^3 n+54\mu^2 n-62\mu n+24n)\mathbf{c_{n,j}}-(j+\mu-3)(2j+\mu-\\
&3)(j-2n+1)(\mu+4n-1)(j^4+2\mu j^3-6j^3+\mu^2 j^2-12n^2 j^2-9\mu j^2-6\mu n j^2+\\
&6nj^2+13j^2-3\mu^2 j-12\mu n^2 j+36n^2 j+13\mu j-6\mu^2 nj+24\mu nj-18nj-12j+\\
&2\mu^2-2\mu^2 n^2+20\mu n^2-24n^2-6\mu-\mu^3 n+11\mu^2 n-22\mu n+12n+4)\mathbf{c_{n,j+1}}+\\
&2(2j+\mu-2)n(2n+1)(-j+2n+1)(-j+2n+2)(j+\mu+2n-1)(\mu+4n-3)(\mu+\\
&4n-1)\mathbf{c_{n+1,j}},-(j+1)(2j+\mu)(j-2n)(j+\mu+2n-3)\mathbf{c_{n,j}}+(4j^4+8\mu j^3-8j^3+\\
&5\mu^2 j^2-8n^2 j^2-5\mu j^2-4\mu n j^2+12nj^2-8j^2+\mu^3 j+2\mu^2 j-8\mu n^2 j+8n^2 j-15\mu j-\\
&4\mu^2 nj+16\mu nj-12nj+12j+\mu^3-3\mu^2-2\mu^2 n^2+16n^2-2\mu-\mu^3 n+3\mu^2 n+\\
&8\mu n-24n+8)\mathbf{c_{n,j+1}}-(j+\mu-2)(2j+\mu-2)(j-2n+2)(j+\mu+2n-1)\mathbf{c_{n,j+2}}\}
\end{aligned}
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Back to $D_{1,1}(n)$

By (DJD) we had the recurrence

$$D_{1,1}(n) = R_{0,0}(n)D_{1,1}(n-1) + \frac{D_{1,0}(n)D_{0,1}(n)}{D_{0,0}(n)}.$$

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$$= R_{0,0}(n)D_{1,1}(n-1) + \frac{(\mu-1)}{2}\prod_{j=1}^{(n-1)/2}\frac{R_{1,0}(j)R_{0,1}(j)}{R_{0,0}(2j-1)R_{0,0}(2j)}.$$

Main Result

Theorem. Let μ be an indeterminate and let ρ_k be defined as $\rho_0(a,b)=a$ and $\rho_k(a,b)=b$ for $k>0$.

If n is an odd positive integer then

$$D_{1,1}(n)=\sum_{k=0}^{(n+1)/2}\rho_k\left(4(\mu-2),\frac{1}{(2k-1)!}\right)\frac{(\mu-1)_{3k-2}}{2\left(\frac{\mu}{2}+k-\frac{1}{2}\right)_{k-1}}$$
$$\times\left(\prod_{j=1}^{k-1}\frac{(\mu+2j+1)_{j-1}\left(\frac{\mu}{2}+2j+\frac{1}{2}\right)_{j-1}}{(j)_{j-1}\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j-1}}\right)^2$$
$$\times\left(\prod_{j=k}^{(n-1)/2}\frac{(\mu+2j)_j^2\left(\frac{\mu}{2}+2j-\frac{1}{2}\right)_j\left(\frac{\mu}{2}+2j+\frac{3}{2}\right)_{j+1}}{(j)_j\,(j+1)_{j+1}\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_j^2}\right)$$

If n is an even positive integer then... [similar formula]

Off-Diagonal Kronecker-Deltas

Now let's look at the situation $s \neq t$.

General formula:

$$D_{s,t}(n) = \begin{cases} \sum_{I \subseteq \{1,\dots,n-s+t\}} (-1)^{(s-t)\cdot|I|} \det\big(B^{I}_{I+s-t}\big) & \text{if } s \geqslant t \\ \sum_{I \subseteq \{1,\dots,n-t+s\}} (-1)^{(s-t)\cdot|I|} \det\big(B^{I+s-t}_{I}\big) & \text{if } s \leqslant t, \end{cases}$$

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Remark:

For a particular (cyclically symmetric) rhombus tiling, the number of unit segments which are not crossed by a horizontal rhombus corresponds to the cardinality of the set I and hence its parity determines whether this tiling is counted with weight $+1$ or with weight -1.

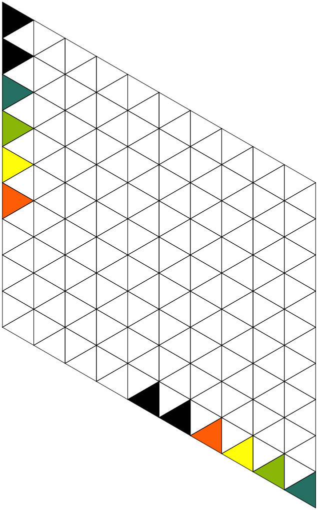
Off-Diagonal Kronecker-Deltas

$s = 1$

$t = 3$

$n = 6$

$\lambda = 3$



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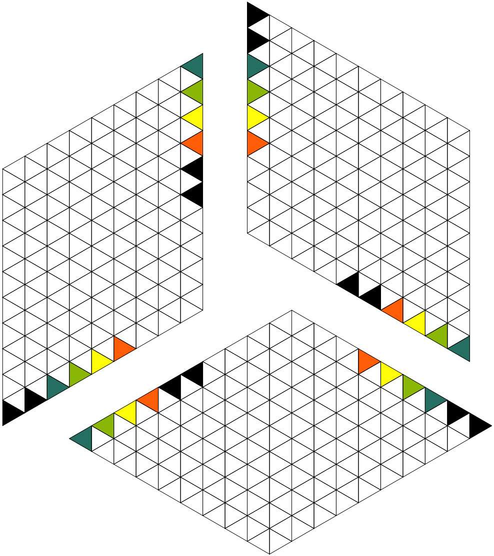

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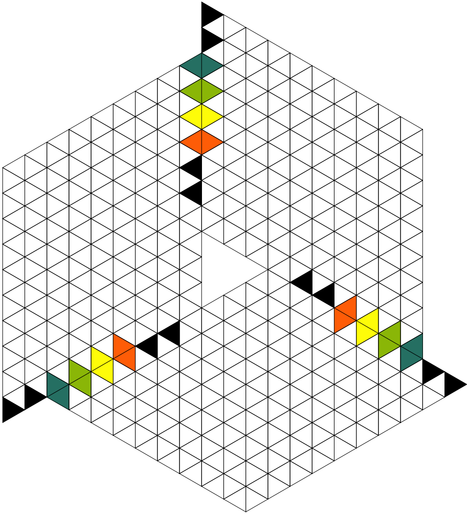

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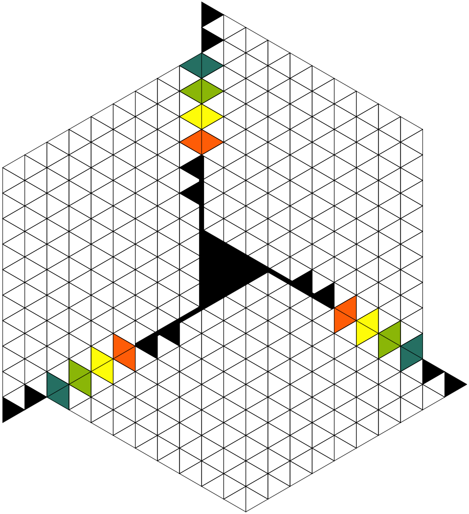

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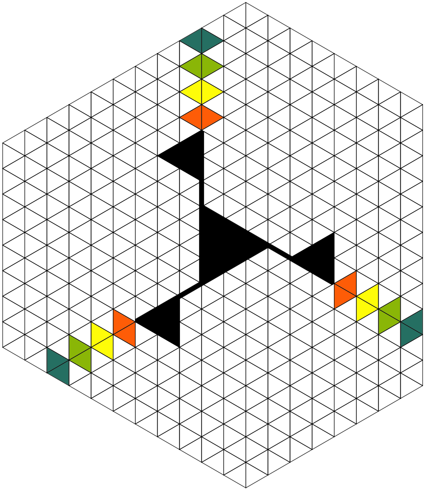

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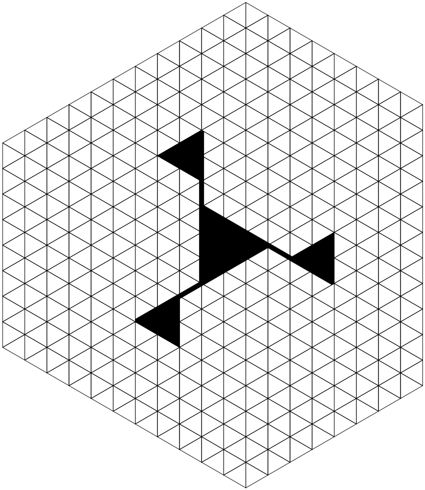
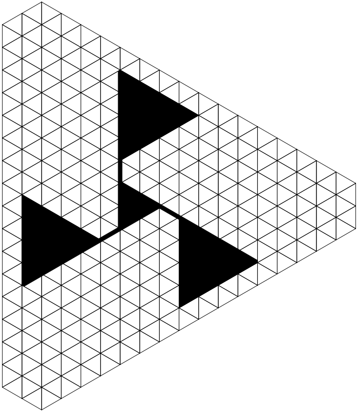
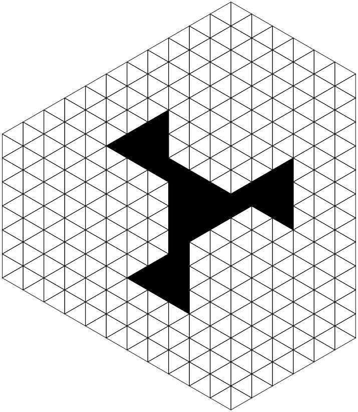


Off-Diagonal Kronecker-Deltas

Example: Shapes for different choices parameters.



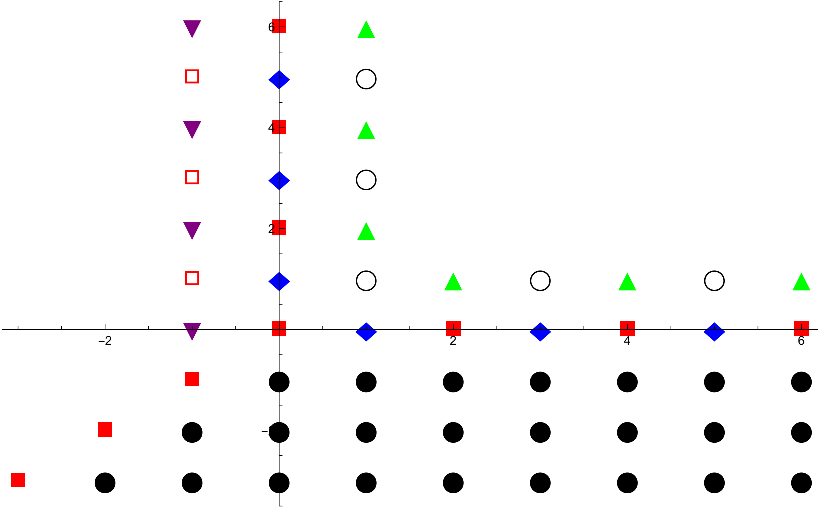
$s = 5$, $t = 1$, $n = 5$, $\lambda = 2$



$s = -1$, $t = 2$, $n = 6$, $\lambda = 4$

More Results

We find closed-form evaluations of some infinite 1-dimensional families of $D_{s,t}(n)$.



Example of an Infinite Family (A): Red Squares

Family A: can be reduced to the base case $D_{0,0}(n)$:

$$D_{2r,0}(n) = D_{0,0}(n-2r)\big|_{\mu \to \mu+6r}$$

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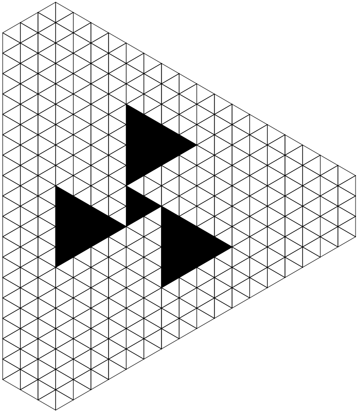

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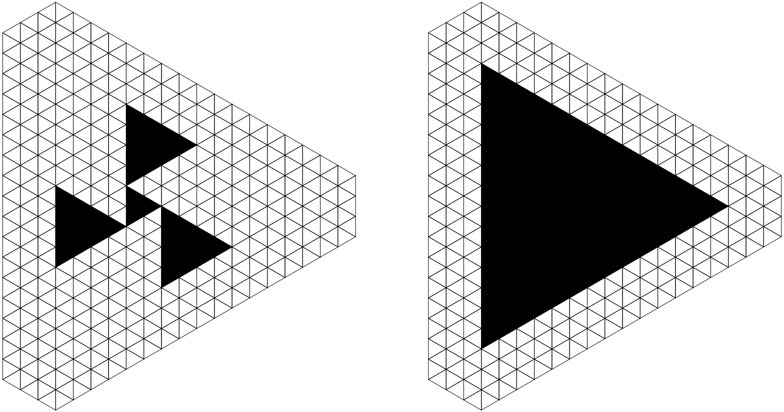

Example of an Infinite Family (B): Blue Diamonds

Family B: If $n \geqslant 2r$ is an even number, then $D_{2r-1,0}(n) = 0$.

$s = 3$

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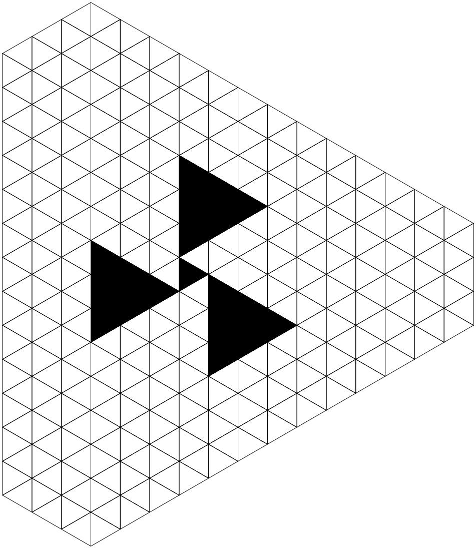
$n = 6$

$\lambda = 1$



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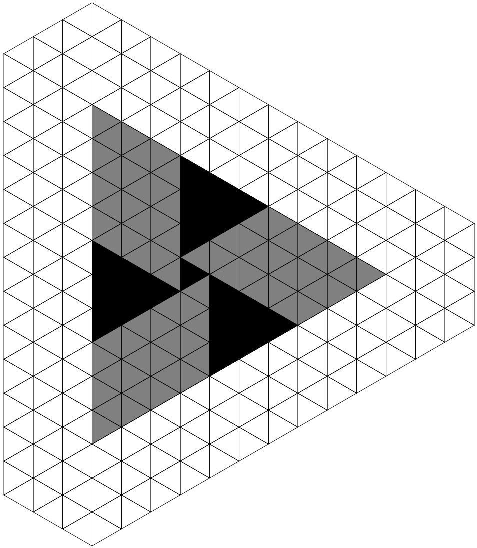

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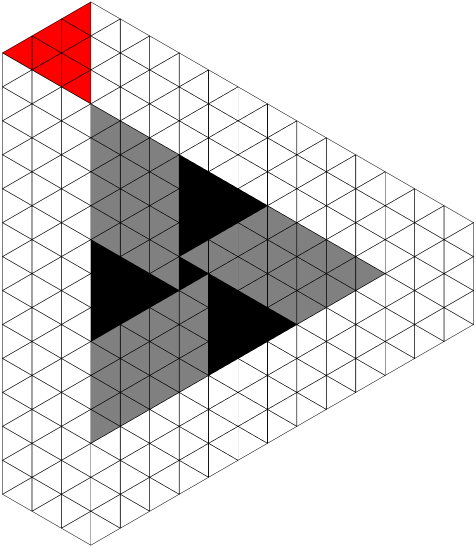

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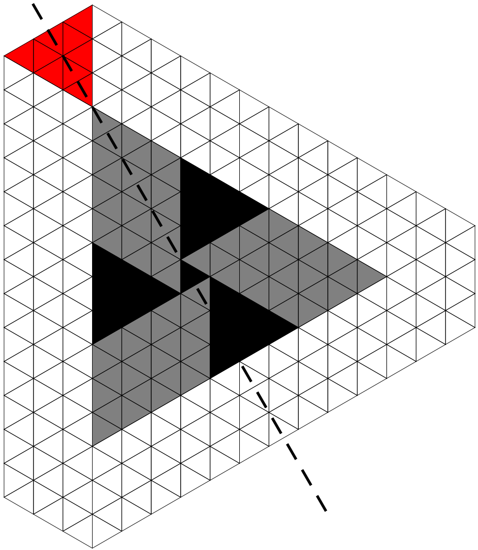


Example of an Infinite Family (B): Blue Diamonds

Theorem. Let μ be an indeterminate, and let r and n be positive integers. If n is an odd number, then

$$D_{2r-1,0}(n) = \prod_{i=r}^{(n-1)/2} \big(-R_{2r-1,0}(i)\big),$$

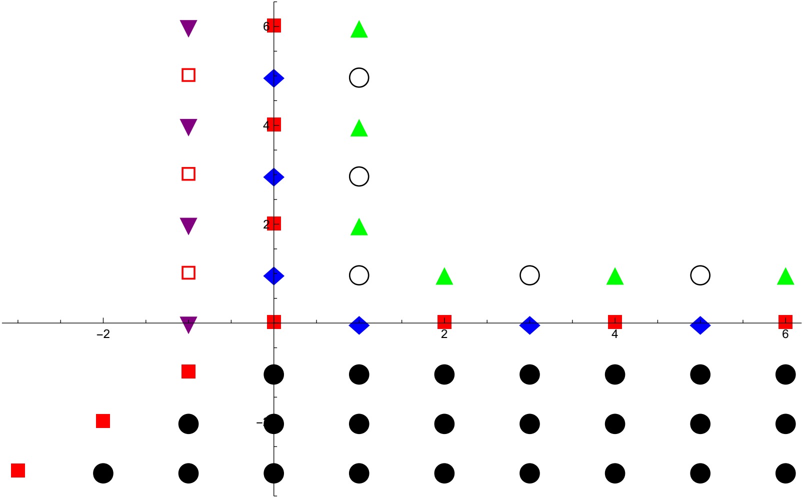
where $R_{2r-1,0}(n) =$

$$\frac{\big(\mu+2n+4r-4\big)_{n-r+1}\,\big(\mu+2n+4r-3\big)_{n-r}\,\big(\frac{\mu}{2}+2n+r-\frac{1}{2}\big)_{n-r}^{2}}{\big(n-r+1\big)_{n-r+1}\,\big(n-r+1\big)_{n-r}\,\big(\frac{\mu}{2}+n+2r-\frac{3}{2}\big)_{n-r}^{2}}$$

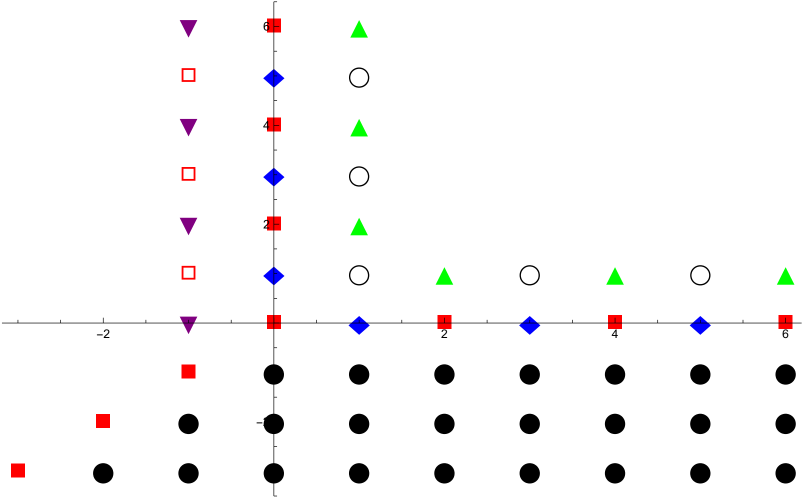
i.e., $R_{2r-1,0}(n) = D_{2r-1,0}(2n+1)/D_{2r-1,0}(2n-1)$ for $n \geqslant r$. If $n \geqslant 2r$ is an even number, then $D_{2r-1,0}(n) = 0$. Moreover,

$$D_{0,2r-1}(n) = \left(\prod_{i=0}^{n-1} \frac{\big(\mu+i-1\big)_{2r-1}}{\big(i+1\big)_{2r-1}}\right) \cdot D_{2r-1,0}(n).$$

To Do List

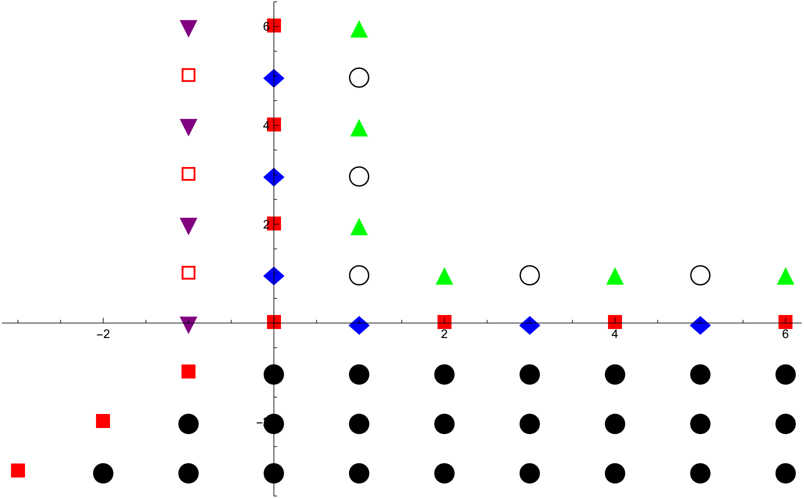


To Do List



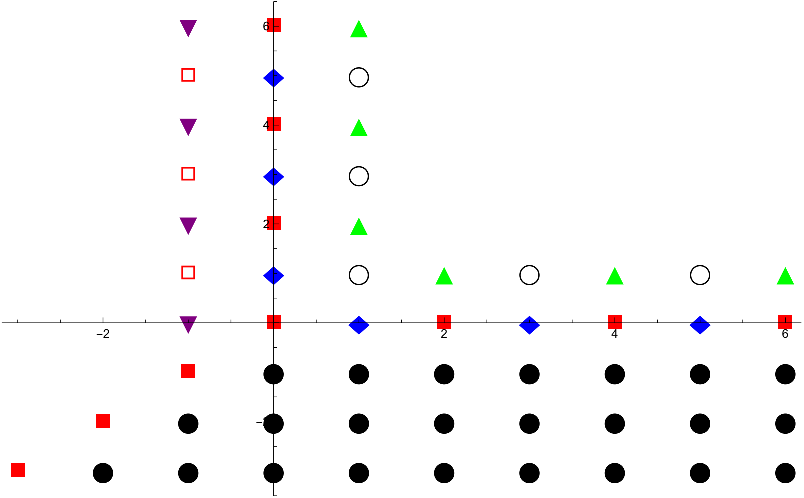
✓ Done: Red Squares and Blue Diamonds (Combinatorics)

To Do List



- ✓ Done: Red Squares and Blue Diamonds (Combinatorics)
- ✓ Follows: Hollow Squares and Hollow Circles (DJD)

To Do List



- ✓ Done: Red Squares and Blue Diamonds (Combinatorics)
- ✓ Follows: Hollow Squares and Hollow Circles (DJD)
- ✓ To Do: Purple and Green Triangles

Reference

Christoph Koutschan and Thotsaporn Thanatipanonda:
A Curious Family of Binomial Determinants That Count Rhombus Tilings of a Holey Hexagon

- arxiv:1709.02616
- http://www.koutschan.de/data/det2/