Spring 2016, Solutions to Complex Variables / Calculus

Question 1 Set

$$f(z) := \frac{e^{i2\pi z\xi}}{1+4z^2}$$

and note that

$$f(x) = \frac{\cos(2\pi z\xi) + i\sin(2\pi z\xi)}{1+4x^2}$$

when $x \in \mathbb{R}$. We can find the integral by (1) integrating

$$\int_{[-R,R]} f(z)\,dz + \int_{C_R} f(z)\,dz$$

using the Residue Theorem where $C_R = \{Re^{i\theta} : 0 < \theta < \pi\}$, (2) taking $R \to \infty$ and noting that the integral along C_R vanishes as $R \to \infty$, and (3) noting that the integral along the real axis becomes the desired integral as $R \to \infty$.

(1) The function $f(z)$ is meromorphic and its two poles occur at $\pm i/2$; its residue at $i/2$ is

$$\frac{e^{i2\pi\cdot i\xi}}{2(i/2)+i} = \frac{e^{i2\pi\cdot i\xi}}{2i}$$

whence the integral of $f(z)$ along the proposed contour is $\pi e^{-2\pi\xi}$.

(2)

$$\begin{aligned} \left|\int_{C_R} f(z)\,dz\right| &\le (\pi R)\frac{|e^{2\pi i\xi R(\cos\theta + i\sin\theta)}|}{4R^2-1} \\ &\le (\pi R)\max_{0<\theta<\pi}\frac{e^{-2\pi\xi R\sin\theta}}{4R^2-1} \\ &\to 0 \text{ as } R \to \infty \end{aligned}$$

Thus we have $\int_{\mathbb{R}} f(x)\,dx = \pi e^{-2\pi\xi}$; taking real parts, this same value is the value of the desired integral.

Question 2 Let

$$\phi(z) := \frac{f(0) - z}{1 - \overline{f(0)}f(z)}$$

It is well-known that ϕ gives a holomorphic function from the unit disc to the unit disc and interchanges the origin and $f(0)$; hence the function

$$g(z) := \frac{f(0) - f(z)}{1 - \overline{f(0)}f(z)}$$

given by $\phi \circ f$ is holomorphic from the unit disc to the unit disc. Since $g(0) = 0$ Schwarz's lemma applies, whence

$$|f(0) - f(z)| \le |z||1 - \overline{f(0)}f(z)|$$

which implies

$$|f(0)| - |f(z)| \leq |z| + |f(0)||f(z)||z|$$

$$|f(0)| - |z| \leq |f(z)|(1 + |f(0)||z|)$$

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)|$$

On the other hand, from the fact that

$$|f(z)| - |f(0)| \leq |z||1 - \overline{f(0)}f(z)|$$

one obtains

$$|f(z)| - |f(0)| \leq |z|(1 + |f(0)f(z)|)$$

$$|f(z)|(1 - |zf(0)|) \leq |z| + |f(0)|$$

$$|f(z)| \leq \frac{|f(0)| + |z|}{(1 - |f(0)||z|)}$$

(the denominators are positive in each case and therefore there is no need to change the sign of the inequality)

Question 3 The vector field $F(x, y, z)$ is equal to the curl of

$$G : (x, y, z) \to (0, xz^4, 0)$$

and so

$$\begin{aligned} \iint_S F(x, y, z)\, dS &= \iint_S \text{curl } G\, dS \\ &= \oint G \cdot dr \end{aligned}$$

where $r(t)$ traces the boundary curve of S with parameterization

$$\begin{aligned} r(\theta) &:= (3\cos\theta, 3\sin\theta, 9) \\ \oint G \cdot dr &= \int_0^{2\pi} (0, 9^4 \cdot 3\cos\theta, 0)(-3\sin\theta, 3\cos\theta, 0)\, d\theta \\ &= \int_0^{2\pi} 9^5 \cdot \cos^2\theta\, d\theta \\ &= \frac{9^5}{2} \int_0^{2\pi} \cos(2\theta) + 1\, d\theta \\ &= \pi \cdot 9^5 \end{aligned}$$

Question 4 We have for arbitrary $z_0 \in \mathbb{C}$, through a change of variables,

$$\begin{aligned}\iint_{\mathbb{C}} |f(z)|^2 \, dx \, dy &= \lim_{R\to\infty} \int_0^R \int_0^{2\pi} |f(z_0 + re^{i\theta})|^2 \, d\theta \cdot r \, dr \\ &\geq |\lim_{R\to\infty} \int_0^R \int_0^{2\pi} f(z_0 + re^{i\theta}) \, d\theta \cdot r \, dr| \\ &\geq |\lim_{R\to\infty} \int_0^R 2\pi \cdot f(z_0) \cdot r \, dr|\end{aligned}$$

This last equality holds by the Mean Value Theorem; we have a contradiction unless $f(z_0) = 0$. Since z_0 was arbitrary the conclusion holds.

Question 5 Consider the conformal mappings

$$\begin{aligned}f_1(z) &:= \frac{(z-1)(i+1)}{(z+1)(i-1)} \\ f_2(z) &: -\frac{1}{4}(z^2+1)\end{aligned}$$

We propose that

$$f_2 \circ f_1 := -\frac{1}{4} \frac{(z-1)^2 2i + (z+1)^2(-2i)}{(z+1)^2(-2i)} = -\frac{z}{(z+1)^2}$$

gives the desired map. The map f_1 takes the unit disc to the upper half-plane by the fact that the unit circle is mapped to a generalized circle (possibly including a line) by Möbius mappings and that we have points 1, i, and -1 taken to 0, 1, and ∞, respectively. The map f_2 takes the upper half-plane to the plane minus the positive reals, then rotates it and shifts it to the left by 1/4.

On the other hand, $f_2 \circ f_1(0) = 0$ and

$$(f_2 \circ f_1)'(0) = \frac{(1)^2(-1) - (0)2(0+1)}{(0+1)^4}$$