Spring 2007 Qualifying Exam, Selected Solutions

DAY I

Question 2 Let $g(z) = 100z^4$; we propose $f \equiv g$. Now $f - g$ is holomorphic on the unit disc and

$$\begin{aligned} g(i/n) &= 100(i/n)^4 \\ &= \frac{100}{n^4} \\ &= f(i/n) \end{aligned}$$

So $f - g = 0$ on a set of points converging within the open unit circle (i.e., to $z = 0$). But this fact indicates that $f - g \equiv 0$ by the uniqueness of analytic continuations; hence $f \equiv g$.

Question 5 Since f is meromorphic in $\mathbb{C}$, f has a Laurent series with centre $z = 0$; this Laurent series may or may not contain negative powers. Suppose that it does; then there is a least n such that $a_{-n} \neq 0$ in the Laurent series representation

$$\sum_{n \in \mathbb{Z}} a_n z^n$$

Because Laurent series are unique, we will have that $g(z) := z^{n-1} f(z)$ will have a non-zero residue at $z = 0$, whence, by the Residue Theorem,

$$\int_{B(0,\epsilon} z^{n-1} f(z)\, dz = L \neq 0$$

where $B(0, \epsilon)$ is the circle of radius ϵ about the origin. By the inequality in the hypothesis, we have

$$\begin{aligned} \Big| \int_{B(0,\epsilon)} z^{n-1} f(z)\, dz \Big| &\leq (2\pi\epsilon)\epsilon^{n-1} \max_{z \in B(0,\epsilon)} |f(z)| \\ &= (2\pi\epsilon)(c\epsilon^{n-\frac{3}{2}} + a\epsilon^{n+\frac{1}{2}}) \\ &= (2\pi)(c\epsilon^{n-\frac{1}{2}} + a\epsilon^{n+\frac{3}{2}}) \end{aligned}$$

But when $n \geq 1$ this last quantity approaches zero as $\epsilon \to 0$, in contradiction of our choice of n.

Thus the Laurent expansion of f at zero has no negative powers; this means that f is holomorphic there, thus entire, whence we simply have to apply the Cauchy inequalities to obtain the conclusion.

Indeed, if $n \geq 2$:

$$\begin{aligned} |f^{(n)}(0)| &\leq \lim_{R\to\infty} n! \frac{c|R|^{-1/2} + a|R|^{3/2}}{R^n} \\ &= \lim_{R\to\infty} n! \left(c|R|^{\frac{-2n-1}{2}} + a|R|^{\frac{-2n+3}{2}} \right) \\ &\to 0 \end{aligned}$$

as $R \to 0$; since f is entire it equals its Taylor series everywhere and the conclusion follows.

DAY II

Question 2 Suppose $z_1 = 0$; then Schwarz' lemma applies. Since $|f(z_2)| = |z_2| \neq 0$ we have that f is a rotation. Finally, since f takes a non-origin point to itself we must have $f(z) \equiv z$. Of course $z_2 = 0$ implies $f(z) \equiv z$ with a nearly identical proof.

On the other hand, consider

$$g(z) := \phi_{z_2} \circ f \circ \phi_{z_2}$$

where

$$\phi_a(z) := \frac{a - z}{1 - \overline{a}z}$$

For $|a| < 1$ one can observe that ϕ_a is its own inverse and that $\phi_a : D \to D$ is holomorphic. Also $g(0) = 0$, meaning that Schwarz applies, and finally

Question 9